CS 5114: Theory of Algorithms

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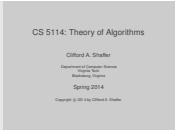
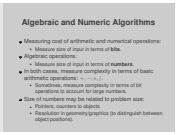


Title page

Students should be familiar with inductive proofs, recursion, data structures, and programming at the CS3114 level.

Algebraic and Numeric Algorithms

- Measuring cost of arithmetic and numerical operations:
  - Measure size of input in terms of **bits**.
- Algebraic operations:
  - Measure size of input in terms of **numbers**.
- In both cases, measure complexity in terms of basic arithmetic operations: $+, -, *, /$.
  - Sometimes, measure complexity in terms of bit operations to account for large numbers.
- Size of numbers may be related to problem size:
  - Pointers, counters to objects.
  - Resolution in geometry/graphics (to distinguish between object positions).



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Exponentiation

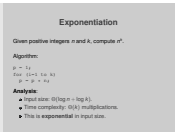
Given positive integers n and k, compute n^k.

Algorithm:

```
p = 1;
for (i=1 to k)
  p = p * n;
```

Analysis:

- Input size: $\Theta(\log n + \log k)$.
- Time complexity: $\Theta(k)$ multiplications.
- This is **exponential** in input size.



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Faster Exponentiation

Write k as:

$$k = b_t 2^t + b_{t-1} 2^{t-1} + \cdots + b_1 2 + b_0, b \in \{0, 1\}.$$

Rewrite as:

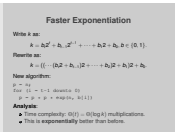
$$k = ((\cdots (b_t 2 + b_{t-1})2 + \cdots + b_2)2 + b_1)2 + b_0.$$

New algorithm:

```
p = n;
for (i = t-1 downto 0)
  p = p * p * exp(n, b[i])
```

Analysis:

- Time complexity: $\Theta(t) = \Theta(\log k)$ multiplications.
- This is **exponentially** better than before.



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Greatest Common Divisor

- The Greatest Common Divisor (GCD) of two integers is the greatest integer that divides both evenly.
- Observation: If k divides n and m, then k divides $n - m$.
- So,

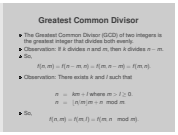
$$f(n, m) = f(n - m, n) = f(m, n - m) = f(m, n).$$

- Observation: There exists k and l such that

$$\begin{aligned} n &= km + l \text{ where } m > l \geq 0. \\ n &= \lfloor n/m \rfloor m + n \mod m. \end{aligned}$$

- So,

$$f(n, m) = f(m, l) = f(m, n \mod m).$$



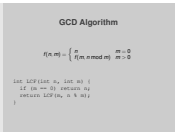
Assuming $n > m$, then $n = ak$, $m = bk$, $n - m = (a - b)k$ for integers a, b.

This comes from definition of mod .

GCD Algorithm

$$f(n, m) = \begin{cases} n & m = 0 \\ f(m, n \bmod m) & m > 0 \end{cases}$$

```
int LCF(int n, int m) {
  if (m == 0) return n;
  return LCF(m, n % m);
}
```

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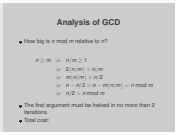
Analysis of GCD

- How big is n mod m relative to n?

$$\begin{aligned} n \geq m &\Rightarrow n/m \geq 1 \\ &\Rightarrow 2\lfloor n/m \rfloor > n/m \\ &\Rightarrow m\lfloor n/m \rfloor > n/2 \\ &\Rightarrow n - n/2 > n - m\lfloor n/m \rfloor = n \bmod m \\ &\Rightarrow n/2 > n \bmod m \end{aligned}$$

- The first argument must be halved in no more than 2 iterations.
- Total cost:



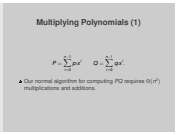
Can split in half log n times. So $2 \log n$ is upper bound.

Note that this is linier on problem size, since problem size is $2 \log n$ (2 numbers).

Multiplying Polynomials (1)

$$P = \sum_{i=0}^{n-1} p_i x^i \qquad Q = \sum_{i=0}^{n-1} q_i x^i.$$

- Our normal algorithm for computing PQ requires $\Theta(n^2)$ multiplications and additions.



