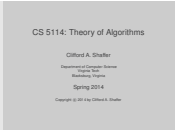
CS 5114: Theory of Algorithms

Clifford A. Shaffer

Department of Computer Science
Virginia Tech
Blacksburg, Virginia

Spring 2014

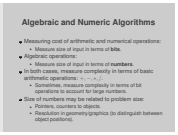


Title page

Students should be familiar with inductive proofs, recursion, data structures, and programming at the CS3114 level.

Algebraic and Numeric Algorithms

- Measuring cost of arithmetic and numerical operations:
  - Measure size of input in terms of **bits**.
- Algebraic operations:
  - Measure size of input in terms of **numbers**.
- In both cases, measure complexity in terms of basic arithmetic operations: $+, -, *, /$.
  - Sometimes, measure complexity in terms of bit operations to account for large numbers.
- Size of numbers may be related to problem size:
  - Pointers, counters to objects.
  - Resolution in geometry/graphics (to distinguish between object positions).



no notes

Exponentiation

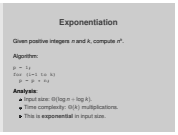
Given positive integers n and k, compute n^k.

Algorithm:

```
p = 1;
for (i=1 to k)
  p = p * n;
```

Analysis:

- Input size: $\Theta(\log n + \log k)$.
- Time complexity: $\Theta(k)$ multiplications.
- This is **exponential** in input size.



no notes

Faster Exponentiation

Write k as:

$$k = b_t 2^t + b_{t-1} 2^{t-1} + \cdots + b_1 2 + b_0, b \in \{0, 1\}.$$

Rewrite as:

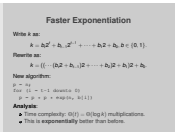
$$k = ((\cdots(b_t 2 + b_{t-1})2 + \cdots + b_2)2 + b_1)2 + b_0.$$

New algorithm:

```
p = n;
for (i = t-1 downto 0)
  p = p * p * exp(n, b[i])
```

Analysis:

- Time complexity: $\Theta(t) = \Theta(\log k)$ multiplications.
- This is **exponentially** better than before.



no notes

Greatest Common Divisor

- The Greatest Common Divisor (GCD) of two integers is the greatest integer that divides both evenly.
- Observation: If k divides n and m, then k divides $n - m$.
- So,

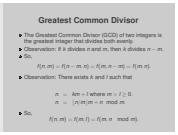
$$f(n, m) = f(n - m, n) = f(m, n - m) = f(m, n).$$

- Observation: There exists k and l such that

$$\begin{aligned} n &= km + l \text{ where } m > l \geq 0. \\ n &= \lfloor n/m \rfloor m + n \mod m. \end{aligned}$$

- So,

$$f(n, m) = f(m, l) = f(m, n \mod m).$$



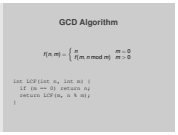
Assuming $n > m$, then $n = ak$, $m = bk$, $n - m = (a - b)k$ for integers a, b.

This comes from definition of mod .

GCD Algorithm

$$f(n, m) = \begin{cases} n & m = 0 \\ f(m, n \bmod m) & m > 0 \end{cases}$$

```
int LCF(int n, int m) {
  if (m == 0) return n;
  return LCF(m, n % m);
}
```

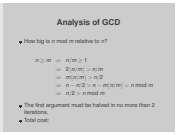
no notes

Analysis of GCD

- How big is n mod m relative to n?

$$\begin{aligned} n \geq m &\Rightarrow n/m \geq 1 \\ &\Rightarrow 2\lfloor n/m \rfloor > n/m \\ &\Rightarrow m\lfloor n/m \rfloor > n/2 \\ &\Rightarrow n - n/2 < n - m\lfloor n/m \rfloor = n \bmod m \\ &\Rightarrow n/2 > n \bmod m \end{aligned}$$

- The first argument must be halved in no more than 2 iterations.
- Total cost:



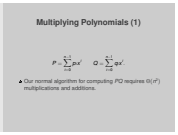
Can split in half $\log n$ times. So $2 \log n$ is upper bound.

