CS 5114: Theory of Algorithms

Clifford A. Shaffer

Department of Computer Science
Virginia Tech
Blacksburg, Virginia

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Tractable Problems

We would like some convention for distinguishing tractable from intractable problems.

A problem is said to be **tractable** if an algorithm exists to solve it with polynomial time complexity: $O(p(n))$.

- It is said to be **intractable** if the best known algorithm requires exponential time.

Examples:

- Sorting: $O(n^2)$
- Convex Hull: $O(n^2)$
- Single source shortest path: $O(n^2)$
- All pairs shortest path: $O(n^3)$
- Matrix multiplication: $O(n^3)$

Tractable Problems (cont)

The technique we will use to classify one group of algorithms is based on two concepts:

1. A special kind of reduction.
2. Nondeterminism.

Decision Problems

(I, S) such that S(X) is always either "yes" or "no."

- Usually formulated as a question.

Example:

- Instance: A weighted graph $G = (V, E)$, two vertices s and t, and an integer K.
- Question: Is there a path from s to t of length $\leq K$? In this example, the answer is "yes."

Decision Problems (cont)

Can also be formulated as a language recognition problem:

- Let L be the subset of I consisting of instances whose answer is "yes." Can we recognize L?

The class of tractable problems $\mathcal{P}$ is the class of languages or decision problems recognizable in polynomial time.

Polynomial Reducibility

Reduction of one language to another language.

Let $L_1 \subset I_1$ and $L_2 \subset I_2$ be languages. L_1 is **polynomially reducible** to L_2 if there exists a transformation $f : I_1 \rightarrow I_2$, computable in polynomial time, such that $f(x) \in L_2$ if and only if $x \in L_1$.
We write: $L_1 \leq_p L_2$ or $L_1 \leq L_2$.

Examples

- CLIQUE $\leq_p$ INDEPENDENT SET.
- An instance I of CLIQUE is a graph $G = (V, E)$ and an integer K.
- The instance $I' = f(I)$ of INDEPENDENT SET is the graph $G' = (V, E')$ and the integer K, were an edge $(u, v) \in E'$ iff $(u, v) \notin E$.
- f is computable in polynomial time.

Transformation Example

- G has a clique of size $\geq K$ iff G' has an independent set of size $\geq K$.
- Therefore, CLIQUE $\leq_p$ INDEPENDENT SET.
- **IMPORTANT WARNING:** The reduction does not **solve** either INDEPENDENT SET or CLIQUE, it merely transforms one into the other.

Nondeterminism

Nondeterminism allows an algorithm to make an arbitrary choice among a finite number of possibilities.

Implemented by the "nd-choice" primitive:

nd-choice(ch_1, ch_2, ..., ch_j)

returns one of the choices ch_1, ch_2, ... **arbitrarily**.

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Alternatively, nondeterminsitic algorithms can be thought of as running on super-parallel machines that make all choices simultaneously and then reports the "right" solution.

Nondeterministic CLIQUE Algorithm

```
procedure nd-CLIQUE(Graph G, int K) {
  VertexSet S = EMPTY;  int size = 0;
  for (v in G.V)
    if (nd-choice(YES, NO) == YES) then {
      S = union(S, v);
      size = size + 1;
    }
  if (size < K) then
    REJECT;      // S is too small
  for (u in S)
    for (v in S)
      if ((u <> v) && ((u, v) not in E))
        REJECT;  // S is missing an edge
  ACCEPT;
}
```

Nondeterministic Acceptance

- (G, K) is in the "language" CLIQUE iff there exists a sequence of nd-choice guesses that causes nd-CLIQUE to accept.
- Definition of acceptance by a nondeterministic algorithm:
  - An instance is accepted iff there exists a sequence of nondeterministic choices that causes the algorithm to accept.

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- An unrealistic model of computation.
  - There are an exponential number of possible choices, but only one must accept for the instance to be accepted.
- Nondeterminism is a useful concept
  - It provides insight into the nature of certain hard problems.

Class $\mathcal{NP}$

- The class of languages accepted by a nondeterministic algorithm in polynomial time is called $\mathcal{NP}$.
- There are an **exponential** number of different executions of nd-CLIQUE on a single instance, but any one execution requires only **polynomial time** in the size of that instance.
- Time complexity of nondeterministic algorithm is greatest amount of time required by any **one** of its executions.

