Coping with NP-Completeness

T. M. Murali

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How Do We Tackle an $\mathcal{NP}$-Complete Problem?

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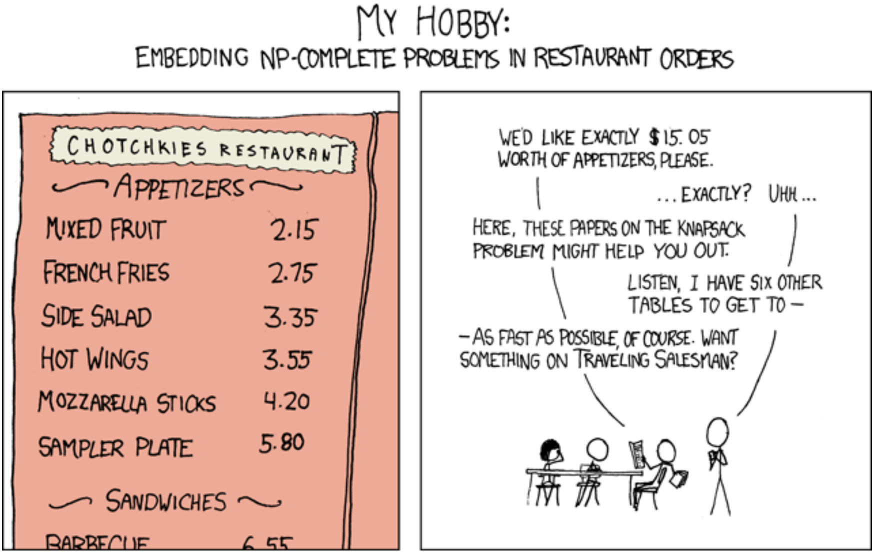

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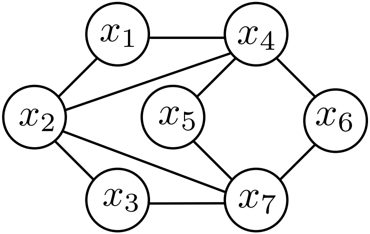
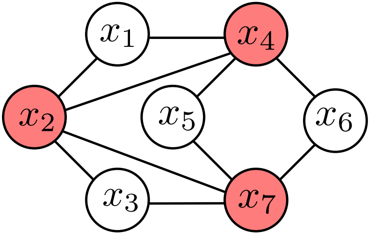
  - Develop algorithms that are exponential in one parameter in the problem.
  - Consider special cases of the input, e.g., graphs that "look like" trees.
  - Develop algorithms that can provably compute a solution close to the optimal.

Vertex Cover Problem





VERTEX COVER

INSTANCE: Undirected graph G and an integer k

QUESTION: Does G contain a vertex cover of size at most k?

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- What is the running time of a brute-force algorithm?

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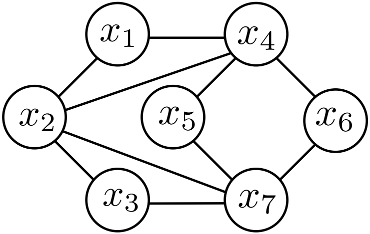

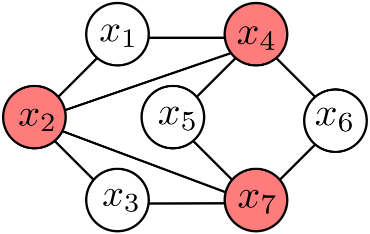

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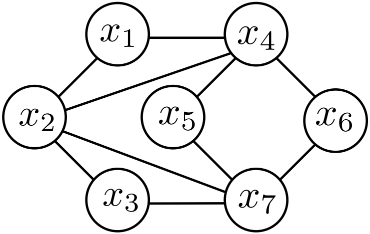

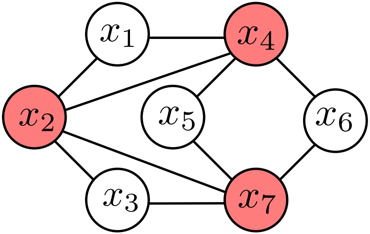

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- What is the running time of a brute-force algorithm? $O(kn\binom{n}{k}) = O(kn^{k+1})$.
- Can we devise an algorithm whose running time is exponential in k but polynomial in n, e.g., $O(2^k n)$?

Designing the Vertex Cover Algorithm

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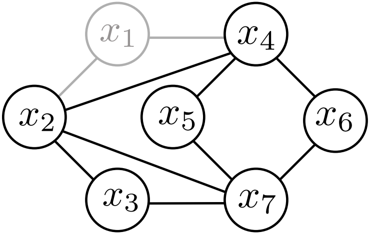

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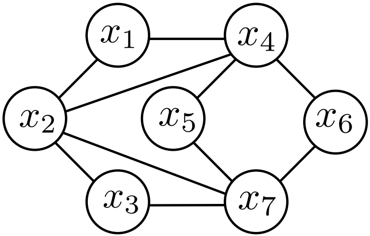

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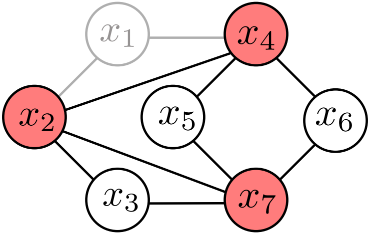
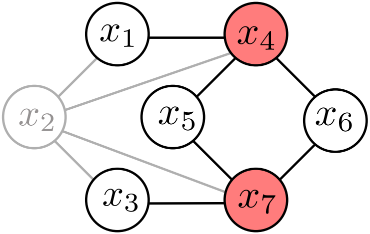
- $G - \{u\}$ is the graph G without node u and the edges incident on u.
- Consider an edge (u, v). Either u or v must be in the vertex cover.
- Claim: G has a vertex cover of size at most k iff for any edge (u, v) either $G - \{u\}$ or $G - \{v\}$ has a vertex cover of size at most $k - 1$.