Note that this is linier on problem size, since problem size is $2 \log n$ (2 numbers).

Multiplying Polynomials (1)

$$P = \sum_{i=0}^{n-1} p_i x^i \qquad Q = \sum_{i=0}^{n-1} q_i x^i.$$

- Our normal algorithm for computing PQ requires $\Theta(n^2)$ multiplications and additions.



no notes

Multiplying Polynomials (2)

- Divide and Conquer:

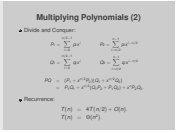
$$P_1 = \sum_{i=0}^{n/2-1} p_i x^i \qquad P_2 = \sum_{i=n/2}^{n-1} p_i x^{i-n/2}$$

$$Q_1 = \sum_{i=0}^{n/2-1} q_i x^i \qquad Q_2 = \sum_{i=n/2}^{n-1} q_i x^{i-n/2}$$

$$\begin{aligned} PQ &= (P_1 + x^{n/2}P_2)(Q_1 + x^{n/2}Q_2) \\ &= P_1Q_1 + x^{n/2}(Q_1P_2 + P_1Q_2) + x^nP_2Q_2. \end{aligned}$$

- Recurrence:

$$\begin{aligned} T(n) &= 4T(n/2) + O(n). \\ T(n) &= \Theta(n^2). \end{aligned}$$



Do this to make the subproblems look the same.

Multiplying Polynomials (3)

Observation:

$$(P_1 + P_2)(Q_1 + Q_2) = P_1Q_1 + (Q_1P_2 + P_1Q_1) + P_2Q_2$$

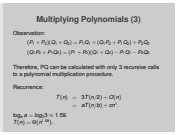
$$(Q_1P_2 + P_1Q_2) = (P_1 + P_2)(Q_1 + Q_2) - P_1Q_1 - P_2Q_2$$

Therefore, PQ can be calculated with only 3 recursive calls to a polynomial multiplication procedure.

Recurrence:

$$\begin{aligned} T(n) &= 3T(n/2) + O(n) \\ &= aT(n/b) + cn^1. \end{aligned}$$

$\log_b a = log_2 3 \approx 1.59$.
$T(n) = \Theta(n^{1.59})$.



In the second equation, the sums in the first term are half the original problem size, and the second two terms were needed for the first equation.

$$PQ = P_1Q_1 + X^{n/2}((P_1 + P_2)(Q_1 + Q_2) - P_1Q_1 - P_2Q_2) + x^nP_2Q_2$$

A significant improvement came from algebraic manipulation to express the product in terms of 3, rather than 4, smaller products.

Matrix Multiplication

Given: $n \times n$ matrices A and B.

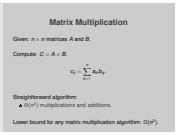
Compute: $C = A \times B$.

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

Straightforward algorithm:

- $\Theta(n^3)$ multiplications and additions.

Lower bound for any matrix multiplication algorithm: $\Omega(n^2)$.



no notes

Strassen's Algorithm

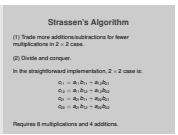
(1) Trade more additions/subtractions for fewer multiplications in 2×2 case.

(2) Divide and conquer.

In the straightforward implementation, 2×2 case is:

$$\begin{aligned} c_{11} &= a_{11}b_{11} + a_{12}b_{21} \\ c_{12} &= a_{11}b_{12} + a_{12}b_{22} \\ c_{21} &= a_{21}b_{11} + a_{22}b_{21} \\ c_{22} &= a_{21}b_{12} + a_{22}b_{22} \end{aligned}$$

Requires 8 multiplications and 4 additions.



no notes

Another Approach (1)

Compute:

$$\begin{aligned} m_1 &= (a_{12} - a_{22})(b_{21} + b_{22}) \\ m_2 &= (a_{11} + a_{22})(b_{11} + b_{22}) \\ m_3 &= (a_{11} - a_{21})(b_{11} + b_{12}) \\ m_4 &= (a_{11} + a_{12})b_{22} \\ m_5 &= a_{11}(b_{12} - b_{22}) \\ m_6 &= a_{22}(b_{21} - b_{11}) \\ m_7 &= (a_{21} + a_{22})b_{11} \end{aligned}$$

