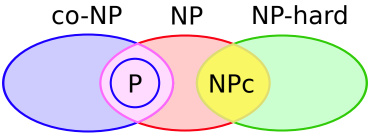
NP-Complete Problems

T. M. Murali

April 25, 30, 2013

Proving Other Problems $\mathcal{NP}$-Complete



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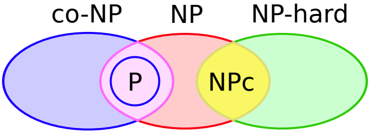

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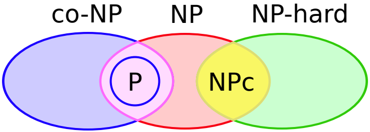

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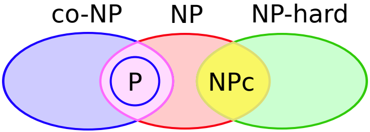

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  3. Prove that $Y \leq_P X$.
- To prove X is $\mathcal{NP}$-Complete, reduce a known $\mathcal{NP}$-Complete problem Y to X. Do not prove reduction in the opposite direction, i.e., $X \leq_P Y$.
- If we use Karp reductions, we can refine the strategy:
  1. Prove that $X \in \mathcal{NP}$.
  2. Select a problem Y known to be $\mathcal{NP}$-Complete.
  3. Consider an arbitrary instance s_Y of problem Y. Show how to construct, in polynomial time, an instance s_X of problem X such that
     (a) If $s_Y \in Y$, then $s_X \in X$ and
     (b) If $s_X \in X$, then $s_Y \in Y$.

3-SAT is $\mathcal{NP}$-Complete

- Why is 3-SAT in NP?

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- Why is 3-SAT in NP?
- CIRCUIT SATISFIABILITY $\leq_P$ 3-SAT.
  1. Given an instance of CIRCUIT SATISFIABILITY, create an instance of SAT, in which each clause has *at most* three variables.
  2. Convert this instance of SAT into one of 3-SAT.

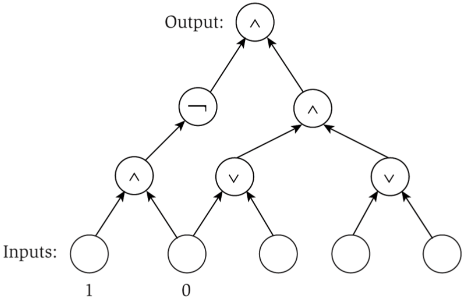

Figure 8.4 A circuit with three inputs, two additional sources that have assigned truth values, and one output.

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- Output: if o is the output node, use the clause (x_o).

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  - If a clause has a two terms t and t', replace the clause with $t \vee t' \vee z_1$.

More $\mathcal{NP}$-Complete problems

- CIRCUIT SATISFIABILITY is $\mathcal{NP}$-Complete.
- We just showed that CIRCUIT SATISFIABILITY $\leq_P$ 3-SAT.
- We know that

3-SAT $\leq_P$ INDEPENDENT SET $\leq_P$ VERTEX COVER $\leq_P$ SET COVER

- All these problems are in $\mathcal{NP}$.
- Therefore, INDEPENDENT SET, VERTEX COVER, and SET COVER are $\mathcal{NP}$-Complete.

Hamiltonian Cycle

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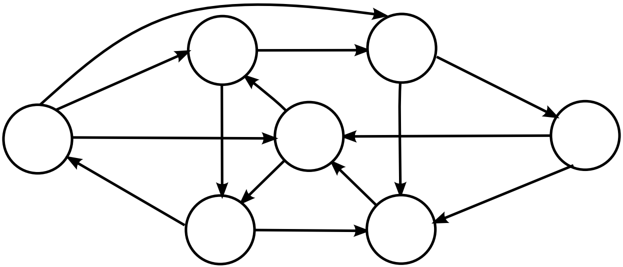
- In a directed graph $G(V, E)$, a cycle C is a *Hamiltonian cycle* if C visits each vertex exactly once.



HAMILTONIAN CYCLE
INSTANCE: A directed graph G.
QUESTION: Does G contain a Hamiltonian cycle?

