CS 5114: Theory of Algorithms

Clifford A. Shaffer

Department of Computer Science
Virginia Tech
Blacksburg, Virginia

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Title page

Students should be familiar with inductive proofs, recursion, data structures, and programming at the CS3114 level.

Graph Algorithms

Graphs are useful for representing a variety of concepts:

- Data Structures
- Relationships
- Families
- Communication Networks
- Road Maps

- A **graph** $G = (V, E)$ consists of a set of **vertices** V, and a set of **edges** E, such that each edge in E is a connection between a pair of vertices in V.
- Directed vs. Undirected
- Labeled graph, weighted graph
- Labels for edges vs. weights for edges
- Multiple edges, loops
- Cycle, Circuit, path, simple path, tours
- Bipartite, acyclic, connected
- Rooted tree, unrooted tree, free tree

A Tree Proof

- **Definition**: A **free tree** is a connected, undirected graph that has no cycles.
- **Theorem**: If T is a free tree having n vertices, then T has exactly $n - 1$ edges.
- **Proof**: By induction on n.
- **Base Case**: $n = 1$. T consists of 1 vertex and 0 edges.
- **Inductive Hypothesis**: The theorem is true for a tree having $n - 1$ vertices.
- **Inductive Step**:
  - If T has n vertices, then T contains a vertex of degree 1.
  - Remove that vertex and its incident edge to obtain T', a free tree with $n - 1$ vertices.
  - By IH, T' has $n - 2$ edges.
  - Thus, T has $n - 1$ edges.

This is close to a satisfactory definition for free tree. There are several equivalent definitions for free trees, with similar proofs to relate them.

Why do we know that some vertex has degree 1? Because the definition says that the Free Tree has no cycles.

Graph Traversals

Various problems require a way to **traverse** a graph – that is, visit each vertex and edge in a systematic way.

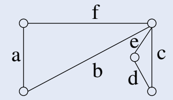
Three common traversals:

1. Eulerian tours
   Traverse each edge exactly once
2. Depth-first search
   Keeps vertices on a stack
3. Breadth-first search
   Keeps vertices on a queue

a vertex may be visited multiple times

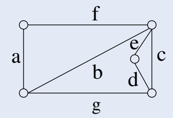
Eulerian Tours

A circuit that contains every edge exactly once.
Example:

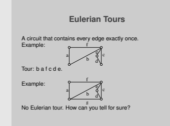


Tour: b a f c d e.

Example:



No Eulerian tour. How can you tell for sure?

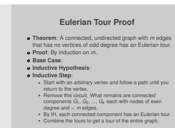


Why no tour? Because some vertices have odd degree.

All even nodes is a necessary condition. Is it sufficient?

Eulerian Tour Proof

- **Theorem**: A connected, undirected graph with m edges that has no vertices of odd degree has an Eulerian tour.
- **Proof**: By induction on m.
- **Base Case**:
- **Inductive Hypothesis**:
- **Inductive Step**:
  - Start with an arbitrary vertex and follow a path until you return to the vertex.
  - Remove this circuit. What remains are connected components G_1, G_2, ..., G_k each with nodes of even degree and $< m$ edges.
  - By IH, each connected component has an Eulerian tour.
  - Combine the tours to get a tour of the entire graph.



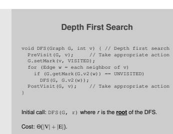
Base case: 0 edges and 1 vertex fits the theorem.
IH: The theorem is true for $< m$ edges.
Always possible to find a circuit starting at any arbitrary vertex, since each vertex has even degree.

Depth First Search

```
void DFS(Graph G, int v) { // Depth first search
  PreVisit(G, v);      // Take appropriate action
  G.setMark(v, VISITED);
  for (Edge w = each neighbor of v)
    if (G.getMark(G.v2(w)) == UNVISITED)
      DFS(G, G.v2(w));
  PostVisit(G, v);     // Take appropriate action
}
```

Initial call: `DFS(G, r)` where r is the **root** of the DFS.

Cost: $\Theta(|V| + |E|)$.