Vertex Cover Algorithm

To search for a k-node vertex cover in G:
  If G contains no edges, then the empty set is a vertex cover
  If G contains $> k\ |V|$ edges, then it has no k-node vertex cover
  Else let $e = (u, v)$ be an edge of G
    Recursively check if either of $G - \{u\}$ or $G - \{v\}$
        has a vertex cover of size $k - 1$
    If neither of them does, then G has no k-node vertex cover
    Else, one of them (say, $G - \{u\}$) has a $(k-1)$-node vertex cover T
      In this case, $T \cup \{u\}$ is a k-node vertex cover of G
    Endif
  Endif

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- Claim: $T(n, k) = O(2^k kn)$.

Solving $\mathcal{NP}$-Hard Problems on Trees

- "$\mathcal{NP}$-Hard": at least as hard as $\mathcal{NP}$-Complete. We will use $\mathcal{NP}$-Hard to refer to optimisation versions of decision problems.

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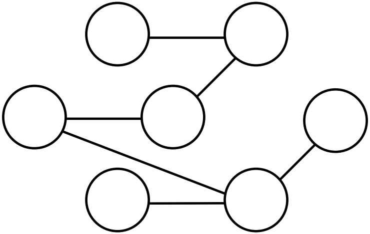
- "$\mathcal{NP}$-Hard": at least as hard as $\mathcal{NP}$-Complete. We will use $\mathcal{NP}$-Hard to refer to optimisation versions of decision problems.
- Many $\mathcal{NP}$-Hard problems can be solved efficiently on trees.
- Intuition: subtree rooted at any node v of the tree "interacts" with the rest of tree only through v. Therefore, depending on whether we include v in the solution or not, we can decouple solving the problem in v's subtree from the rest of the tree.

Designing Greedy Algorithm for Independent Set



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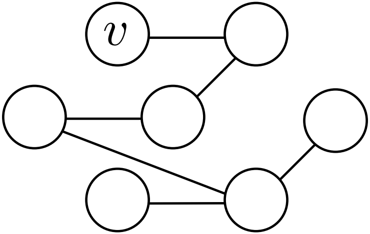

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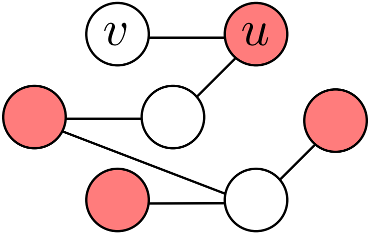

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  - Let S be a maximum-size independent set that does not contain v.
  - Let v be connected to u.
  - u must be in S; otherwise, we can add v to S, which means S is not maximum size.
  - Since u is in S, we can swap u and v.

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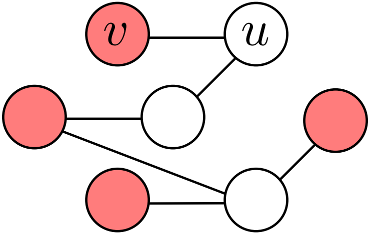

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- Claim: If a tree T has a a leaf v, then a maximum-size independent set in T is v and a maximum-size independent set in $T - \{v\}$.

Greedy Algorithm for Independent Set

- A *forest* is a graph where every connected component is a tree.

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To find a maximum-size independent set in a forest F:
  Let S be the independent set to be constructed (initially empty)
  While F has at least one edge
     Let e = (u, v) be an edge of F such that v is a leaf
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Greedy Algorithm for Independent Set

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- The algorithm works correctly on any graph for which we can repeatedly find a leaf.

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Maximum Weight Independent Set

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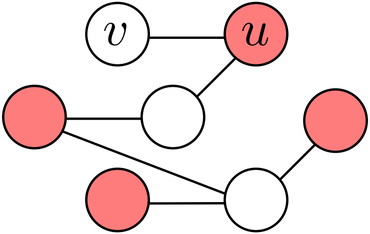

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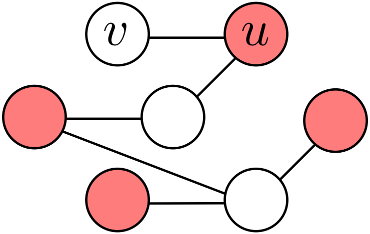

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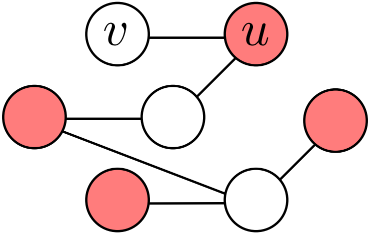

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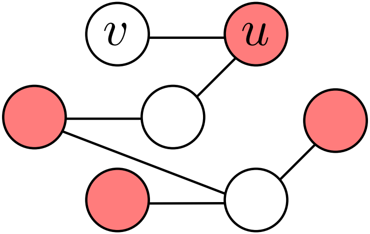

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- Suggests dynamic programming algorithm.

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- Dynamic programming algorithm needs a set of sub-problems, recursion to combine sub-problems, and order over sub-problems.
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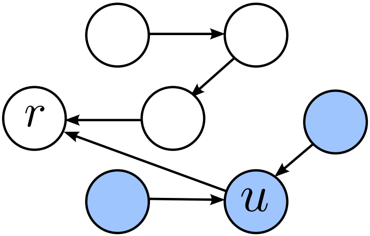

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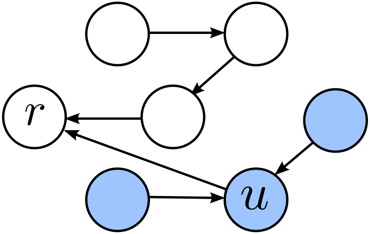
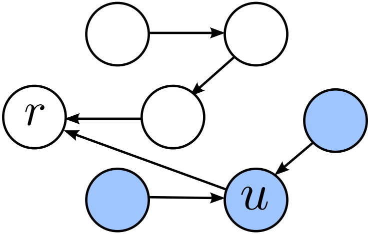
- Ordering the sub-problems: start at leaves and work our way up to the root.