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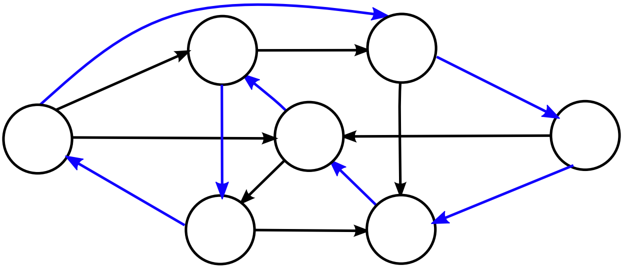

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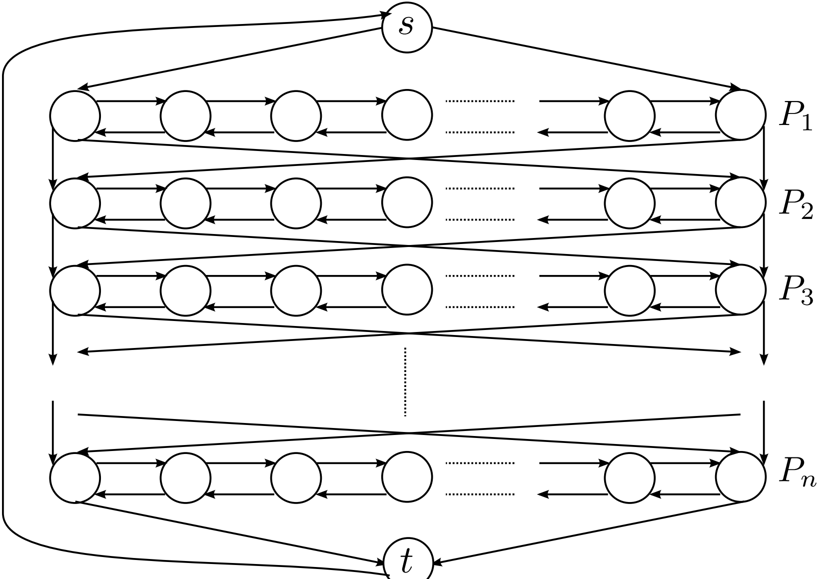
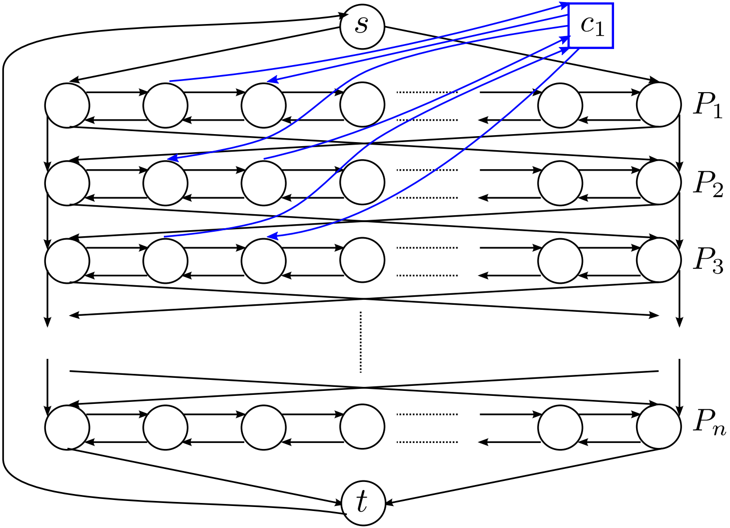
- Why is the problem in $\mathcal{NP}$?
- Claim: $\text{3-SAT} \leq_P$ Hamiltonian Cycle. Jump to TSP
- Consider an arbitrary instance of 3-SAT with variables $x_1, x_2, \ldots, x_n$ and clauses $C_1, C_2, \ldots C_k$.
- Strategy:
  1. Construct a graph G with $O(nk)$ nodes and edges and 2^n Hamiltonian cycles with a one-to-one correspondence with 2^n truth assignments.
  2. Add nodes to impose constraints arising from clauses.
  3. Construction takes $O(nk)$ time.
- G contains n paths $P_1, P_2, \ldots P_n$, one for each variable.
- Each P_i contains $b = 3k + 3$ nodes $v_{i,1}, v_{i,2}, \ldots v_{i,b}$, three for each clause and some extra nodes.