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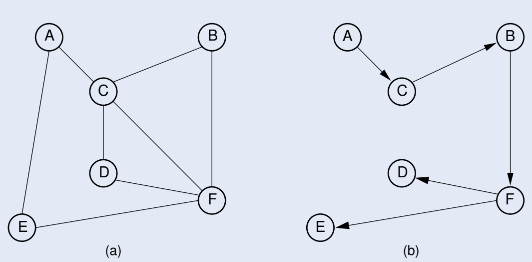
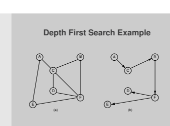
Depth First Search Example





The directions are imposed by the traversal. This is the Depth First Search Tree.

DFS Tree

If we number the vertices in the order that they are marked, we get **DFS numbers**.

Lemma 7.2: Every edge $e \in E$ is either in the DFS tree T, or connects two vertices of G, one of which is an ancestor of the other in T.

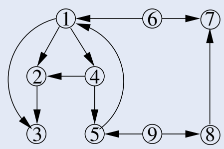
Proof: Consider the first time an edge (v, w) is examined, with v the current vertex.

- If w is unmarked, then (v, w) is in T.
- If w is marked, then w has a smaller DFS number than v AND (v, w) is an unexamined edge of w.
- Thus, w is still on the stack. That is, w is on a path from v.

Results: No "cross edges." That is, no edges connecting vertices sideways in the tree.

DFS for Directed Graphs

- Main problem: A connected graph may not give a single DFS tree.



- Forward edges: (1, 3)
- Back edges: (5, 1)
- Cross edges: (6, 1), (8, 7), (9, 5), (9, 8), (4, 2)
- **Solution**: Maintain a list of unmarked vertices.
  - Whenever one DFS tree is complete, choose an arbitrary unmarked vertex as the root for a new tree.

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Directed Cycles

Lemma 7.4: Let G be a directed graph. G has a directed cycle iff every DFS of G produces a back edge.

Proof:

1. Suppose a DFS produces a back edge (v, w).
   - v and w are in the same DFS tree, w an ancestor of v.
   - (v, w) and the path in the tree from w to v form a directed cycle.
2. Suppose G has a directed cycle C.
   - Do a DFS on G.
   - Let w be the vertex of C with smallest DFS number.
   - Let (v, w) be the edge of C coming into w.
   - v is a descendant of w in a DFS tree.
   - Therefore, (v, w) is a back edge.

See earlier lemma.

Breadth First Search

- Like DFS, but replace stack with a queue.
- Visit vertex's neighbors before going deeper in tree.

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Breadth First Search Algorithm

```
void BFS(Graph G, int start) {
  Queue Q(G.n());
  Q.enqueue(start);
  G.setMark(start, VISITED);
  while (!Q.isEmpty()) {
    int v = Q.dequeue();
    PreVisit(G, v);   // Take appropriate action
    for (Edge w = each neighbor of v)
      if (G.getMark(G.v2(w)) == UNVISITED) {
        G.setMark(G.v2(w), VISITED);
        Q.enqueue(G.v2(w));
      }
    PostVisit(G, v);  // Take appropriate action
}}
```


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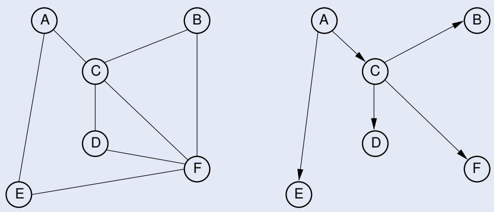
Breadth First Search Example



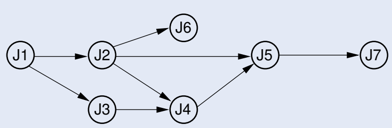
(a) (b)

Non-tree edges connect vertices at levels differing by 0 or 1.

We know this because if an edge had connected to a deeper level, then that target node would have been placed on the queue when the edge was encountered.

Topological Sort

Problem: Given a set of jobs, courses, etc. with prerequisite constraints, output the jobs in an order that does not violate any of the prerequisites.