Recursion for Dynamic Programming Algorithm



- Either we include u in an optimal solution or exclude u.
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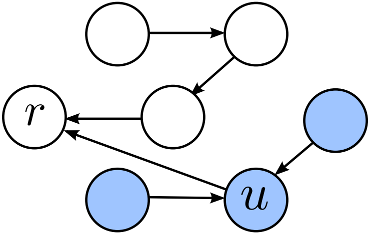

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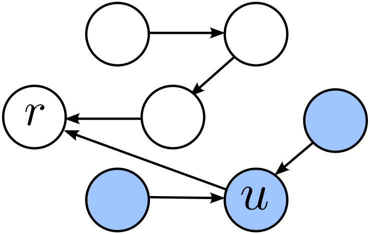

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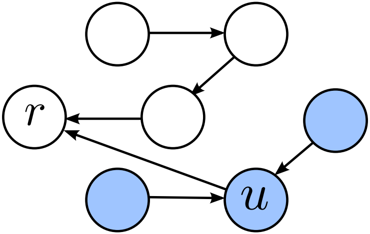

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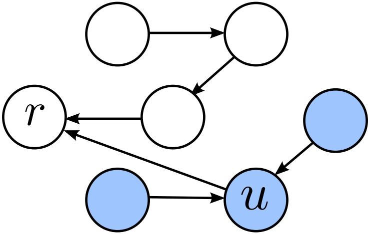

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     $\mathrm{OPT}_{\mathrm{in}}(u) = w_u + \sum_{v \in \mathrm{children}(u)} \mathrm{OPT}_{\mathrm{out}}(v)$
  2. If we exclude u, a child may or may not be excluded.
     $\mathrm{OPT}_{\mathrm{out}}(u) = \sum_{v \in \mathrm{children}(u)} \max\left(\mathrm{OPT}_{\mathrm{in}}(v), \mathrm{OPT}_{\mathrm{in}}(v)\right)$

Dynamic Programming Algorithm

To find a maximum-weight independent set of a tree T:

Root the tree at a node r

For all nodes u of T in post-order

If u is a leaf then set the values:

$$M_{out}[u] = 0$$

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- Trees are only a very specific sub-class of graphs. What use are algorithms for $\mathcal{NP}$-Hard problems that work well on trees?

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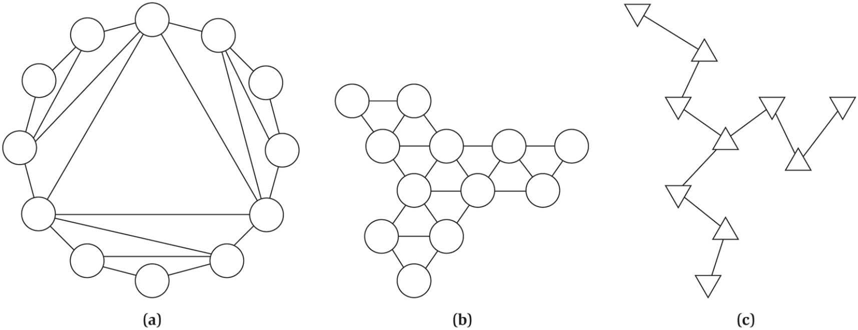
- Trees are only a very specific sub-class of graphs. What use are algorithms for $\mathcal{NP}$-Hard problems that work well on trees?
- These ideas can be generalised to graphs that "look like" trees: graphs with bounded treewidth.

Example of Tree Decomposition



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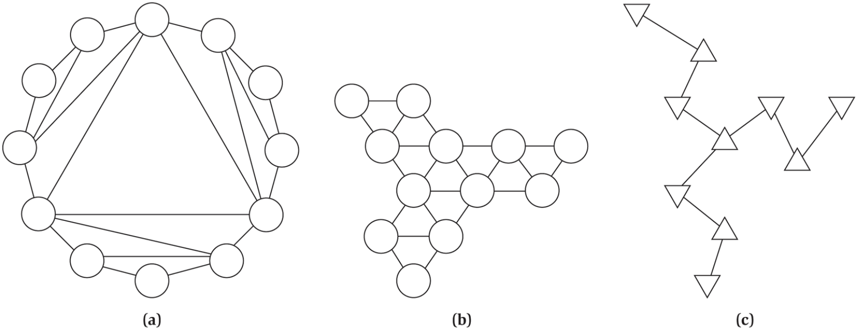

Figure 10.5 Parts (a) and (b) depict the same graph drawn in different ways. The drawing in (b) emphasizes the way in which it is composed of ten interlocking triangles. Part (c) illustrates schematically how these ten triangles "fit together."

- Definition of "tree-like" should capture graphs that we can decompose into disconnected pieces by removing a small number of nodes.
- Definition should make precise the notion of "tree-like" structures in the figure.

Tree Decompositions

A *Tree decomposition* of a graph $G(V, E)$ consists of

1. a tree T (whose nodes are different from V)
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(Node coverage): Every node of G belongs to at least one piece V_t

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A *Tree decomposition* of a graph $G(V, E)$ consists of

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Properties of Tree Decompositions

- Trees have two nice separation properties:
  1. If we delete an edge from a tree, the tree splits into two connected components.
  2. If we delete a node and all incident edges from a tree, the tree splits into a number of connected components equal to the degree of the node.
- Tree decompositions have analogous properties.

Uses of Tree Decompositions

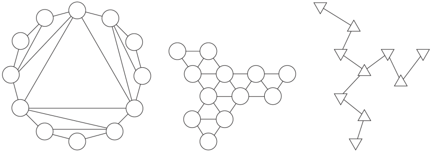

Figure 10.5 Parts (a) and (b) depict the same graph drawn in different ways. The drawing in (b) emphasizes the way in which it is composed of ten interlocking triangles. Part (c) illustrates schematically how these ten triangles "fit together."

- *Width* of a tree decomposition is the size of the largest piece.
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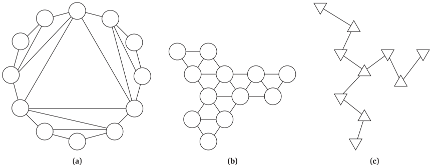

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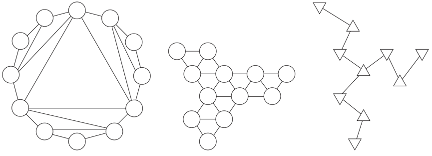

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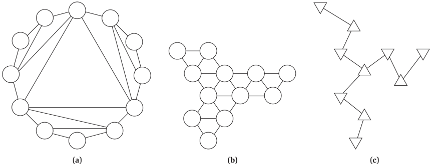

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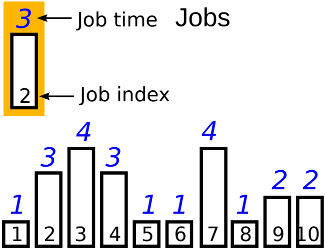
- Does a graph a tree decomposition with width at most w? $\mathcal{NP}$-Complete!
- (Chapter 10.5): Given a graph and a parameter w, there is an algorithm that runs in $O(f(w)mn)$ time and either
  1. produces a tree decomposition of width at most $4w$ or
  2. reports correctly that G does not have a tree decomposition with width less than w.

