Using the *Larch Prover* to Derive Relationships for Cubical Complexes

A statement of project intent

by: Robert Ellis
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Project Summary

In the proof of the computational formula for the optimal number of faces of a cubical complex which lies in a cube, there are a number of steps which use a complex induction or iteration process in order to achieve a desired goal. It is my intent to use the *Larch Prover*(LP) to verify or expand these processes in a formal setting.

Description of the *Larch Prover*

The *Larch Prover* is a symbolic logic system for axiomatizing and proving theorems. Theorem conditions are written as a list of axioms encoded in the program file. Desired results of theorems are also written, along with optional user-defined routes of achieving those results. LP is an interactive theorem prover designed with the notion that most attempts at proofs will be unsuccessful. Thus the user interface allows user guidance of the theorem proving process, especially in directing LP to follow a new structure of proof, splitting a proof into cases, and adding more information. LP can be useful for processing theorems with many conditions and results, especially as a guide through the pathways of a particularly complex proof.

Intended Material for Proof

Bernt Lindström obtains in [3] a formula for the optimal number of faces of cubical complexes which lie in cubes, and presents a mostly complete proof of this result. The result can be stated as follows: given a cubical complex C which has m faces of dimension r, the maximum(minimum) number of faces of dimension s that C can have is $m_{(s/r)} + (m - m_{(r/r)})^{(s/r)}$ if $s > r (s < r)$. The proof, which is elucidated and corected in [1], contains several sections of somewhat complex inductions and iterative manipulations for which LP might be a handy organization tool. A significant part of the proof rests upon a theorem called the *crucial inclusion*, which is stated as

$$\partial R S_r \subseteq R \partial S_r,$$

where S_r is a set of r-faces of a cubical complex C; R is the face-replacement operator which returns the same number of faces, but lowest possible in a total ordering defined on the faces; and ∂ is the boundary operator which yields the $(r-1)$-faces in the boundary of a set of r-faces. The first section of the proof of the crucial inclusion involves the development of an eventually constant sequence of r-faces closely related to S_r but consecutively lower in the total order. It is necesary to prove that the result of this sequence has a smaller boundary set than the first element S_r; this requirement is proved via inclusion-exclusion and induction, and is a good candidate for LP's machinery. The second section of the proof concerns an iterated face-substitution operation, which starts with the final element of the sequence and terminates with RS_r, the lowest $|S_r|$ r-faces in the total order. Once again, LP could be used to ascertain that the size of the boundary set does not increase. The final section of [1] proves, using the optimality conditions, a complete characterization of cubical complexes which lie in cubes in terms of the number of faces of a given dimension. This proof can be viewed as iterative or inductive; either way it is appropriate for investigation using LP.

References

1. ELLIS, R. B. , A Kruskal-Katona Theorem for Cubical Complexes (1996).

2. GARLAND, S. J. , and GUTTAG, J. V. , A Guide to LP, The Larch Prover. Massachusetts Institute of Technology. 1993.

3. LINDSTRÖM, B. , The optimal number of faces in cubical complexes. Arkiv för Matematik, Bd. 8, nr. 24, pp. 245-257 (1972).