CS 5114: Theory of Algorithms

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Title page

Students should be familiar with inductive proofs, recursion, data structures, and programming at the CS3114 level.

Tractable Problems

We would like some convention for distinguishing tractable from intractable problems.
A problem is said to be **tractable** if an algorithm exists to solve it with polynomial time complexity: $O(p(n))$.

- It is said to be **intractable** if the best known algorithm requires exponential time.

Examples:

- Sorting: $O(n^2)$
- Convex Hull: $O(n^2)$
- Single source shortest path: $O(n^2)$
- All pairs shortest path: $O(n^3)$
- Matrix multiplication: $O(n^3)$

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└Tractable Problems

Log-polynomial is $O(n \log n)$

Like any simple rule of thumb for catagorizing, in some cases the distinction between polynomial and exponential could break down. For example, one can argue that, for practical problems, 1.01^n is preferable to n^{25}. But the reality is that very few polynomial-time algorithms have high degree, and exponential-time algorithms nearly always have a constant of 2 or greater. Nearly all algorithms are either low-degree polynomials or “real” exponentials, with very little in between.

Tractable Problems (cont)

The technique we will use to classify one group of algorithms is based on two concepts:

1. A special kind of reduction.
2. Nondeterminism.

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└Tractable Problems (cont)

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Decision Problems

(I, S) such that S(X) is always either “yes” or “no.”

- Usually formulated as a question.

Example:

- Instance: A weighted graph $G = (V, E)$, two vertices s and t, and an integer K.
- Question: Is there a path from s to t of length $\leq K$? In this example, the answer is “yes.”

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└Decision Problems

Need a graph here.

Decision Problems (cont)

Can also be formulated as a language recognition problem:

- Let L be the subset of I consisting of instances whose answer is “yes.” Can we recognize L?

The class of tractable problems $\mathcal{P}$ is the class of languages or decision problems recognizable in polynomial time.

Following our graph example: It is possible to translate from a graph to a string representation, and to define a subset of such strings as corresponding to graphs with a path from s to t. This subset defines a language to “recognize.”

Polynomial Reducibility

Reduction of one language to another language.

Let $L_1 \subset I_1$ and $L_2 \subset I_2$ be languages. L_1 is **polynomially reducible** to L_2 if there exists a transformation $f : I_1 \to I_2$, computable in polynomial time, such that $f(x) \in L_2$ if and only if $x \in L_1$.
We write: $L_1 \leq_p L_2$ or $L_1 \leq L_2$.

Or one decision problem to another.

Specialized case of reduction from Chapter 10.

Examples

- CLIQUE $\leq_p$ INDEPENDENT SET.
- An instance I of CLIQUE is a graph $G = (V, E)$ and an integer K.
- The instance $I' = f(I)$ of INDEPENDENT SET is the graph $G' = (V, E')$ and the integer K, were an edge $(u, v) \in E'$ iff $(u, v) \notin E$.
- f is computable in polynomial time.

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Transformation Example

- G has a clique of size $\geq K$ iff G' has an independent set of size $\geq K$.
- Therefore, CLIQUE $\leq_p$ INDEPENDENT SET.
- **IMPORTANT WARNING:** The reduction does not **solve** either INDEPENDENT SET or CLIQUE, it merely transforms one into the other.

Need a graph here.

If nodes in G' are independent, then no connections. Thus, in G they all connect.

Nondeterminism

Nondeterminism allows an algorithm to make an arbitrary choice among a finite number of possibilities.

Implemented by the “nd-choice” primitive:
nd-choice(ch_1, ch_2, ..., ch_j)
returns one of the choices ch_1, ch_2, ... **arbitrarily**.

Nondeterministic algorithms can be thought of as “correctly guessing” (choosing nondeterministically) a solution.

Alternatively, nondeterminsitic algorithms can be thought of as running on super-parallel machines that make all choices simultaneously and then reports the “right” solution.