3-SAT $\leq_P$ Hamiltonian Cycle: Constructing G

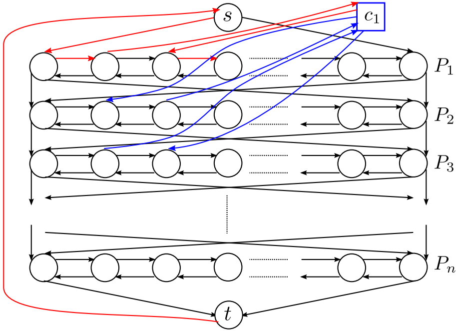


3-SAT $\leq_P$ Hamiltonian Cycle: Modelling clauses

- Consider the clause $C_1 = x_1 \vee \overline{x_2} \vee x_3$.



3-SAT $\leq_P$ Hamiltonian Cycle: Proof Part 1



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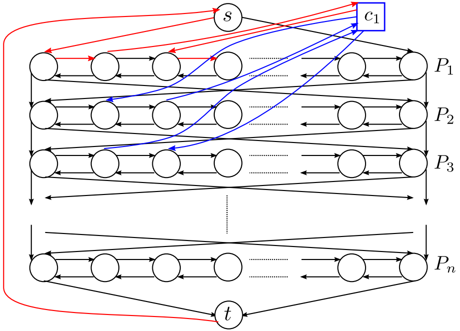

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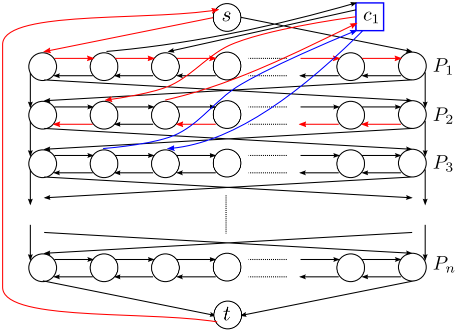
  - Construct a Hamiltonian cycle $\mathcal{C}$ as follows:
  - If $x_i = 1$, traverse P_i from left to right in $\mathcal{C}$.
  - Otherwise, traverse P_i from right to left in $\mathcal{C}$.
  - For each clause C_j, there is at least one term set to 1. If the term is x_i, splice c_j into $\mathcal{C}$ using edge from $v_{i,3j}$ and edge to $v_{i,3j+1}$. Analogous construction if term is $\overline{x_i}$.

3-SAT $\leq_P$ Hamiltonian Cycle: Proof Part 2



- G has a Hamiltonian cycle $\mathcal{C} \rightarrow$ 3-SAT instance is satisfiable.
  - If $\mathcal{C}$ enters c_j on an edge from $v_{i,3j}$, it must leave c_j along the edge to $v_{i,3j+1}$.
  - Analogous statement if $\mathcal{C}$ enters c_j on an edge from $v_{i,3j+1}$.

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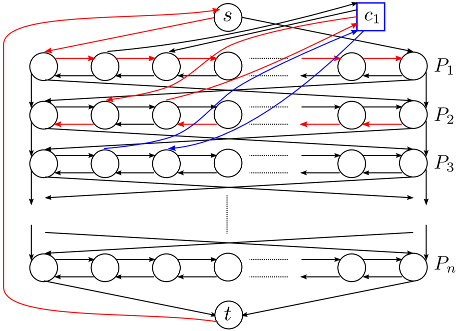

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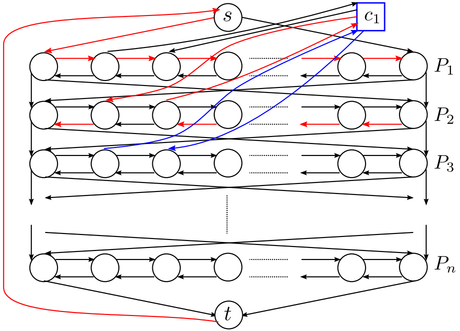

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  - Nodes immediately before and after c_j in $\mathcal{C}$ are themselves connected by an edge e in G.
  - If we remove all such edges e from $\mathcal{C}$, we get a Hamiltonian cycle $\mathcal{C}'$ in $G - \{c_1, c_2, \ldots, c_k\}$.
  - Use $\mathcal{C}'$ to construct truth assignment to variables; prove assignment is satisfying.