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Topological Sort Algorithm

```
void topsort(Graph G) { // Top sort: recursive
  for (int i=0; i<G.n(); i++) // Initialize Mark
    G.setMark(i, UNVISITED);
  for (i=0; i<G.n(); i++)     // Process vertices
    if (G.getMark(i) == UNVISITED)
      tophelp(G, i);          // Call helper
}
void tophelp(Graph G, int v) { // Helper function
  G.setMark(v, VISITED);
  for (Edge w = each neighbor of v)
    if (G.getMark(G.v2(w)) == UNVISITED)
      tophelp(G, G.v2(w));
  printout(v);         // PostVisit for Vertex v
}
```


Prints in reverse order.

Queue-based Topological Sort

```
void topsort(Graph G) { // Top sort: Queue
  Queue Q(G.n());  int Count[G.n()];
  for (int v=0; v<G.n(); v++) Count[v] = 0;
  for (v=0; v<G.n(); v++) // Process every edge
    for (Edge w each neighbor of v)
       Count[G.v2(w)]++;  // Add to v2's count
  for (v=0; v<G.n(); v++) // Initialize Queue
    if (Count[v] == 0) Q.enqueue(v);
  while (!Q.isEmpty()) {  // Process the vertices
    int v = Q.dequeue();
    printout(v);          // PreVisit for v
    for (Edge w = each neighbor of v) {
      Count[G.v2(w)]--;   // One less prereq
      if (Count[G.v2(w)]==0) Q.enqueue(G.v2(w));
}}}
```


└Queue-based Topological Sort

no notes

Shortest Paths Problems

Input: A graph with **weights** or **costs** associated with each edge.

Output: The list of edges forming the shortest path.

Sample problems:

- Find the shortest path between two specified vertices.
- Find the shortest path from vertex S to all other vertices.
- Find the shortest path between all pairs of vertices.

Our algorithms will actually calculate only **distances**.

└Shortest Paths Problems

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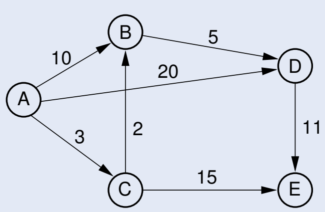
Shortest Paths Definitions

d(A, B) is the **shortest distance** from vertex A to B.

w(A, B) is the **weight** of the edge connecting A to B.

- If there is no such edge, then w(A, B) $= \infty$.



└Shortest Paths Definitions

w(A, D) $=$ 20; d(A, D) $=$ 10 (through ACBD).

Single Source Shortest Paths

Given start vertex s, find the shortest path from s to all other vertices.

Try 1: Visit all vertices in some order, compute shortest paths for all vertices seen so far, then add the shortest path to next vertex x.

Problem: Shortest path to a vertex already processed might go through x.
Solution: Process vertices in order of distance from s.

└Single Source Shortest Paths

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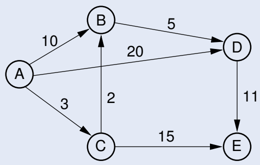
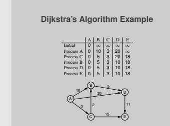
Dijkstra's Algorithm Example

	A	B	C	D	E
Initial	0	∞	∞	∞	∞
Process A	0	10	3	20	∞
Process C	0	5	3	20	18
Process B	0	5	3	10	18
Process D	0	5	3	10	18
Process E	0	5	3	10	18





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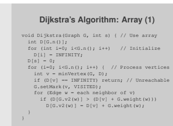
Dijkstra's Algorithm: Array (1)

```
void Dijkstra(Graph G, int s) { // Use array
  int D[G.n()];
  for (int i=0; i<G.n(); i++)   // Initialize
    D[i] = INFINITY;
  D[s] = 0;
  for (i=0; i<G.n(); i++) {  // Process vertices
    int v = minVertex(G, D);
    if (D[v] == INFINITY) return; // Unreachable
    G.setMark(v, VISITED);
    for (Edge w = each neighbor of v)
      if (D[G.v2(w)] > (D[v] + G.weight(w)))
        D[G.v2(w)] = D[v] + G.weight(w);
  }
}
```