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Multiplying Polynomials (2)

- Divide and Conquer:

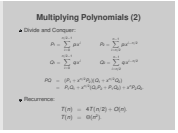
$$P_1 = \sum_{i=0}^{n/2-1} p_i x^i \qquad P_2 = \sum_{i=n/2}^{n-1} p_i x^{i-n/2}$$

$$Q_1 = \sum_{i=0}^{n/2-1} q_i x^i \qquad Q_2 = \sum_{i=n/2}^{n-1} q_i x^{i-n/2}$$

$$\begin{aligned} PQ &= (P_1 + x^{n/2}P_2)(Q_1 + x^{n/2}Q_2) \\ &= P_1Q_1 + x^{n/2}(Q_1P_2 + P_1Q_2) + x^nP_2Q_2. \end{aligned}$$

- Recurrence:

$$\begin{aligned} T(n) &= 4T(n/2) + O(n). \\ T(n) &= \Theta(n^2). \end{aligned}$$



Do this to make the subproblems look the same.

Multiplying Polynomials (3)

Observation:

$$(P_1 + P_2)(Q_1 + Q_2) = P_1Q_1 + (Q_1P_2 + P_1Q_1) + P_2Q_2$$

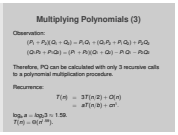
$$(Q_1P_2 + P_1Q_2) = (P_1 + P_2)(Q_1 + Q_2) - P_1Q_1 - P_2Q_2$$

Therefore, PQ can be calculated with only 3 recursive calls to a polynomial multiplication procedure.

Recurrence:

$$\begin{aligned} T(n) &= 3T(n/2) + O(n) \\ &= aT(n/b) + cn^1. \end{aligned}$$

$\log_b a = log_2 3 \approx 1.59$.
$T(n) = \Theta(n^{1.59})$.



In the second equation, the sums in the first term are half the original problem size, and the second two terms were needed for the first equation.

$$PQ = P_1Q_1 + X^{n/2}((P_1 + P_2)(Q_1 + Q_2) - P_1Q_1 - P_2Q_2) + x^nP_2Q_2$$

A significant improvement came from algebraic manipulation to express the product in terms of 3, rather than 4, smaller products.

Matrix Multiplication

Given: $n \times n$ matrices A and B.

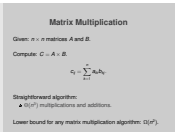
Compute: $C = A \times B$.

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

Straightforward algorithm:

- $\Theta(n^3)$ multiplications and additions.

Lower bound for any matrix multiplication algorithm: $\Omega(n^2)$.



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Strassen's Algorithm

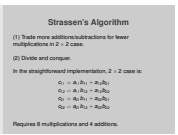
(1) Trade more additions/subtractions for fewer multiplications in 2×2 case.

(2) Divide and conquer.

In the straightforward implementation, 2×2 case is:

$$\begin{aligned} c_{11} &= a_{11}b_{11} + a_{12}b_{21} \\ c_{12} &= a_{11}b_{12} + a_{12}b_{22} \\ c_{21} &= a_{21}b_{11} + a_{22}b_{21} \\ c_{22} &= a_{21}b_{12} + a_{22}b_{22} \end{aligned}$$

Requires 8 multiplications and 4 additions.



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Another Approach (1)

Compute:

$$\begin{aligned} m_1 &= (a_{12} - a_{22})(b_{21} + b_{22}) \\ m_2 &= (a_{11} + a_{22})(b_{11} + b_{22}) \\ m_3 &= (a_{11} - a_{21})(b_{11} + b_{12}) \\ m_4 &= (a_{11} + a_{12})b_{22} \\ m_5 &= a_{11}(b_{12} - b_{22}) \\ m_6 &= a_{22}(b_{21} - b_{11}) \\ m_7 &= (a_{21} + a_{22})b_{11} \end{aligned}$$

Another Approach (1)

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$m_1 = (a_{12} - a_{22})(b_{21} + b_{22})$
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$m_3 = (a_{11} - a_{21})(b_{11} + b_{12})$
$m_4 = (a_{11} + a_{12})b_{22}$
$m_5 = a_{11}(b_{12} - b_{22})$
$m_6 = a_{22}(b_{21} - b_{11})$
$m_7 = (a_{21} + a_{22})b_{11}$

no notes

Another Approach (2)

Then:

$$\begin{aligned} c_{11} &= m_1 + m_2 - m_4 + m_6 \\ c_{12} &= m_4 + m_5 \\ c_{21} &= m_6 + m_7 \\ c_{22} &= m_2 - m_3 + m_5 - m_7 \end{aligned}$$

7 multiplications and 18 additions/subtractions.