Another Approach (1)

Compute:

$m_1 = (a_{12} - a_{22})(b_{21} + b_{22})$
$m_2 = (a_{11} + a_{22})(b_{11} + b_{22})$
$m_3 = (a_{11} - a_{21})(b_{11} + b_{12})$
$m_4 = (a_{11} + a_{12})b_{22}$
$m_5 = a_{11}(b_{12} - b_{22})$
$m_6 = a_{22}(b_{21} - b_{11})$
$m_7 = (a_{21} + a_{22})b_{11}$

no notes

Another Approach (2)

Then:

$$\begin{aligned} c_{11} &= m_1 + m_2 - m_4 + m_6 \\ c_{12} &= m_4 + m_5 \\ c_{21} &= m_6 + m_7 \\ c_{22} &= m_2 - m_3 + m_5 - m_7 \end{aligned}$$

7 multiplications and 18 additions/subtractions.

Another Approach (2)

Then:

$c_{11} = m_1 + m_2 - m_4 + m_6$
$c_{12} = m_4 + m_5$
$c_{21} = m_6 + m_7$
$c_{22} = m_2 - m_3 + m_5 - m_7$

7 multiplications and 18 additions/subtractions.

$$\begin{aligned} c_{12} &= m_4 + m_5 \\ &= (a_{11} + a_{12})b_{22} + a_{11}(b_{12} + b_{22}) \\ &= a_{11}b_{22} + a_{11}b_{12} - a_{11}b_{22} \\ &= a_{12}b_{22} + b_{11}b_{12} \end{aligned}$$

Strassen's Algorithm (cont)

Divide and conquer step:

Assume n is a power of 2.

Express $C = A \times B$ in terms of $\frac{n}{2} \times \frac{n}{2}$ matrices.

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

Strassen's Algorithm (cont)

Divide and conquer step:

Assume n is a power of 2.

Express $C = A \times B$ in terms of $\frac{n}{2} \times \frac{n}{2}$ matrices.

$$\left[\begin{array}{cc|cc} c_{11} & c_{12} & a_{11} & a_{12} \\ c_{21} & c_{22} & a_{21} & a_{22} \end{array}\right] = \left[\begin{array}{cc|cc} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \end{array}\right]$$

no notes

Strassen's Algorithm (cont)

By Strassen's algorithm, this can be computed with 7 multiplications and 18 additions/subtractions of $n/2 \times n/2$ matrices.

Recurrence:

$$\begin{aligned} T(n) &= 7T(n/2) + 18(n/2)^2 \\ T(n) &= \Theta(n^{\log_2 7}) = \Theta(n^{2.81}). \end{aligned}$$

Current "fastest" algorithm is $\Theta(n^{2.376})$
Open question: Can matrix multiplication be done in $O(n^2)$ time?

Strassen's Algorithm (cont)

By Strassen's algorithm, this can be computed with 7 multiplications and 18 additions/subtractions of $n/2 \times n/2$ matrices.

Recurrence:

$T(n) = 7T(n/2) + 18(n/2)^2$
$T(n) = \Theta(n^{\log_2 7}) = \Theta(n^{2.81})$.

Current "fastest" algorithm is $\Theta(n^{2.376})$.
Open question: Can matrix multiplication be done in $O(n^2)$ time?

But, this has a high constant due to the additions. This makes it rather impractical in real applications.

But this "fastest" algorithm is even more impractical due to overhead.

Introduction to the Sliderule

Compared to addition, multiplication is hard.

In the physical world, addition is merely concatenating two lengths.

Observation:

$$\log nm = \log n + \log m.$$

Therefore,

$$nm = \text{ antilog}(\log n + \log m).$$

What if taking logs and antilogs were easy?

Introduction to the Sliderule
Compared to addition, multiplication is hard.
In the physical world, addition is merely concatenating two lengths.
Observation:
$\log nm = \log n + \log m$.
Therefore,
$nm =$ antilog$(\log n + \log m)$.
What if taking logs and antilogs were easy?

no notes

Introduction to the Sliderule (2)

The sliderule does exactly this!