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Nondeterministic CLIQUE Algorithm

```
procedure nd-CLIQUE(Graph G, int K) {
  VertexSet S = EMPTY;  int size = 0;
  for (v in G.V)
    if (nd-choice(YES, NO) == YES) then {
      S = union(S, v);
      size = size + 1;
    }
  if (size < K) then
    REJECT;      // S is too small
  for (u in S)
    for (v in S)
      if ((u <> v) && ((u, v) not in E))
        REJECT;  // S is missing an edge
  ACCEPT;
}
```


What makes this different than random guessing is that **all** choices happen “in parallel.”

Nondeterministic Acceptance

- (G, K) is in the “language” CLIQUE iff there exists a sequence of nd-choice guesses that causes nd-CLIQUE to accept.
- Definition of acceptance by a nondeterministic algorithm:
  - An instance is accepted iff there exists a sequence of nondeterministic choices that causes the algorithm to accept.
- An unrealistic model of computation.
  - There are an exponential number of possible choices, but only one must accept for the instance to be accepted.
- Nondeterminism is a useful concept
  - It provides insight into the nature of certain hard problems.

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Class $\mathcal{NP}$

- The class of languages accepted by a nondeterministic algorithm in polynomial time is called $\mathcal{NP}$.
- There are an **exponential** number of different executions of nd-CLIQUE on a single instance, but any one execution requires only **polynomial time** in the size of that instance.
- Time complexity of nondeterministic algorithm is greatest amount of time required by any **one** of its executions.

Note that Towers of Hanoi is not in $\mathcal{NP}$.

Class $\mathcal{NP}$(cont)

Alternative Interpretation:

- $\mathcal{NP}$ is the class of algorithms that — never mind how we got the answer — can check if the answer is correct in polynomial time.
- If you cannot verify an answer in polynomial time, you cannot hope to find the right answer in polynomial time!

This is worded a bit loosely. Specifically, we assume that we can get the answer fast enough – that is, in polynomial time non-deterministically.

How to Get Famous

Clearly, $\mathcal{P} \subset \mathcal{NP}$.

Extra Credit Problem:

- Prove or disprove: $\mathcal{P} = \mathcal{NP}$.

This is important because there are many natural decision problems in $\mathcal{NP}$ for which no $\mathcal{P}$ (tractable) algorithm is known.

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$\mathcal{NP}$-completeness

A theory based on identifying problems that are as hard as any problems in $\mathcal{NP}$.

The next best thing to knowing whether $\mathcal{P}$= $\mathcal{NP}$ or not.

A decision problem A is **$\mathcal{NP}$-hard** if every problem in $\mathcal{NP}$ is polynomially reducible to A, that is, for all

$$B \in \mathcal{NP}, \quad B \leq_p A.$$

A decision problem A is **$\mathcal{NP}$-complete** if $A \in \mathcal{NP}$ and A is $\mathcal{NP}$-hard.

A is not permitted to be harder than $\mathcal{NP}$. For example, Tower of Hanoi is not in $\mathcal{NP}$. It requires exponential time to verify a set of moves.

Satisfiability

Let E be a Boolean expression over variables $x_1, x_2, \cdots, x_n$ in conjunctive normal form (CNF), that is, an AND of ORs.

$$E = (x_5 + x_7 + \overline{x_8} + x_{10}) \cdot (\overline{x_2} + x_3) \cdot (x_1 + \overline{x_3} + x_6).$$

A variable or its negation is called a **literal**.
Each sum is called a **clause**.

SATISFIABILITY (SAT):

- Instance: A Boolean expression E over variables $x_1, x_2, \cdots, x_n$ in CNF.
- Question: Is E satisfiable?

Cook's Theorem: SAT is $\mathcal{NP}$-complete.

Is there a truth assignment for the variables that makes E true?

Cook won a Turing award for this work.

Proof Sketch

SAT $\in \mathcal{NP}$:

- A non-deterministic algorithm **guesses** a truth assignment for $x_1, x_2, \cdots, x_n$ and **checks** whether E is true in polynomial time.
- It accepts iff there is a satisfying assignment for E.