Another Approach (2)

Then:

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7 multiplications and 18 additions/subtractions.

$$\begin{aligned} c_{12} &= m_4 + m_5 \\ &= (a_{11} + a_{12})b_{22} + a_{11}(b_{12} + b_{22}) \\ &= a_{11}b_{22} + a_{11}b_{12} - a_{11}b_{22} \\ &= a_{12}b_{22} + b_{11}b_{12} \end{aligned}$$

Strassen's Algorithm (cont)

Divide and conquer step:

Assume n is a power of 2.

Express $C = A \times B$ in terms of $\frac{n}{2} \times \frac{n}{2}$ matrices.

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

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no notes

Strassen's Algorithm (cont)

By Strassen's algorithm, this can be computed with 7 multiplications and 18 additions/subtractions of $n/2 \times n/2$ matrices.

Recurrence:

$$\begin{aligned} T(n) &= 7T(n/2) + 18(n/2)^2 \\ T(n) &= \Theta(n^{\log_2 7}) = \Theta(n^{2.81}). \end{aligned}$$

Current "fastest" algorithm is $\Theta(n^{2.376})$
Open question: Can matrix multiplication be done in $O(n^2)$ time?

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But, this has a high constant due to the additions. This makes it rather impractical in real applications.

But this "fastest" algorithm is even more impractical due to overhead.

Introduction to the Sliderule

Compared to addition, multiplication is hard.

In the physical world, addition is merely concatenating two lengths.

Observation:

$$\log nm = \log n + \log m.$$

Therefore,

$$nm = \text{ antilog}(\log n + \log m).$$

What if taking logs and antilogs were easy?

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no notes

Introduction to the Sliderule (2)

The sliderule does exactly this!

- It is essentially two rulers in log scale.
- Slide the scales to add the lengths of the two numbers (in log form).
- The third scale shows the value for the total length.

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This is an example of a transform. We do transforms to convert a hard problem into a (relatively) easy problem.

Representing Polynomials

A vector **a** of n values can uniquely represent a polynomial of degree $n-1$

$$P_{\mathbf{a}}(x) = \sum_{i=0}^{n-1} \mathbf{a}_i x^i.$$

Alternatively, a degree $n-1$ polynomial can be uniquely represented by a list of its values at n distinct points.

- Finding the value for a polynomial at a given point is called **evaluation**.
- Finding the coefficients for the polynomial given the values at n points is called **interpolation**.

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That is, a polynomial can be represented by it coefficients.

Multiplication of Polynomials

To multiply two $n-1$-degree polynomials A and B normally takes $\Theta(n^2)$ coefficient multiplications.

However, if we evaluate both polynomials, we can simply multiply the corresponding pairs of values to get the values of polynomial AB.

Process:

- Evaluate polynomials A and B at enough points.
- Pairwise multiplications of resulting values.
- Interpolation of resulting values.

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Multiplication of Polynomials (2)

This can be faster than $\Theta(n^2)$ IF a fast way can be found to do evaluation/interpolation of $2n-1$ points (normally this takes $\Theta(n^2)$ time).

Note that evaluating a polynomial at 0 is easy, and that if we evaluate at 1 and -1, we can share a lot of the work between the two evaluations.

Can we find enough such points to make the process cheap?

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An Example

Polynomial A: x^2+1.
Polynomial B: $2x^2-x+1$.
Polynomial AB: $2x^4-x^3+3x^2-x+1$.

Notice:

$$\begin{aligned} AB(-1) &= (2)(4) = 8 \\ AB(0) &= (1)(1) = 1 \\ AB(1) &= (2)(2) = 4 \end{aligned}$$

But: We need 5 points to nail down Polynomial AB. And, we also need to interpolate the 5 values to get the coefficients back.

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	-1	0	1
A	2	1	2
B	4	1	2
AB	8	1	4

Nth Root of Unity

The key to fast polynomial multiplication is finding the right points to use for evaluation/interpolation to make the process efficient.

Complex number ω is a **primitive nth root of unity** if

1. $\omega^n = 1$ and
2. $\omega^k \neq 1$ for $0 < k < n$.

$\omega^0, \omega^1, ..., \omega^{n-1}$ are the **nth roots of unity**.

Example:

- For $n = 4$, $\omega = i$ or $\omega = -i$.

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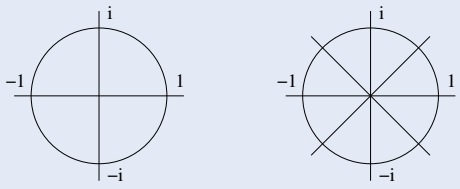
Example:

- For $n = 4$, $\omega = i$ or $\omega = -i$.