The Traveling Salesman Problem

- A salesman must visit n cities $v_1, v_2, \ldots v_n$ starting at home city v_1.
- Salesman must find a *tour*, an order in which to visit each city exactly once, and return home.
- Goal is to find as short a tour as possible.

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- For every pair of cities v_i and v_j, $d(v_i, v_j) > 0$ is the distance from v_i to v_j.
- A *tour* is a permutation $v_{i_1} = v_1, v_{i_2}, \ldots v_{i_n}$.
- The *length* of the tour is $\sum_{j=1}^{n-1} d(v_{i_j} v_{i_{j+1}}) + d(v_{i_n}, v_{i_1})$.

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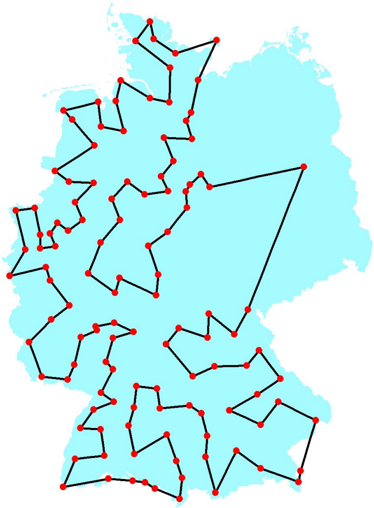
TRAVELLING SALESMAN

INSTANCE: A set V of n cities, a function $d : V \times V \rightarrow \mathbb{R}^+$, and a number $D > 0$.

QUESTION: Is there a tour of length at most D?

Examples of Travelling Salesman



(1977) 120 cities, Groetschel

Images taken from `http://tsp.gatech.edu`

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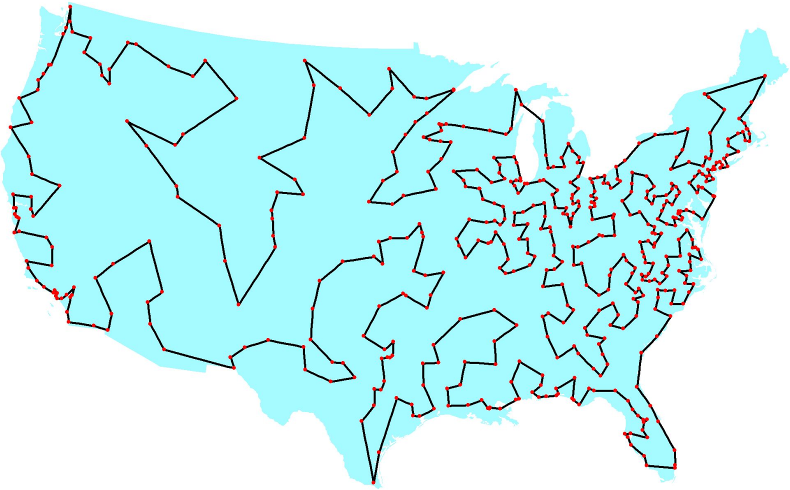

(1987) 532 AT&T switch locations, Padberg and Rinaldi
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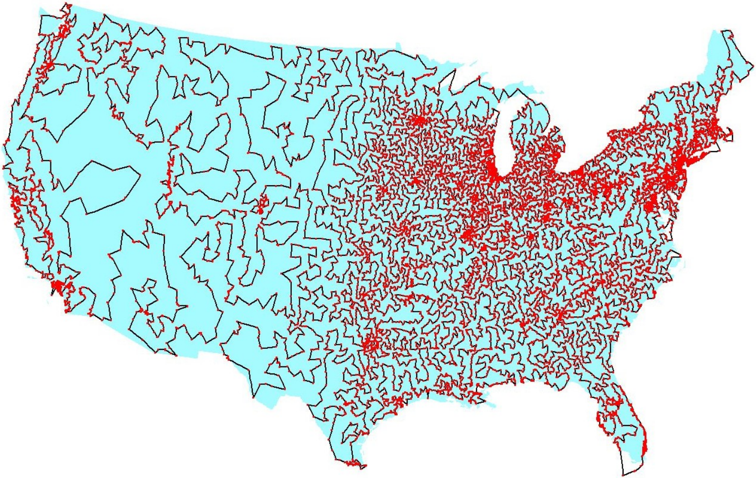