Approximation Algorithms

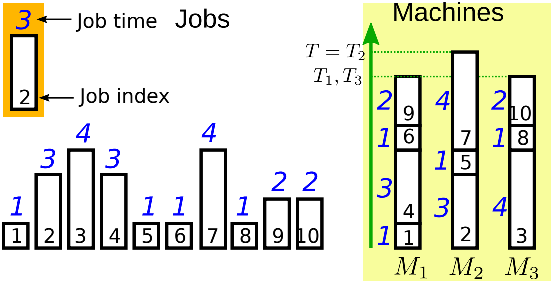
- Methods for optimisation versions of $\mathcal{NP}$-Complete problems.
- Run in polynomial time.
- Solution returned is guaranteed to be within a small factor of the optimal solution

Load Balancing Problem



- Given set of m machines $M_1, M_2, \ldots M_m$.
- Given a set of n jobs: job j has processing time t_j.
- Assign each job to one machine so that the total time spent is minimised.

Load Balancing Problem



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- Total time spent on machine i $T_i = \sum_{k \in A(i)} t_k$.
- Minimise *makespan* $T = \max_i T_i$, the largest load on any machine.

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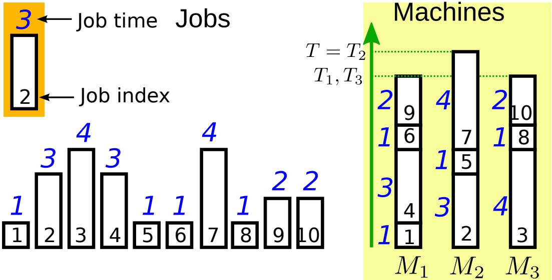

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- Minimise *makespan* $T = \max_i T_i$, the largest load on any machine.
- Minimising makespan is $\mathcal{NP}$-Complete.

Greedy-Balance Algorithm

- Adopt a greedy approach.
- Process jobs in *any* order.
- Assign next job to the processor that has smallest total load so far.

Greedy-Balance:

Start with no jobs assigned

Set $T_i = 0$ and $A(i) = \emptyset$ for all machines M_i

For $j = 1, \ldots, n$

  Let M_i be a machine that achieves the minimum $\min_k T_k$

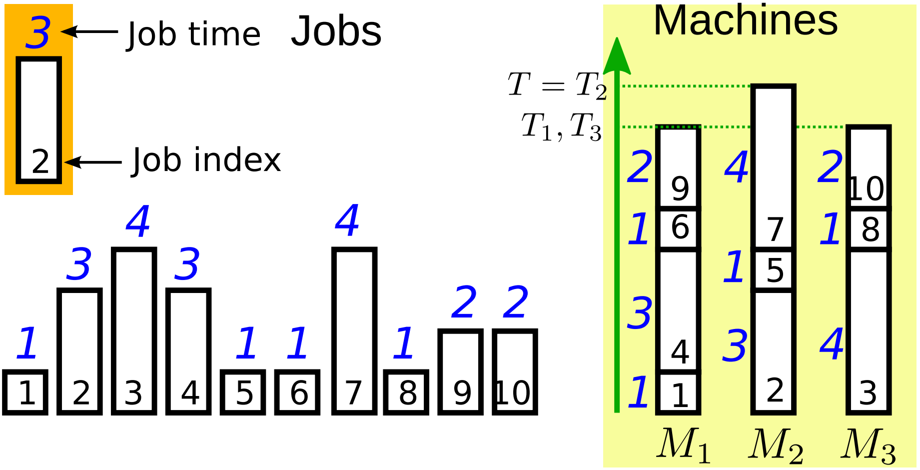
  Assign job j to machine M_i

  Set $A(i) \leftarrow A(i) \cup \{j\}$

  Set $T_i \leftarrow T_i + t_j$

EndFor

Example of Greedy-Balance Algorithm



Lower Bounds on the Optimal Makespan

- We need a lower bound on the optimum makespan T^*.

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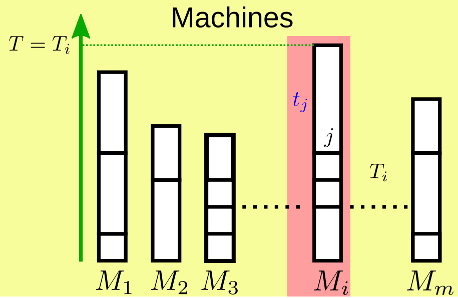
- The two bounds below will suffice:

$$T^* \geq \frac{1}{m} \sum_j t_j$$

$$T^* \geq \max_j t_j$$

Analysing Greedy-Balance



- Claim: Computed makespan $T \leq 2T^*$.

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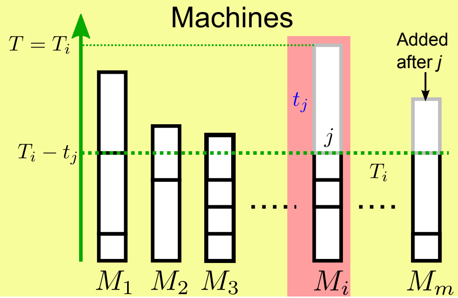

- Claim: Computed makespan $T \leq 2T^*$.
- Let M_i be the machine whose load is T and j be the last job placed on M_i.
- What was the situation just before placing this job?