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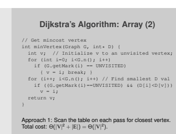
Dijkstra's Algorithm: Array (2)

```
// Get mincost vertex
int minVertex(Graph G, int* D) {
  int v;  // Initialize v to an unvisited vertex;
  for (int i=0; i<G.n(); i++)
    if (G.getMark(i) == UNVISITED)
      { v = i; break; }
  for (i++; i<G.n(); i++) // Find smallest D val
    if ((G.getMark(i)==UNVISITED) && (D[i]<D[v]))
      v = i;
  return v;
}
```

Approach 1: Scan the table on each pass for closest vertex.
Total cost: $\Theta(|V|^2 + |E|) = \Theta(|V|^2)$.



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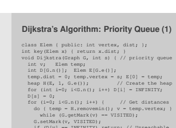
Dijkstra's Algorithm: Priority Queue (1)

```
class Elem { public: int vertex, dist; };
int key(Elem x) { return x.dist; }
void Dijkstra(Graph G, int s) { // priority queue
  int v;   Elem temp;
  int D[G.n()];  Elem E[G.e()];
  temp.dist = 0; temp.vertex = s; E[0] = temp;
  heap H(E, 1, G.e());         // Create the heap
  for (int i=0; i<G.n(); i++) D[i] = INFINITY;
  D[s] = 0;
  for (i=0; i<G.n(); i++) {     // Get distances
    do { temp = H.removemin(); v = temp.vertex; }
      while (G.getMark(v) == VISITED);
    G.setMark(v, VISITED);
    if (D[v] == INFINITY) return; // Unreachable
```

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Dijkstra's Algorithm: Priority Queue (2)

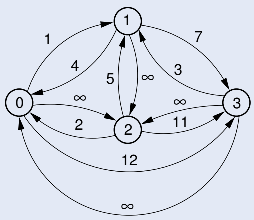
```
    for (Edge w = each neighbor of v)
      if (D[G.v2(w)] > (D[v] + G.weight(w))) {
        D[G.v2(w)] = D[v] + G.weight(w);
        temp.dist = D[G.v2(w)];
        temp.vertex = G.v2(w);
        H.insert(temp);  // Insert new distance
}}}
```

- Approach 2: Store unprocessed vertices using a min-heap to implement a priority queue ordered by D value. Must update priority queue for each edge.
- Total cost: $\Theta((|\mathrm{V}| + |\mathrm{E}|) \log |\mathrm{V}|)$.

no notes

All Pairs Shortest Paths

- For every vertex $u, v \in \mathrm{V}$, calculate d(u, v).
- Could run Dijkstra's Algorithm |V| times.
- Better is **Floyd's Algorithm**.
- Define a **k-path** from u to v to be any path whose intermediate vertices all have indices less than k.



Multiple runs of Dijkstra's algorithm Cost: $|V||E| \log |V| = |V|^3 \log |V|$ for dense graph.

The issue driving the concept of "k paths" is how to efficiently check all the paths without computing any path more than once.

0,3 is a 0-path. 2,0,3 is a 1-path. 0,2,3 is a 3-path, but not a 2 or 1 path. Everything is a 4 path.

Floyd's Algorithm

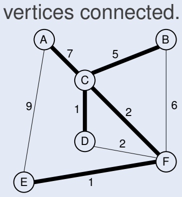
```
void Floyd(Graph G) { // All-pairs shortest paths
  int D[G.n()][G.n()]; // Store distances
  for (int i=0; i<G.n(); i++) // Initialize D
    for (int j=0; j<G.n(); j++)
      D[i][j] = G.weight(i, j);
  for (int k=0; k<G.n(); k++) // Compute k paths
    for (int i=0; i<G.n(); i++)
      for (int j=0; j<G.n(); j++)
        if (D[i][j] > (D[i][k] + D[k][j]))
          D[i][j] = D[i][k] + D[k][j];
}
```


no notes

Minimum Cost Spanning Trees

Minimum Cost Spanning Tree (MST) Problem:

- Input: An undirected, connected graph G.
- Output: The subgraph of G that
  1. has minimum total cost as measured by summing the values for all of the edges in the subset, and
  2. keeps the vertices connected.