(1987) 13,509 cities with population $\geq$ 500, Applegate, Bixby, Chvátal, and Cook
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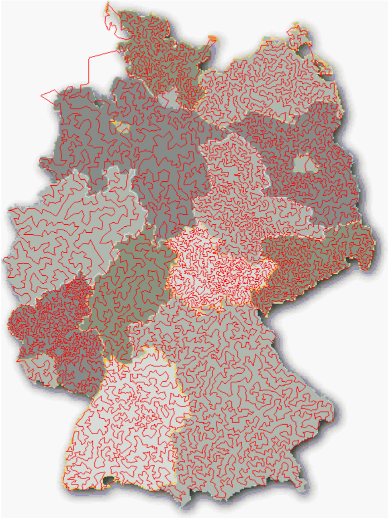

(2001) 15,112 cities, Applegate, Bixby, Chváthal, and Cook
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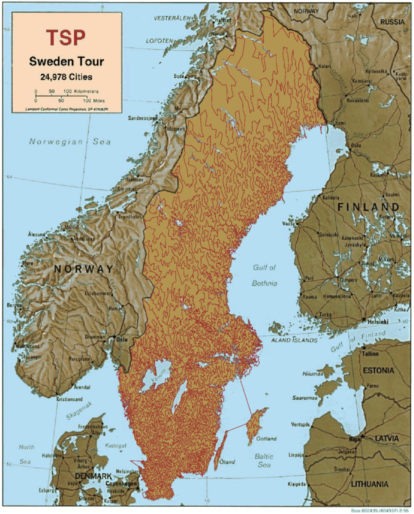

(2004) 24978, cities, Applegate, Bixby, Chváthal, Cook, and Helsgaum

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- Claim: Hamiltonian Cycle $\leq_P$ Travelling Salesman.

Hamiltonian Cycle	Travelling Salesman
Directed graph $G(V, E)$	Cities
Edges have identical weights	Distances between cities can vary
Not all pairs of nodes are connected in G	*Every pair* of cities has a distance
(u, v) and (v, u) may both be edges	$d(v_i, v_j) \neq d(v_j, v_i)$, in general
Does a cycle exist?	Does a tour of length $\leq D$ exist?

Travelling Salesman is $\mathcal{NP}$-Complete

- Why is the problem in $\mathcal{NP}$?
- Why is the problem $\mathcal{NP}$-Complete?
- Claim: HAMILTONIAN CYCLE $\leq_P$ TRAVELLING SALESMAN.

HAMILTONIAN CYCLE	TRAVELLING SALESMAN
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(u, v) and (v, u) may both be edges	$d(v_i, v_j) \neq d(v_j, v_i)$, in general
Does a cycle exist?	Does a tour of length $\leq D$ exist?

- Given a directed graph $G(V, E)$ (instance of HAMILTONIAN CYCLE),
  - Create a city v_i for each node $i \in V$.
  - Define $d(v_i, v_j) = 1$ if $(i, j) \in E$.
  - Define $d(v_i, v_j) = 2$ if $(i, j) \notin E$.

Travelling Salesman is $\mathcal{NP}$-Complete

- Why is the problem in $\mathcal{NP}$?
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- Claim: HAMILTONIAN CYCLE $\leq_P$ TRAVELLING SALESMAN.

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Directed graph $G(V, E)$	Cities
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(u, v) and (v, u) may both be edges	$d(v_i, v_j) \neq d(v_j, v_i)$, in general
Does a cycle exist?	Does a tour of length $\leq D$ exist?

- Given a directed graph $G(V, E)$ (instance of HAMILTONIAN CYCLE),
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- Claim: G has a Hamiltonian cycle iff the instance of Travelling Salesman has a tour of length at most

Travelling Salesman is $\mathcal{NP}$-Complete

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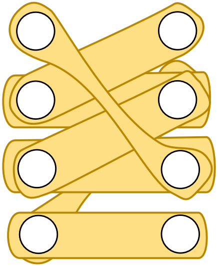
HAMILTONIAN CYCLE	TRAVELLING SALESMAN
Directed graph $G(V, E)$	Cities
Edges have identical weights	Distances between cities can vary
Not all pairs of nodes are connected in G	*Every pair* of cities has a distance
(u, v) and (v, u) may both be edges	$d(v_i, v_j) \neq d(v_j, v_i)$, in general
Does a cycle exist?	Does a tour of length $\leq D$ exist?