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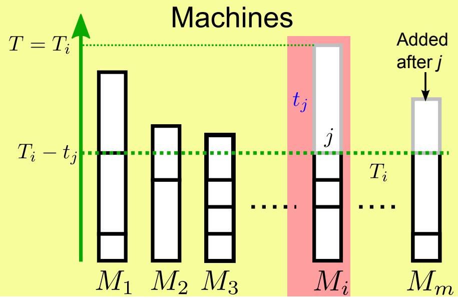
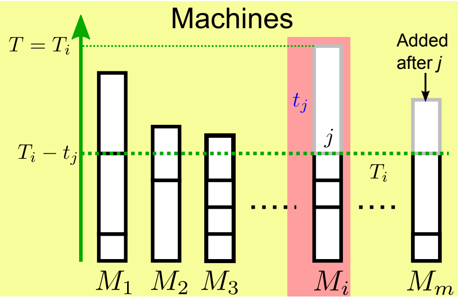


- Claim: Computed makespan $T \leq 2T^*$.
- Let M_i be the machine whose load is T and j be the last job placed on M_i.
- What was the situation just before placing this job?
- M_i had the smallest load and its load was $T - t_j$.
- For every machine M_k, load $T_k \geq T - t_j$.

Analysing Greedy-Balance



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- For every machine M_k, load $T_k \geq T - t_j$.

$$\sum_k T_k \geq m(T - t_j), \text{ where } k \text{ ranges over machines}$$

$$\sum_j t_j \geq m(T - t_j), \text{ where } j \text{ ranges over jobs}$$

$$T - t_j \leq 1/m \sum_j t_j \leq T^*$$

$$T \leq 2T^*, \text{ since } t_j \leq T^*$$

Improving the Bound

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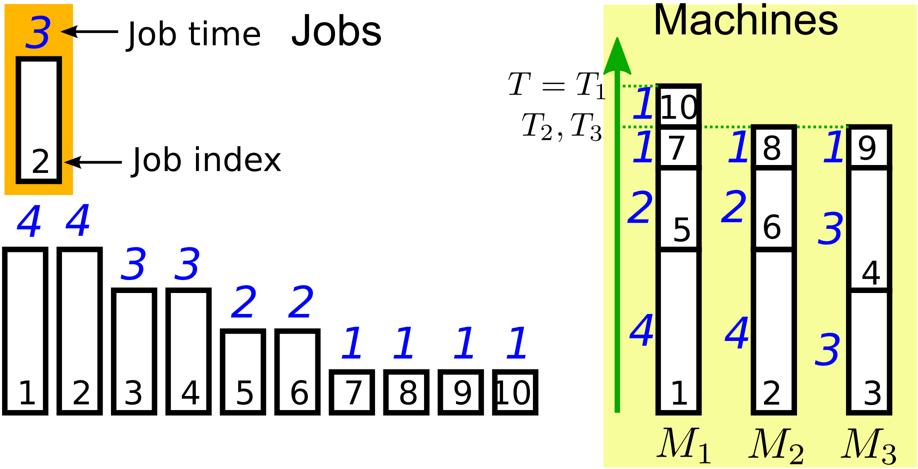
- How can we improve the algorithm?
- What if we process the jobs in decreasing order of processing time?

Sorted-Balance Algorithm

```
Sorted-Balance:
Start with no jobs assigned
Set T_i = 0 and A(i) = ∅ for all machines M_i
Sort jobs in decreasing order of processing times t_j
Assume that t_1 ≥ t_2 ≥ ... ≥ t_n
For j = 1, ..., n
  Let M_i be the machine that achieves the minimum min_k T_k
  Assign job j to machine M_i
  Set A(i) ← A(i) ∪ {j}
  Set T_i ← T_i + t_j
EndFor
```

Example of Sorted-Balance Algorithm



Analyzing Sorted-Balance

- Claim: if there are fewer than m jobs, algorithm is optimal.
- Claim: if there are more than m jobs, then $T^* \geq 2t_{m+1}$.

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  - Consider *any* assignment of these $m + 1$ jobs to machines.
  - Some machine must be assigned two jobs, each with processing time at least t_{m+1}.
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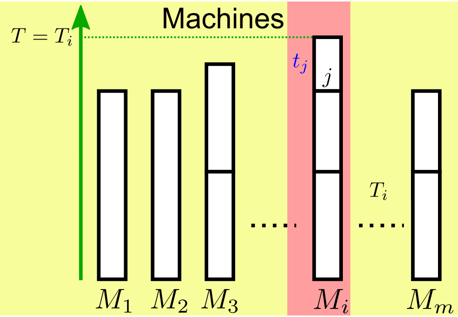

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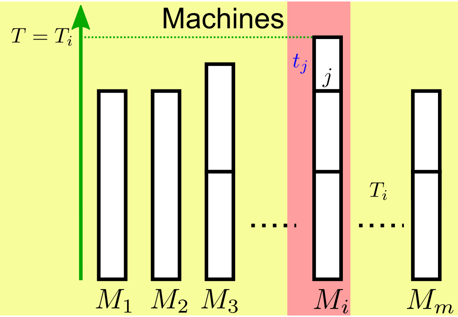
$$t_j \leq t_{m+1} \leq T^*/2, \text{ since } j \geq m+1$$
$$T - t_j \leq T^*, \text{ Greedy-Balance proof}$$
$$T \leq 3T^*/2$$

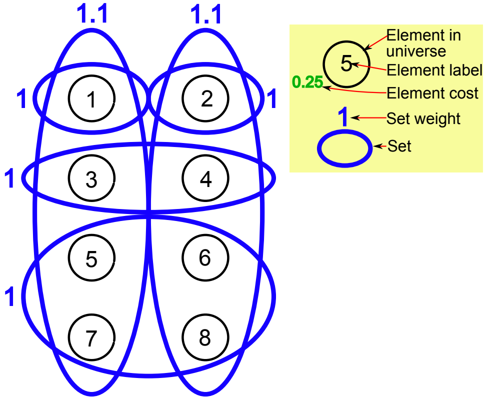


Set Cover

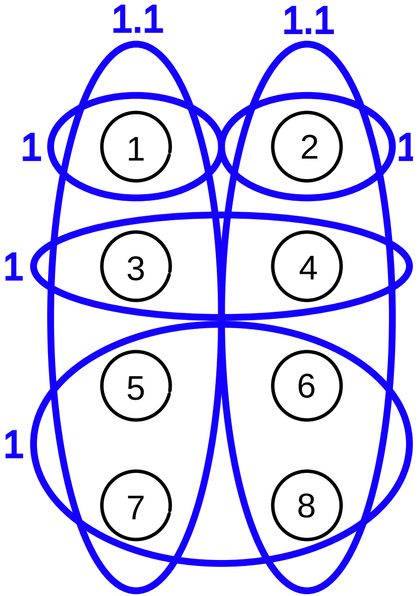
SET COVER

INSTANCE: A set U of n elements, a collection $S_1, S_2, \ldots, S_m$ of subsets of U, each with an associated weight w.