SAT is $\mathcal{NP}$-hard:

- Start with an arbitrary problem $B \in \mathcal{NP}$.
- We know there is a polynomial-time, nondeterministic algorithm to accept B.
- Cook showed how to transform an instance X of B into a Boolean expression E that is satisfiable if the algorithm for B accepts X.

The proof of this last step is usually several pages long. One approach is to develop a nondeterministic Turing Machine program to solve an arbitrary problem B in $\mathcal{NP}$.

Implications

(1) Since SAT is $\mathcal{NP}$-complete, we have not defined an empty concept.

(2) If SAT $\in \mathcal{P}$, then $\mathcal{P} = \mathcal{NP}$.

(3) If $\mathcal{P} = \mathcal{NP}$, then SAT $\in \mathcal{P}$.

(4) If $A \in \mathcal{NP}$ and B is $\mathcal{NP}$-complete, then $B \leq_p A$ implies A is $\mathcal{NP}$-complete.
Proof:

- Let $C \in \mathcal{NP}$.
- Then $C \leq_p B$ since B is $\mathcal{NP}$-complete.
- Since $B \leq_p A$ and $\leq_p$ is transitive, $C \leq_p A$.
- Therefore, A is $\mathcal{NP}$-hard and, finally, $\mathcal{NP}$-complete.

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Implications (cont)

(5) This gives a simple two-part strategy for showing a decision problem A is $\mathcal{NP}$-complete.

(a) Show $A \in \mathcal{NP}$.
(b) Pick an $\mathcal{NP}$-complete problem B and show $B \leq_p A$.

Proving $A \in \mathcal{NP}$ is usually easy.

Don't get the reduction backwards!

$\mathcal{NP}$-completeness Proof Template

To show that decision problem B is $\mathcal{NP}$-complete:

1. $B \in \mathcal{NP}$
   - Give a polynomial time, non-deterministic algorithm that accepts B.
     1. Given an instance X of B, **guess** evidence Y.
     2. **Check** whether Y is evidence that $X \in B$. If so, accept X.
2. B is $\mathcal{NP}$-hard.
   - Choose a known $\mathcal{NP}$-complete problem, A.
   - Describe a polynomial-time transformation T of an **arbitrary** instance of A to a [not necessarily arbitrary] instance of B.
   - Show that $X \in A$ if and only if $T(X) \in B$.

$B \in \mathcal{NP}$ is usually the easy part.

The first two steps of the $\mathcal{NP}$-hard proof are usually the hardest.

3-SATISFIABILITY (3SAT)

Instance: A Boolean expression E in CNF such that each clause contains exactly 3 literals.

Question: Is there a satisfying assignment for E?

A special case of SAT.

One might hope that 3SAT is easier than SAT.

What about 2SAT? This is in $\mathcal{P}$.

Effectively a 2-coloring graph problem. Join 2 vertices if they are in same clause, also join x_i and $\overline{x_i}$. Then, try to 2-color the graph with a DFS.

How to solve 1SAT? Answer is "yes" iff x_i and $\overline{x_i}$ are not both in list for any i.

3SAT is $\mathcal{NP}$-complete

(1) 3SAT $\in \mathcal{NP}$.

```
procedure nd-3SAT(E) {
  for (i = 1 to n)
    x[i] = nd-choice(TRUE, FALSE);
  Evaluate E for the guessed truth assignment.
  if (E evaluates to TRUE)
    ACCEPT;
  else
    REJECT;
}
```

nd-3SAT is a polynomial-time nondeterministic algorithm that accepts 3SAT.

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Proving 3SAT $\mathcal{NP}$-hard

1. Choose SAT to be the known $\mathcal{NP}$-complete problem.
   - We need to show that SAT $\leq_p$ 3SAT.
2. Let $E = C_1 \cdot C_2 \cdots C_k$ be any instance of SAT.