Class $\mathcal{NP}$(cont)

Alternative Interpretation:

- $\mathcal{NP}$ is the class of algorithms that — never mind how we got the answer — can check if the answer is correct in polynomial time.
- If you cannot verify an answer in polynomial time, you cannot hope to find the right answer in polynomial time!

How to Get Famous

Clearly, $\mathcal{P} \subset \mathcal{NP}$.

Extra Credit Problem:

- Prove or disprove: $\mathcal{P} = \mathcal{NP}$.

This is important because there are many natural decision problems in $\mathcal{NP}$ for which no $\mathcal{P}$ (tractable) algorithm is known.

$\mathcal{NP}$-completeness

A theory based on identifying problems that are as hard as any problems in $\mathcal{NP}$.

The next best thing to knowing whether $\mathcal{P} = \mathcal{NP}$ or not.

A decision problem A is **$\mathcal{NP}$-hard** if every problem in $\mathcal{NP}$ is polynomially reducible to A, that is, for all

$$B \in \mathcal{NP}, \quad B \leq_p A.$$

A decision problem A is **$\mathcal{NP}$-complete** if $A \in \mathcal{NP}$ and A is $\mathcal{NP}$-hard.

Satisfiability

Let E be a Boolean expression over variables $x_1, x_2, \cdots, x_n$ in conjunctive normal form (CNF), that is, an AND of ORs.

$$E = (x_5 + x_7 + \overline{x_8} + x_{10}) \cdot (\overline{x_2} + x_3) \cdot (x_1 + \overline{x_3} + x_6).$$

A variable or its negation is called a **literal**.
Each sum is called a **clause**.

SATISFIABILITY (SAT):

- Instance: A Boolean expression E over variables $x_1, x_2, \cdots, x_n$ in CNF.
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Cook's Theorem: SAT is $\mathcal{NP}$-complete.

Proof Sketch

SAT $\in \mathcal{NP}$:

- A non-deterministic algorithm **guesses** a truth assignment for $x_1, x_2, \cdots, x_n$ and **checks** whether E is true in polynomial time.
- It accepts iff there is a satisfying assignment for E.

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SAT is $\mathcal{NP}$-hard:

- Start with an arbitrary problem $B \in \mathcal{NP}$.
- We know there is a polynomial-time, nondeterministic algorithm to accept B.
- Cook showed how to transform an instance X of B into a Boolean expression E that is satisfiable if the algorithm for B accepts X.

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(4) If $A \in \mathcal{NP}$ and B is $\mathcal{NP}$-complete, then $B \leq_p A$ implies A is $\mathcal{NP}$-complete.

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Proof:

- Let $C \in \mathcal{NP}$.
- Then $C \leq_p B$ since B is $\mathcal{NP}$-complete.
- Since $B \leq_p A$ and $\leq_p$ is transitive, $C \leq_p A$.
- Therefore, A is $\mathcal{NP}$-hard and, finally, $\mathcal{NP}$-complete.

Implications (cont)

(5) This gives a simple two-part strategy for showing a decision problem A is $\mathcal{NP}$-complete.

(a) Show $A \in \mathcal{NP}$.

(b) Pick an $\mathcal{NP}$-complete problem B and show $B \leq_p A$.

$\mathcal{NP}$-completeness Proof Template

To show that decision problem B is $\mathcal{NP}$-complete:

1. $B \in \mathcal{NP}$
   - Give a polynomial time, non-deterministic algorithm that accepts B.
     1. Given an instance X of B, **guess** evidence Y.
     2. **Check** whether Y is evidence that $X \in B$. If so, accept X.

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   - Give a polynomial time, non-deterministic algorithm that accepts B.
     1. Given an instance X of B, **guess** evidence Y.
     2. **Check** whether Y is evidence that $X \in B$. If so, accept X.
2. B is $\mathcal{NP}$-hard.
   - Choose a known $\mathcal{NP}$-complete problem, A.
   - Describe a polynomial-time transformation T of an **arbitrary** instance of A to a [not necessarily arbitrary] instance of B.
   - Show that $X \in A$ if and only if $T(X) \in B$.