SOLUTION: A collection $\mathcal{C}$ of sets in the collection such that $\bigcup_{S_i \in C} S_i = U$ and $\sum_{S_i \in C} w_i$ is minimised.



Greedy Approach



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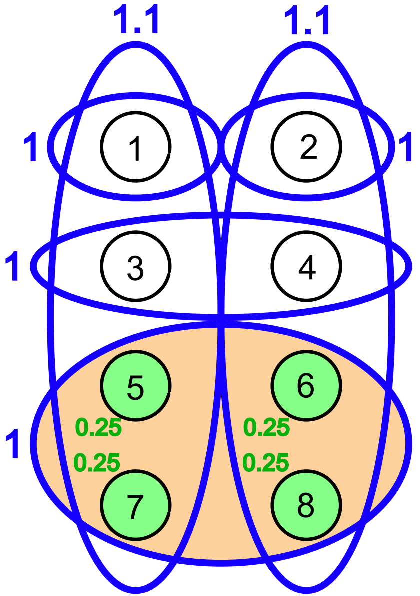

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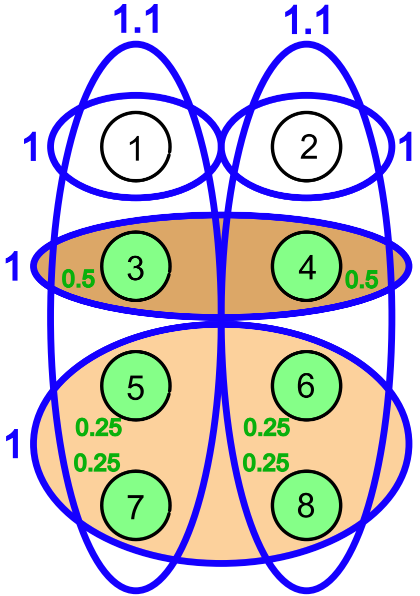

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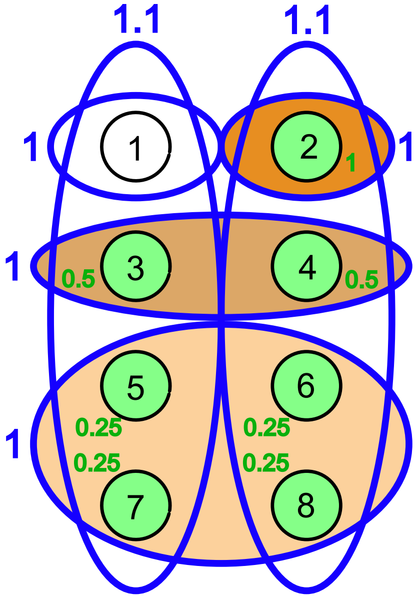

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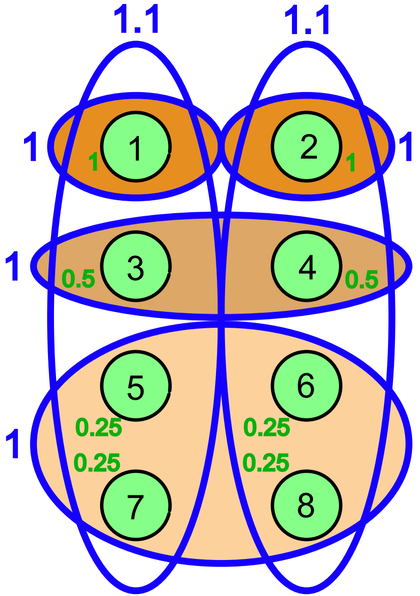

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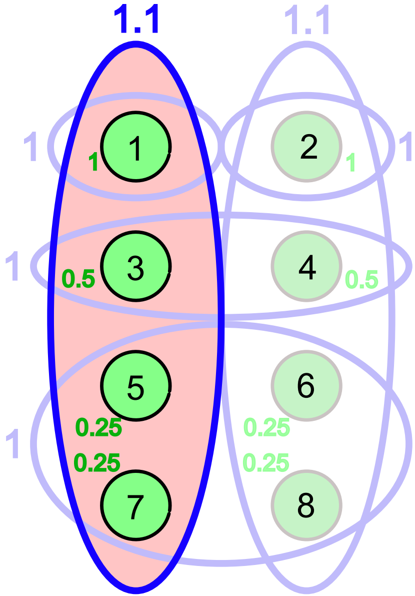

Greedy-Set-Cover

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Greedy-Set-Cover:

Start with $R = U$ and no sets selected

While $R \neq \emptyset$

  Select set S_i that minimizes $w_i/|S_i \cap R|$

  Delete set S_i from R

EndWhile

Return the selected sets

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- The algorithm computes a set cover whose weight is at most $O(\log n)$ times the optimal weight (Johnson 1974, Lovász 1975, Chvatal 1979).

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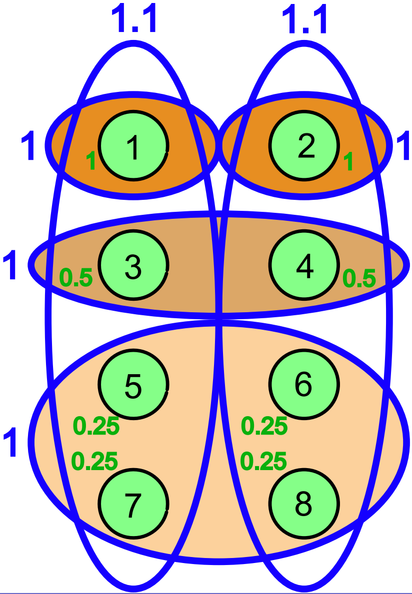
- In the algorithm, after selecting S_i, add the line

  `Define` $c_s = w_i/|S_i \cap R|$ `for all` $s \in S_i \cap R$.
- As each set S_i is selected, distribute its weight over the costs c_s of the *newly*-covered elements.
- Each element in the universe assigned cost exactly once.