- Given a directed graph $G(V, E)$ (instance of HAMILTONIAN CYCLE),
  - Create a city v_i for each node $i \in V$.
  - Define $d(v_i, v_j) = 1$ if $(i, j) \in E$.
  - Define $d(v_i, v_j) = 2$ if $(i, j) \notin E$.
- Claim: G has a Hamiltonian cycle iff the instance of Travelling Salesman has a tour of length at most n.

Special Cases and Extensions that are $\mathcal{NP}$-Complete

- Hamiltonian Cycle for undirected graphs.
- Hamiltonian Path for directed and undirected graphs.
- Travelling Salesman with symmetric distances (by reducing Hamiltonian Cycle for undirected graphs to it).
- Travelling Salesman with distances defined by points on the plane.

3-Dimensional Matching

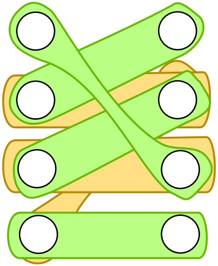


BIPARTITE MATCHING

INSTANCE: Disjoint sets X, Y, each of size n, and a set $T \subseteq X \times Y$ of pairs

QUESTION: Is there a set of n pairs in T such that each element of $X \cup Y$ is contained in exactly one of these pairs?

3-Dimensional Matching



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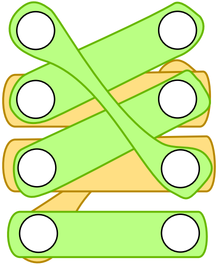

- 3-Dimensional Matching is a harder version of Bipartite Matching.

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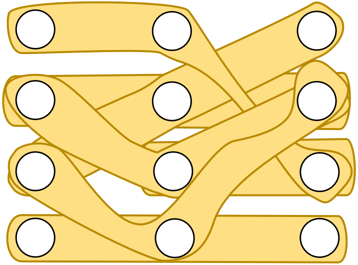

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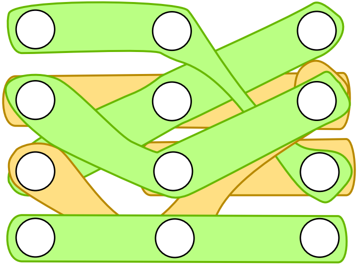
3-Dimensional Matching

INSTANCE: Disjoint sets X, Y, and Z, each of size n, and a set $T \subseteq X \times Y \times Z$ of triples

QUESTION: Is there a set of n triples in T such that each element of $X \cup Y \cup Z$ is contained in exactly one of these triples?

3-Dimensional Matching



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  QUESTION: Is there a set of n triples in T such that each element of $X \cup Y \cup Z$ is contained in exactly one of these triples?

- Easy to show 3-Dimensional Matching $\leq_P$ Set Cover and 3-Dimensional Matching $\leq_P$ Set Packing.

3-Dimensional Matching is $\mathcal{NP}$-Complete

- Why is the problem in $\mathcal{NP}$?

3-Dimensional Matching is $\mathcal{NP}$-Complete

- Why is the problem in $\mathcal{NP}$?
- Show that 3-SAT $\leq_P$ 3-DIMENSIONAL MATCHING. Jump to Colouring
- Strategy:
  - Start with an instance of 3-SAT with n variables and k clauses.
  - Create a gadget for each variable x_i that encodes the choice of truth assignment to x_i.
  - Add gadgets that encode constraints imposed by clauses.

3-SAT $\leq_P$ 3-Dimensional Matching: Variables

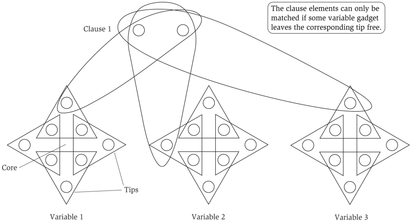
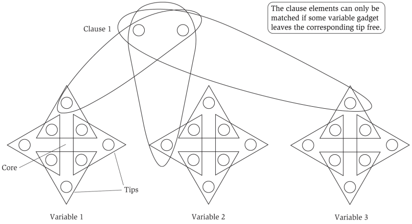


Figure 8.9 The reduction from 3-SAT to 3-Dimensional Matching.