3-SATISFIABILITY (3SAT)

Instance: A Boolean expression E in CNF such that each clause contains exactly 3 literals.

Question: Is there a satisfying assignment for E?

A special case of SAT.

One might hope that 3SAT is easier than SAT.

3SAT is $\mathcal{NP}$-complete

(1) 3SAT $\in \mathcal{NP}$.

```
procedure nd-3SAT(E) {
  for (i = 1 to n)
    x[i] = nd-choice(TRUE, FALSE);
  Evaluate E for the guessed truth assignment.
  if (E evaluates to TRUE)
    ACCEPT;
  else
    REJECT;
}
```

nd-3SAT is a polynomial-time nondeterministic algorithm that accepts 3SAT.

Proving 3SAT $\mathcal{NP}$-hard

1. Choose SAT to be the known $\mathcal{NP}$-complete problem.
   - We need to show that SAT $\leq_p$ 3SAT.
2. Let $E = C_1 \cdot C_2 \cdots C_k$ be any instance of SAT.

Strategy: Replace any clause C_i that does not have exactly 3 literals with two or more clauses having exactly 3 literals.

Let $C_i = y_1 + y_2 + \cdots + y_j$ where $y_1, \cdots, y_j$ are literals.
(a) $j = 1$

- Replace (y_1) with

$$(y_1 + v + w) \cdot (y_1 + \overline{v} + w) \cdot (y_1 + v + \overline{w}) \cdot (y_1 + \overline{v} + \overline{w})$$

where v and w are new variables.

Proving 3SAT $\mathcal{NP}$-hard (cont)

(b) $j = 2$

- Replace $(y_1 + y_2)$ with $(y_1 + y_2 + z) \cdot (y_1 + y_2 + \overline{z})$ where z is a new variable.

(c) $j > 3$

- Relace $(y_1 + y_2 + \cdots + y_j)$ with

$$(y_1 + y_2 + z_1) \cdot (y_3 + \overline{z_1} + z_2) \cdot (y_4 + \overline{z_2} + z_3) \cdots$$

$$(y_{j-2} + \overline{z_{j-4}} + z_{j-3}) \cdot (y_{j-1} + y_j + \overline{z_{j-3}})$$

  where $z_1, z_2, \cdots, z_{j-3}$ are new variables.

- After replacements made for each C_i, a Boolean expression E' results that is an instance of 3SAT.
- The replacement clearly can be done by a polynomial-time deterministic algorithm.

Proving 3SAT $\mathcal{NP}$-hard (cont)

(3) Show E is satisfiable iff E' is satisfiable.

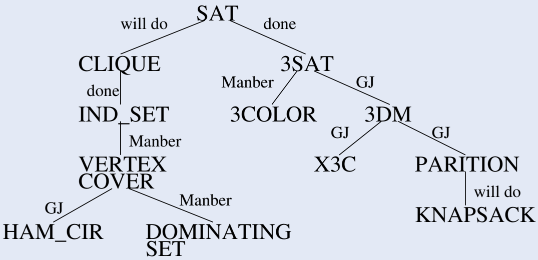
- Assume E has a satisfying truth assignment.
- Then that extends to a satisfying truth assignment for cases (a) and (b).
- In case (c), assume y_m is assigned "true".
- Then assign $z_t, t \leq m - 2$, true and $z_k, t \geq m - 1$, false.
- Then all the clauses in case (c) are satisfied.

Proving 3SAT $\mathcal{NP}$-hard (cont)

- Assume E' has a satisfying assignment.
- By restriction, we have truth assignment for E.
  (a) y_1 is necessarily true.
  (b) $y_1 + y_2$ is necessarily true.
  (c) Proof by contradiction:
    - ⋆ If $y_1, y_2, \cdots, y_j$ are all false, then $z_1, z_2, \cdots, z_{j-3}$ are all true.
    - ⋆ But then $(y_{j-1} + y_{j-2} + \overline{z_{j-3}})$ is false, a contradiction.

We conclude SAT $\leq$ 3SAT and 3SAT is $\mathcal{NP}$-complete.

Tree of Reductions



Reductions go down the tree.

Proofs that each problem $\in \mathcal{NP}$ are straightforward.

Perspective

The reduction tree gives us a collection of 12 diverse $\mathcal{NP}$-complete problems.