- It is essentially two rulers in log scale.
- Slide the scales to add the lengths of the two numbers (in log form).
- The third scale shows the value for the total length.

Introduction to the Sliderule (2)
The sliderule does exactly this!
- It is essentially two rulers in log scale.
- Slide the scales to add the lengths of the two numbers (in log form).
- The third scale shows the value for the total length.

This is an example of a transform. We do transforms to convert a hard problem into a (relatively) easy problem.

Representing Polynomials

A vector **a** of n values can uniquely represent a polynomial of degree $n-1$

$$P_{\mathbf{a}}(x) = \sum_{i=0}^{n-1} \mathbf{a}_i x^i.$$

Alternatively, a degree $n-1$ polynomial can be uniquely represented by a list of its values at n distinct points.

- Finding the value for a polynomial at a given point is called **evaluation**.
- Finding the coefficients for the polynomial given the values at n points is called **interpolation**.

Representing Polynomials
A vector **a** of n values can uniquely represent a polynomial of degree $n-1$
$P_{\mathbf{a}}(x) = \sum_{i=0}^{n-1} \mathbf{a}_i x^i$.
Alternatively, a degree $n-1$ polynomial can be uniquely represented by a list of its values at n distinct points.
- Finding the value for a polynomial at a given point is called evaluation.
- Finding the coefficients for the polynomial given the values at n points is called interpolation.

That is, a polynomial can be represented by it coefficients.

Multiplication of Polynomials

To multiply two $n-1$-degree polynomials A and B normally takes $\Theta(n^2)$ coefficient multiplications.

However, if we evaluate both polynomials, we can simply multiply the corresponding pairs of values to get the values of polynomial AB.

Process:

- Evaluate polynomials A and B at enough points.
- Pairwise multiplications of resulting values.
- Interpolation of resulting values.

Multiplication of Polynomials
To multiply two $n-1$-degree polynomials A and B normally takes $\Theta(n^2)$ coefficient multiplications.
However, if we evaluate both polynomials, we can simply multiply the corresponding pairs of values to get the values of polynomial AB.
Process:
- Evaluate polynomials A and B at enough points.
- Pairwise multiplications of resulting values.
- Interpolation of resulting values.

no notes

Multiplication of Polynomials (2)

This can be faster than $\Theta(n^2)$ IF a fast way can be found to do evaluation/interpolation of $2n-1$ points (normally this takes $\Theta(n^2)$ time).

Note that evaluating a polynomial at 0 is easy, and that if we evaluate at 1 and -1, we can share a lot of the work between the two evaluations.

Can we find enough such points to make the process cheap?

Multiplication of Polynomials (2)
This can be faster than $\Theta(n^2)$ IF a fast way can be found to do evaluation/interpolation of $2n-1$ points (normally this takes $\Theta(n^2)$ time).
Note that evaluating a polynomial at 0 is easy, and that if we evaluate at 1 and -1, we can share a lot of the work between the two evaluations.
Can we find enough such points to make the process cheap?

no notes

An Example

Polynomial A: x^2+1.
Polynomial B: $2x^2-x+1$.
Polynomial AB: $2x^4-x^3+3x^2-x+1$.

Notice:

$$\begin{aligned} AB(-1) &= (2)(4) = 8 \\ AB(0) &= (1)(1) = 1 \\ AB(1) &= (2)(2) = 4 \end{aligned}$$

But: We need 5 points to nail down Polynomial AB. And, we also need to interpolate the 5 values to get the coefficients back.

An Example
Polynomial A: x^2+1.
Polynomial B: $2x^2-x+1$.
Polynomial AB: $2x^4-x^3+3x^2-x+1$.
Notice:
$$\begin{aligned} AB(-1) &= (2)(4) = 8 \\ AB(0) &= (1)(1) = 1 \\ AB(1) &= (2)(2) = 4 \end{aligned}$$
But: We need 5 points to nail down Polynomial AB. And, we also need to interpolate the 5 values to get the coefficients back.

	-1	0	1
A	2	1	2
B	4	1	2
AB	8	1	4