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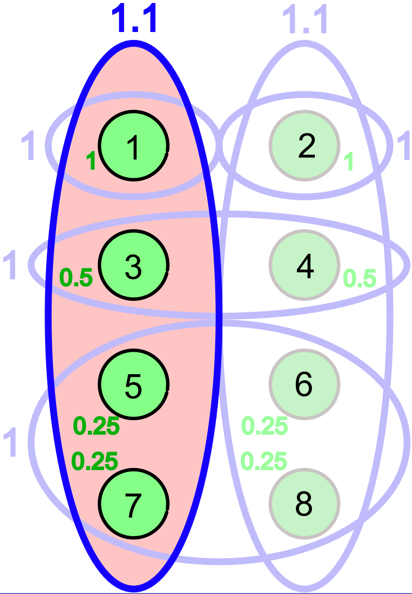

Starting the Analysis of Greedy-Set-Cover

- Let $\mathcal{C}$ be the set cover computed by Greedy-Set-Cover.
- Claim: $\sum_{S_i \in \mathcal{C}} w_i = \sum_{s \in U} c_s$.

$$\sum_{S_i \in \mathcal{C}} w_i = \sum_{S_i \in \mathcal{C}} \left(\sum_{s \in S_i \cap R} c_s \right), \text{ by definition of } c_s$$
$$= \sum_{s \in U} c_s, \text{ since each element in the universe contributes exactly once}$$

- In other words, the total weight of the solution computed by Greedy-Set-Cover is the total costs it assigns to the elements in the universe.
- Can "switch" between set-based weight of solution and element-based costs.
- Note: sets have weights whereas Greedy-Set-Cover assigns costs to elements.

Intuition Behind the Proof

- Suppose $\mathcal{C}^*$ is the optimal set cover: $w^* = \sum_{S_j \in \mathcal{C}^*} w_j$.
- Goal is to relate total weight of sets in $\mathcal{C}$ to total weight of sets in $\mathcal{C}^*$.

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- Since $\mathcal{C}^*$ is a set cover, $\displaystyle\sum_{S_j \in \mathcal{C}^*} \left(\sum_{s \in S_j} c_s \right) \geq \sum_{s \in U} c_s = \sum_{S_i \in \mathcal{C}} w_i = w.$

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- In the sum on the left, S_j is a set in $\mathcal{C}^*$ (need not be a set in $\mathcal{C}$). How large can total cost of elements in such a set be?
- For *any* set S_k, suppose we can prove $\sum_{s \in S_k} c_s \leq \alpha w_k$, for some fixed $\alpha > 0$, i.e., total cost assigned by GREEDY-SET-COVER to the elements in S_k cannot be much larger than the weight of s_k.

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- For *any* set S_k, suppose we can prove $\sum_{s \in S_k} c_s \leq \alpha w_k$, for some fixed $\alpha > 0$, i.e., total cost assigned by GREEDY-SET-COVER to the elements in S_k cannot be much larger than the weight of s_k.
- Then $$w \leq \sum_{S_j \in \mathcal{C}^*} \left(\sum_{s \in S_j} c_s \right) \leq \sum_{S_j \in \mathcal{C}^*} \alpha w_j = \alpha w^*.$$

Intuition Behind the Proof

- Suppose $\mathcal{C}^*$ is the optimal set cover: $w^* = \sum_{S_j \in \mathcal{C}^*} w_j$.
- Goal is to relate total weight of sets in $\mathcal{C}$ to total weight of sets in $\mathcal{C}^*$.
- What is the total cost assigned by GREEDY-SET-COVER to the elements in the sets in the optimal cover $\mathcal{C}^*$?
- Since $\mathcal{C}^*$ is a set cover, $\displaystyle\sum_{S_j \in \mathcal{C}^*} \left(\sum_{s \in S_j} c_s \right) \geq \sum_{s \in U} c_s = \sum_{S_i \in \mathcal{C}} w_i = w.$
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- Then $\displaystyle w \leq \sum_{S_j \in \mathcal{C}^*} \left(\sum_{s \in S_j} c_s \right) \leq \sum_{S_j \in \mathcal{C}^*} \alpha w_j = \alpha w^*.$
- For every set S_k in the input, goal is to prove an upper bound on $\dfrac{\sum_{s \in S_k} c_s}{w_k}$.

Upper Bounding Cost-by-Weight Ratio

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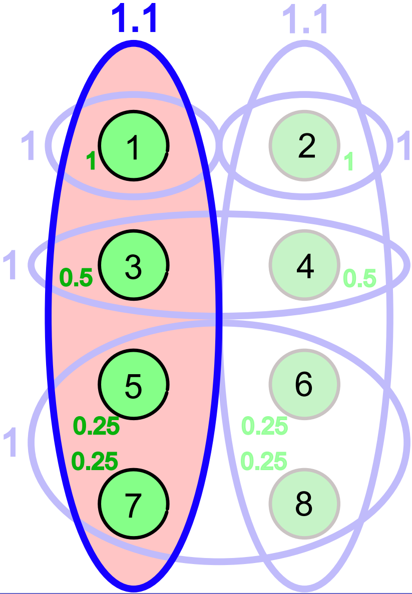
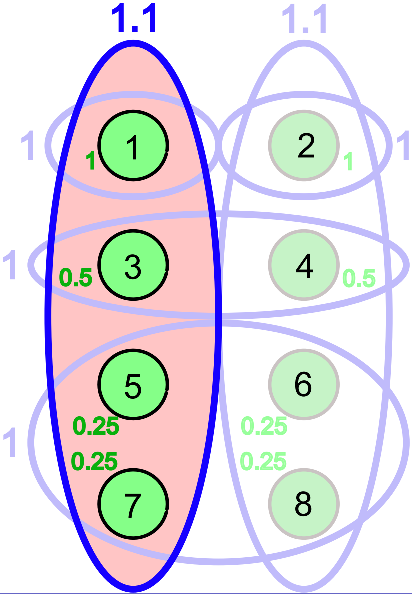
- Claim: For every set S_k, the sum $\sum_{s \in S_k} c_s \leq H(|S_K|)w_k$.



Renumbering Elements in S_k

- Renumber elements in U so that elements in S_k are the first $d = |S_k|$ elements of U, i.e., $S_k = \{s_1, s_2, \ldots, s_d\}$.
- Order elements of S in the order they get covered by the algorithm (i.e., when they get assigned a cost by GREEDY-SET-COVER).