The complexity of all these problems depends on the complexity of any one:

- If any $\mathcal{NP}$-complete problem is tractable, then they all are.

This collection is a good place to start when attempting to show a decision problem is $\mathcal{NP}$-complete.

Observation: If we find a problem is $\mathcal{NP}$-complete, then we should do something other than try to find a $\mathcal{P}$-time algorithm.

SAT $\leq_p$ CLIQUE

(1) Easy to show CLIQUE in $\mathcal{NP}$.
(2) An instance of SAT is a Boolean expression

$$B = C_1 \cdot C_2 \cdots C_m,$$

where

$$C_i = y[i,1] + y[i,2] + \cdots + y[i,k_i].$$

Transform this to an instance of CLIQUE $G = (V, E)$ and K.

$$V = \{v[i,j] | 1 \leq i \leq m, 1 \leq j \leq k_i\}$$

Two vertices $v[i_1, j_1]$ and $v[i_2, j_2]$ are adjacent in G if $i_1 \neq i_2$
AND EITHER $y[i_1, j_1]$ and $y[i_2, j_2]$ are the same literal
OR $y[i_1, j_1]$ and $y[i_2, j_2]$ have different underlying variables.
$K = m$.

SAT $\leq_p$ CLIQUE (cont)

Example: $B = (x + y + \bar{(}z)) \cdot (\overline{x} + \overline{y} + z) \cdot (y + \overline{z})$.
K = 3.

(3) B is satisfiable iff G has clique of size $\geq K$.

- B is satisfiable implies there is a truth assignment such that $y[i, j_i]$ is true for each i.
- But then $v[i, j_i]$ must be in a clique of size $K = m$.
- If G has a clique of size $\geq K$, then the clique must have size exactly K and there is one vertex $v[i, j_i]$ in the clique for each i.
- There is a truth assignment making each $y[i, j_i]$ true. That truth assignment satisfies B.

We conclude that CLIQUE is $\mathcal{NP}$-hard, therefore $\mathcal{NP}$-complete.

Co-$\mathcal{NP}$

- Note the asymmetry in the definition of $\mathcal{NP}$.
  - The non-determinism can identify a clique, and you can verify it.
  - But what if the correct answer is "NO"? How do you verify that?
- Co-$\mathcal{NP}$: The complements of problems in $\mathcal{NP}$.
  - Is a boolean expression **always** false?
  - Is there no clique of size k?
- It seems unlikely that $\mathcal{NP}=$ co-$\mathcal{NP}$.

Is $\mathcal{NP}$-complete $= \mathcal{NP}$?

- It has been proved that if $\mathcal{P} \neq \mathcal{NP}$, then $\mathcal{NP}$-complete $\neq \mathcal{NP}$.
- The following problems are not known to be in $\mathcal{P}$ or $\mathcal{NP}$, but seem to be of a type that makes them unlikely to be in $\mathcal{NP}$.
  - GRAPH ISOMORPHISM: Are two graphs isomorphic?
  - COMPOSITE NUMBERS: For positive integer K, are there integers $m, n > 1$ such that $K = mn$?
  - LINEAR PROGRAMMING

PARTITION $\leq_p$ KNAPSACK

PARTITION is a special case of KNAPSACK in which

$$K = \frac{1}{2}\sum_{a \in A} s(a)$$

assuming $\sum s(a)$ is even.

Assuming PARTITION is $\mathcal{NP}$-complete, KNAPSACK is $\mathcal{NP}$-complete.

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  - Thus $O(KN)$ is exponential in $N \log K$.
- The dynamic programming algorithm counts through numbers $1, \cdots, K$. Takes exponential time when measured by number of bits to represent K.
- If K is "small" ($K = O(p(N))$), then algorithm has complexity polynomial in N and is truly polynomial in input size.
- An algorithm that is polynomial-time if the numbers IN the input are "small" (as opposed to number OF inputs) is called a **pseudo-polynomial** time algorithm.

"Practical" Problems (cont)

- Lesson: While KNAPSACK is $\mathcal{NP}$-complete, it is often not that hard.
- Many $\mathcal{NP}$-complete problems have no pseudo-polynomial time algorithm unless $\mathcal{P} = \mathcal{NP}$.