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Key Theorem for MST

Let V_1, V_2 be an arbitrary, non-trivial partition of V. Let (v_1, v_2), $v_1 \in V_1, v_2 \in V_2$, be the cheapest edge between V_1 and V_2. Then (v_1, v_2) is in some MST of G.

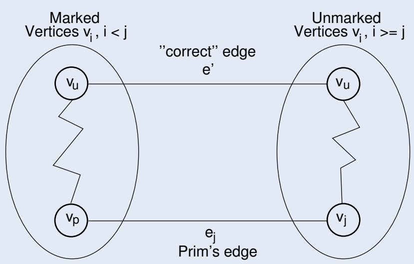
Proof:

- Let T be an arbitrary MST of G.
- If (v_1, v_2) is in T, then we are done.
- Otherwise, adding (v_1, v_2) to T creates a cycle C.
- At least one edge (u_1, u_2) of C other than (v_1, v_2) must be between V_1 and V_2.
- $c(u_1, u_2) \geq c(v_1, v_2)$.
- Let $T' = T \cup \{(v_1, v_2)\} - \{(u_1, u_2)\}$.
- Then, T' is a spanning tree of G and $c(T') \leq c(T)$.
- But $c(T)$ is minimum cost.

Therefore, $c(T') = c(T)$ and T' is a MST containing (v_1, v_2).

There can only be multiple MSTs when there are edges with equal cost.

Key Theorem Figure



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Prim's MST Algorithm (1)

```
void Prim(Graph G, int s) {     // Prim's MST alg
  int D[G.n()];  int V[G.n()];  // Distances
  for (int i=0; i<G.n(); i++)   // Initialize
    D[i] = INFINITY;
  D[s] = 0;
  for (i=0; i<G.n(); i++) {   // Process vertices
    int v = minVertex(G, D);
    G.setMark(v, VISITED);
    if (v != s) AddEdgetoMST(V[v], v);
    if (D[v] == INFINITY) return; //v unreachable
    for (Edge w = each neighbor of v)
      if (D[G.v2(w)] > G.weight(w)) {
        D[G.v2(w)] = G.weight(w); // Update dist
        V[G.v2(w)] = v;          // who came from
}}}
```


no notes

Prim's MST Algorithm (2)

```
int minVertex(Graph G, int* D) {
  int v;  // Initialize v to any unvisited vertex
  for (int i=0; i<G.n(); i++)
    if (G.getMark(i) == UNVISITED)
      { v = i; break; }
  for (i=0; i<G.n(); i++) // Find smallest value
    if ((G.getMark(i)==UNVISITED) && (D[i]<D[v]))
      v = i;
  return v;
}
```

This is an example of a **greedy** algorithm.

no notes

Alternative Prim's Implementation (1)

Like Dijkstra's algorithm, can implement with priority queue.

```
void Prim(Graph G, int s) {
  int v;                  // The current vertex
  int D[G.n()];           // Distance array
  int V[G.n()];           // Who's closest
  Elem temp;
  Elem E[G.e()];          // Heap array
  temp.distance = 0; temp.vertex = s;
  E[0] = temp;            // Initialize heap array
  heap H(E, 1, G.e());    // Create the heap
  for (int i=0; i<G.n(); i++) D[i] = INFINITY;
  D[s] = 0;
```


Alternative Prim's Implementation (1)

no notes

Alternative Prim's Implementation (2)

```
  for (i=0; i<G.n(); i++) { // Now build MST
    do { temp = H.removemin(); v = temp.vertex; }
      while (G.getMark(v) == VISITED);
    G.setMark(v, VISITED);
    if (v != s) AddEdgetoMST(V[v], v);
    if (D[v] == INFINITY) return; // Unreachable
    for (Edge w = each neighbor of v)
      if (D[G.v2(w)] > G.weight(w)) { // Update D
        D[G.v2(w)] = G.weight(w);
        V[G.v2(w)] = v;          // Who came from
        temp.distance = D[G.v2(w)];
        temp.vertex = G.v2(w);
        H.insert(temp);    // Insert dist in heap
      }
}}
```


Alternative Prim's Implementation (2)

no notes