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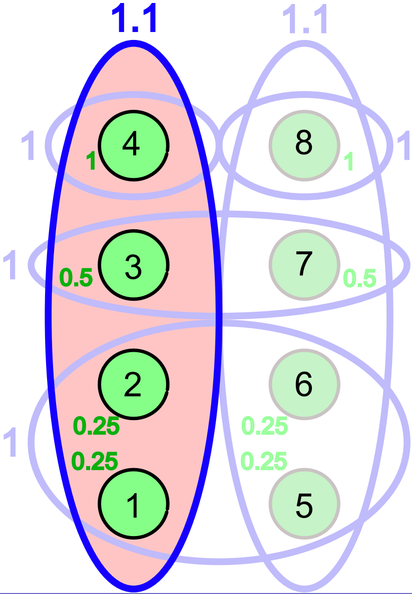

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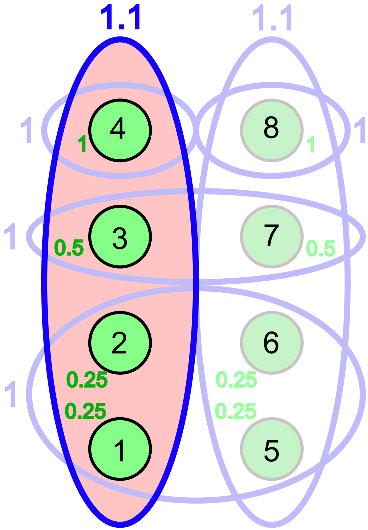

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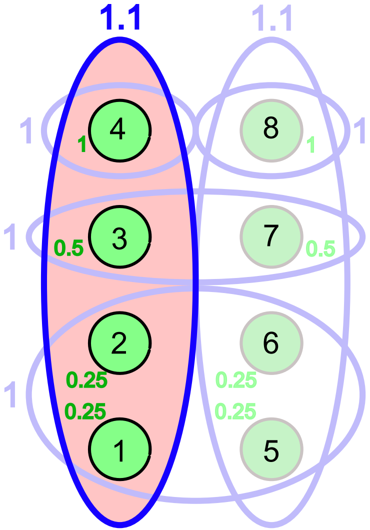

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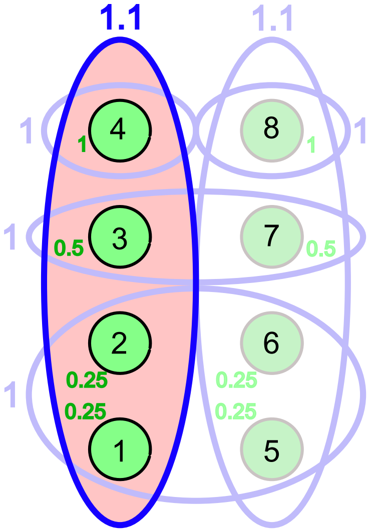

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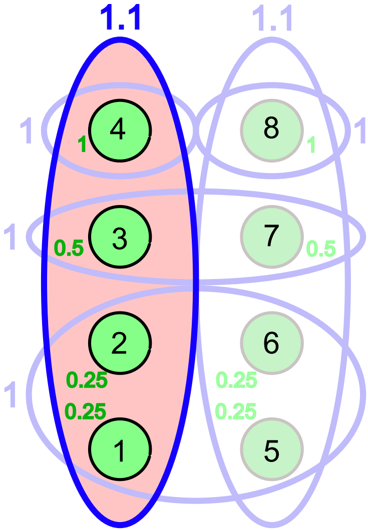
- We are done!

$$\sum_{s \in S_k} c_s = \sum_{j=1}^{d} c_{s_j} \leq \sum_{j=1}^{d} \frac{w_k}{d - j + 1} = H(d) w_k.$$



Proving Upper Bound on Cost of Greedy-Set-Cover

- Let us assume $\sum_{s \in S_k} c_s \leq H(|S_K|)w_k$.
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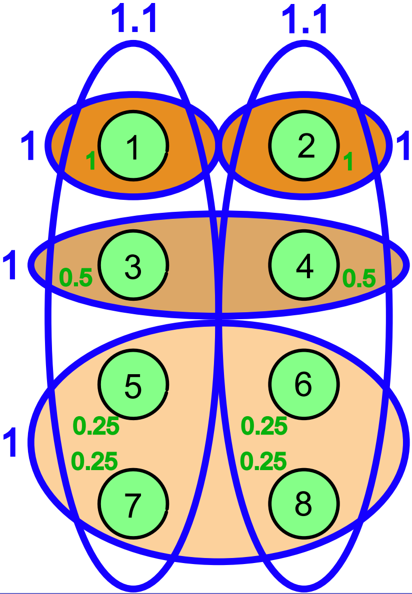
- We have proven that Greedy-Set-Cover computes a set cover whose weight is at most $H(d^*)$ times the optimal weight.

How Badly Can Greedy-Set-Cover Perform?



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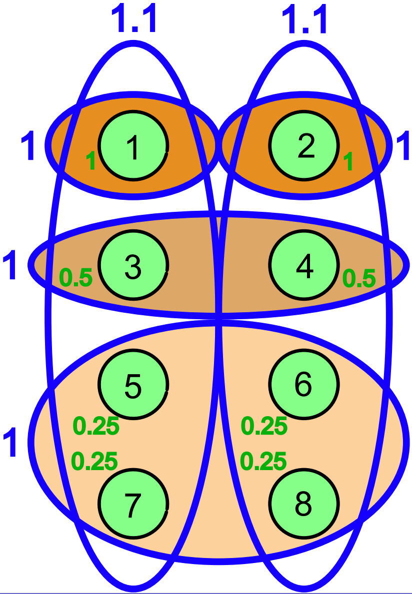

- Generalise this example to show that algorithm produces a set cover of weight $\Omega(\log n)$ even though optimal weight is $2 + \varepsilon$.
- More complex constructions show greedy algorithm incurs a weight close to $H(n)$ times the optimal weight.
- No polynomial time algorithm can achieve an approximation bound better than $H(n)$ times optimal unless $\mathcal{P} = \mathcal{NP}$ (Lund and Yannakakis, 1994).