For the first circle, $n = 4, \omega = i$.

For the second circle, $n = 8, \omega = \sqrt{i}$.

Nth Root of Unity (cont)



$n = 4, \omega = i$.
$n = 8, \omega = \sqrt{i}$.

Nth Root of Unity (cont)

$n = 4, \omega = i$.
$n = 8, \omega = \sqrt{i}$.

no notes

Discrete Fourier Transform

Define an $n \times n$ matrix $V(\omega)$ with row i and column j as

$$V(\omega) = (\omega^{ij}).$$

Example: $n = 4$, $\omega = i$:

$$V(\omega) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

Let $\overline{a} = [a_0, a_1, ..., a_{n-1}]^T$ be a vector.
The **Discrete Fourier Transform** (DFT) of $\overline{a}$ is:

$$F_\omega = V(\omega)\overline{a} = \overline{v}.$$

This is equivalent to evaluating the polynomial at the nth roots of unity.

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In the array, indexing begins with 0.

Example:
$1 + 2x + 3x^2 + 4x^3$
Values to evaluate at: $1, i, -1, -i$.

Array example

For $n = 8$, $\omega = \sqrt{i}$, $V(\omega) =$

$$\begin{array}{cccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \sqrt{i} & i & i\sqrt{i} & -1 & -\sqrt{i} & -i & -i\sqrt{i} \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & i\sqrt{i} & -i & \sqrt{i} & -1 & -i\sqrt{i} & i & -\sqrt{i} \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -\sqrt{i} & i & -i\sqrt{i} & -1 & \sqrt{i} & -i & i\sqrt{i} \\ 1 & -i & -1 & i & 1 & -i & -1 & i \\ 1 & -i\sqrt{i} & -i & -\sqrt{i} & -1 & i\sqrt{i} & i & \sqrt{i} \end{array}$$

Array example
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no notes

Inverse Fourier Transform

The inverse Fourier Transform to recover $\overline{a}$ from $\overline{v}$ is:

$$F_\omega^{-1} = \overline{a} = [V(\omega)]^{-1} \cdot \overline{v}.$$

$$[V(\omega)]^{-1} = \frac{1}{n} V(\frac{1}{\omega}).$$

This is equivalent to interpolating the polynomial at the nth roots of unity.

An efficient divide and conquer algorithm can perform both the DFT and its inverse in $\Theta(n \lg n)$ time.

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An efficient divide and conquer algorithm can perform both the DFT and its inverse in $\Theta(n \lg n)$ time.

Just replace each ω with $1/\omega$

After substituting $1/\omega$ for ω.

Observe the sharable parts in the matrix.

Fast Polynomial Multiplication

Polynomial multiplication of A and B:

- Represent an $n-1$-degree polynomial as $2n-1$ coefficients:
$$[a_0, a_1, ..., a_{n-1}, 0, ..., 0]$$
- Perform DFT on representations for A and B.
- Pairwise multiply results to get $2n-1$ values.
- Perform inverse DFT on result to get $2n-1$ degree polynomial AB.

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Polynomial multiplication of A and B:
- Represent an $n-1$-degree polynomial as $2n-1$ coefficients:
$[a_0, a_1, ..., a_{n-1}, 0, ..., 0]$
- Perform DFT on representations for A and B.
- Pairwise multiply results to get $2n-1$ values.
- Perform inverse DFT on result to get $2n-1$ degree polynomial AB.

$\Theta(n \log n)$

$\Theta(n)$

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Total time: $\Theta(n \log n)$.