Strategy: Replace any clause C_i that does not have exactly 3 literals with two or more clauses having exactly 3 literals.

Let $C_i = y_1 + y_2 + \cdots + y_j$ where $y_1, \cdots, y_j$ are literals.

(a) $j = 1$

- Replace (y_1) with

$$(y_1 + v + w) \cdot (y_1 + \overline{v} + w) \cdot (y_1 + v + \overline{w}) \cdot (y_1 + \overline{v} + \overline{w})$$

  where v and w are new variables.

SAT is the only choice that we have so far!

Replacing (y_1) with $(y_1 + y_1 + y_1)$ seems like a reasonable alternative. But some of the theory behind the definitions rejects clauses with duplicated literals.

Proving 3SAT $\mathcal{NP}$-hard (cont)

(b) $j = 2$

- Replace $(y_1 + y_2)$ with $(y_1 + y_2 + z) \cdot (y_1 + y_2 + \overline{z})$ where z is a new variable.

(c) $j > 3$

- Relace $(y_1 + y_2 + \cdots + y_j)$ with

$$(y_1 + y_2 + z_1) \cdot (y_3 + \overline{z_1} + z_2) \cdot (y_4 + \overline{z_2} + z_3) \cdots$$
$$(y_{j-2} + \overline{z_{j-4}} + z_{j-3}) \cdot (y_{j-1} + y_j + \overline{z_{j-3}})$$

  where $z_1, z_2, \cdots, z_{j-3}$ are new variables.

- After replacements made for each C_i, a Boolean expression E' results that is an instance of 3SAT.
- The replacement clearly can be done by a polynomial-time deterministic algorithm.

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Proving 3SAT $\mathcal{NP}$-hard (cont)

(3) Show E is satisfiable iff E' is satisfiable.

- Assume E has a satisfying truth assignment.
- Then that extends to a satisfying truth assignment for cases (a) and (b).
- In case (c), assume y_m is assigned "true".
- Then assign $z_t, t \leq m-2$, true and $z_k, t \geq m-1$, false.
- Then all the clauses in case (c) are satisfied.

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└Proving 3SAT $\mathcal{NP}$-hard (cont)

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Proving 3SAT $\mathcal{NP}$-hard (cont)

- Assume E' has a satisfying assignment.
- By restriction, we have truth assignment for E.
  - (a) y_1 is necessarily true.
  - (b) $y_1 + y_2$ is necessarily true.
  - (c) Proof by contradiction:
    - ⋆ If $y_1, y_2, \cdots, y_j$ are all false, then $z_1, z_2, \cdots, z_{j-3}$ are all true.
    - ⋆ But then $(y_{j-1} + y_{j-2} + \overline{z_{j-3}})$ is false, a contradiction.

We conclude SAT $\leq$ 3SAT and 3SAT is $\mathcal{NP}$-complete.

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└Proving 3SAT $\mathcal{NP}$-hard (cont)

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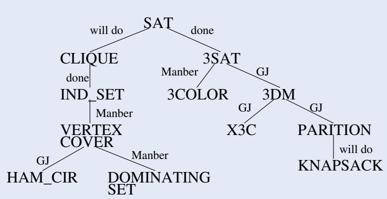
Tree of Reductions



Reductions go down the tree.

Proofs that each problem $\in \mathcal{NP}$ are straightforward.

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└Tree of Reductions

Refer to handout of $\mathcal{NP}$-complete problems

Perspective

The reduction tree gives us a collection of 12 diverse $\mathcal{NP}$-complete problems.
The complexity of all these problems depends on the complexity of any one:

- If any $\mathcal{NP}$-complete problem is tractable, then they all are.

This collection is a good place to start when attempting to show a decision problem is $\mathcal{NP}$-complete.

Observation: If we find a problem is $\mathcal{NP}$-complete, then we should do something other than try to find a $\mathcal{P}$-time algorithm.

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└Perspective

Hundreds of problems, from many fields, have been shown to be $\mathcal{NP}$-complete.

More on this observation later.