FFT Algorithm

```
FFT(n, a0, a1, ..., an-1, omega, var V);
Output: V[0..n-1] of output elements.
begin
  if n=1 then V[0] = a0;
  else
    FFT(n/2, a0, a2, ... an-2, omega^2, U);
    FFT(n/2, a1, a3, ... an-1, omega^2, W);
    for j=0 to n/2-1 do
      V[j] = U[j] + omega^j W[j];
      V[j+n/2] = U[j] - omega^j W[j];
end
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no notes

Parallel Algorithms

- **Running time**: $T(n, p)$ where n is the problem size, p is number of processors.
- **Speedup**: $S(p) = T(n, 1)/T(n, p)$.
  - A comparison of the time for a (good) sequential algorithm vs. the parallel algorithm in question.
- Problem: Best sequential algorithm might not be the same as the best algorithm for p processors, which might not be the best for ∞ processors.
- Efficiency: $E(n, p) = S(p)/p = T(n, 1)/(pT(n, p))$.
- Ratio of the time taken for 1 processor vs. the total time required for p processors.
  - Measure of how much the p processors are used (not wasted).
  - Optimal efficiency $= 1 =$ speedup by factor of p.

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As opposed to $T(n)$ for sequential algorithms.

Question: What algorithms should be compared?

$pT(n, p)$ is total amount of "processor power" put into the problem.

If $E(n, p) > 1$ then the sequential form of the parallel algorithm would be faster than the sequential algorithm being compared against – very suspicious!

So there are differing goals possible: Absolute fastest speedup vs. efficiency.

Parallel Algorithm Design

Approach (1): Pick p and write best algorithm.
- Would need a new algorithm for every p!

Approach (2): Pick best algorithm for $p = \infty$, then convert to run on p processors.

Hopefully, if $T(n, p) = X$, then $T(n, p/k) \approx kX$ for $k > 1$.

Using one processor to **emulate** k processors is called the **parallelism folding principle**.

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no notes

Parallel Algorithm Design (2)

Some algorithms are only good for a large number of processors.

$$\begin{aligned} T(n, 1) &= n \\ T(n, n) &= \log n \\ S(n) &= n/\log n \\ E(n, n) &= 1/\log n \end{aligned}$$

For $p = 256$, $n = 1024$.
$T(1024, 256) = 4 \log 1024 = 40$.
For $p = 16$, running time $= (1024/16) * \log 1024 = 640$.
Speedup < 2, efficiency $= 1024/(16 * 640) = 1/10$.

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Good in terms of speedup.

$1024/256$, assuming one processor emulates 4 in 4 times the time.
$E(1024, 256) = 1024/(256 * 40) = 1/10$.

But note that efficiency goes down as the problem size grows.

Amdahl's Law

Think of an algorithm as having a **parallelizable** section and a **serial** section.

Example: 100 operations.

- 80 can be done in parallel, 20 must be done in sequence.

Then, the best speedup possible leaves the 20 in sequence, or a speedup of $100/20 = 5$.

Amdahl's law:

$$\begin{aligned} \text{Speedup} &= (\mathcal{S} + \mathcal{P})/(\mathcal{S} + \mathcal{P}/N) \\ &= 1/(\mathcal{S} + \mathcal{P}/N) \leq 1/\mathcal{S}, \end{aligned}$$

for $\mathcal{S}$ = serial fraction, $\mathcal{P}$ = parallel fraction, $\mathcal{S} + \mathcal{P} = 1$.

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Amdahl's law:
Speedup = (S + P)/(S + P/N)
= 1/(S + P/N) ≤ 1/S.
for S = serial fraction, P = parallel fraction, S + P = 1.

See John L. Gustafson "Reevaluating Amdahl's Law," CACM 5/88 and follow-up technical correspondance in CACM 8/89.

Speedup is Serial / Parallel.
Draw graph, speed up is Y axis, Sequential is X axis. You will see a nonlinear curve going down.

Amdahl's Law Revisited

However, this version of Amdahl's law applies to a fixed problem size.

What happens as the problem size grows?
Hopefully, $\mathcal{S} = f(n)$ with $\mathcal{S}$ shrinking as n grows.

Instead of fixing problem size, fix execution time for increasing number N processors (and thus, increasing problem size).

$$\begin{aligned} \text{Scaled Speedup} &= (\mathcal{S} + \mathcal{P} \times N)/(\mathcal{S} + \mathcal{P}) \\ &= \mathcal{S} + \mathcal{P} \times N \\ &= \mathcal{S} + (1 - \mathcal{S}) \times N \\ &= N + (1 - N) \times \mathcal{S}. \end{aligned}$$

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What happens as the problem size grows?
Hopefully, S = f(n) with S shrinking as n grows.
Instead of fixing problem size, fix execution time for increasing number N processors (and thus, increasing problem size).
Scaled Speedup = (S + P × N)/(S + P)
= S + P × N
= S + (1 − S) × N
= N + (1 − N) × S.

How long sequential process would take / How long for N processors.

Since $S + P = 1$ and $P = 1 - S$.

The point is that this equation drops off much less slowly in N: Graphing (sequential fraction for fixed N) vs. speedup, you get a line with slope $1 - N$.

All of this seems to assume the same algorithm for sequential and parallel. But that's OK – we want to see how much parallelism is possible for the parallel algorithm.