- Each x_i corresponds to a *variable gadget* i with $2k$ *core* elements $A_i = \{a_{i,1}, a_{i,2}, \ldots a_{i,2k}\}$ and $2k$ *tips* $B_i = \{b_{i,1}, b_{i,2}, \ldots b_{i,2k}\}$.
- For each $1 \leq j \leq 2k$, variable gadget i includes a triple $t_{ij} = (a_{i,j}, a_{i,j+1}, b_{i,j})$.
- A triple (tip) is *even* if j is even. Otherwise, the triple (tip) is *odd*.
- Only these triples contain elements in A_i.

3-SAT $\leq_P$ 3-Dimensional Matching: Variables



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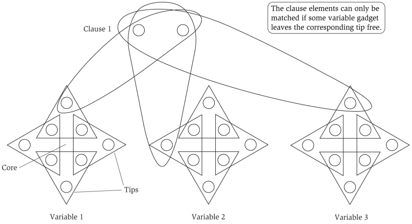
- In any perfect matching, we can cover the elements in A_i

3-SAT $\leq_P$ 3-Dimensional Matching: Variables



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- For each $1 \leq j \leq 2k$, variable gadget i includes a triple $t_{ij} = (a_{i,j}, a_{i,j+1}, b_{i,j})$.
- A triple (tip) is *even* if j is even. Otherwise, the triple (tip) is *odd*.
- Only these triples contain elements in A_i.
- In any perfect matching, we can cover the elements in A_i either using all the even triples in gadget i or all the odd triples in the gadget.
- Even triples used, odd tips free $\equiv x_i = 0$; odd triples used, even tips free $\equiv x_i = 1$.

3-SAT $\leq_P$ 3-Dimensional Matching: Clauses

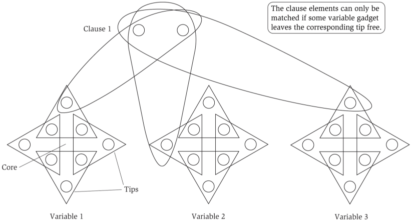


Figure 8.9 The reduction from 3-SAT to 3-Dimensional Matching.

- Consider the clause $C_1 = x_1 \vee \overline{x_2} \vee x_3$.

3-SAT $\leq_P$ 3-Dimensional Matching: Clauses

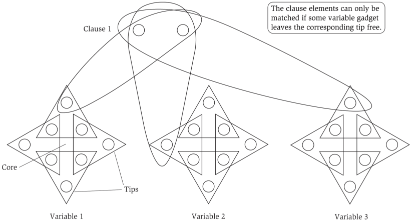
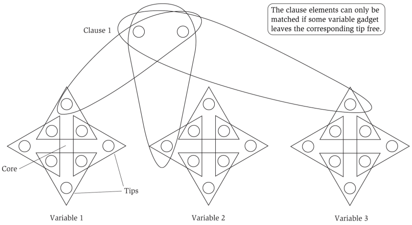


Figure 8.9 The reduction from 3-SAT to 3-Dimensional Matching.

- Consider the clause $C_1 = x_1 \vee \overline{x_2} \vee x_3$.
- C_1 says "The matching on the cores of the gadgets should leave the even tips of gadget 1 free; or it should leave the odd tips of gadget 2 free; or it should leave the even tips of gadget 3 free."

3-SAT $\leq_P$ 3-Dimensional Matching: Clauses



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- C_1 says "The matching on the cores of the gadgets should leave the even tips of gadget 1 free; or it should leave the odd tips of gadget 2 free; or it should leave the even tips of gadget 3 free."
- *Clause gadget* j for clause C_j contains two core elements $P_j = \{p_j, p'_j\}$ and three triples:
  - C_j contains x_i: add triple $(p_j, p'_j, b_{i,2j})$.
  - C_j contains $\overline{x_i}$: add triple $(p_j, p'_j, b_{i,2j-1})$.

3-SAT $\leq_P$ 3-Dimensional Matching: Example

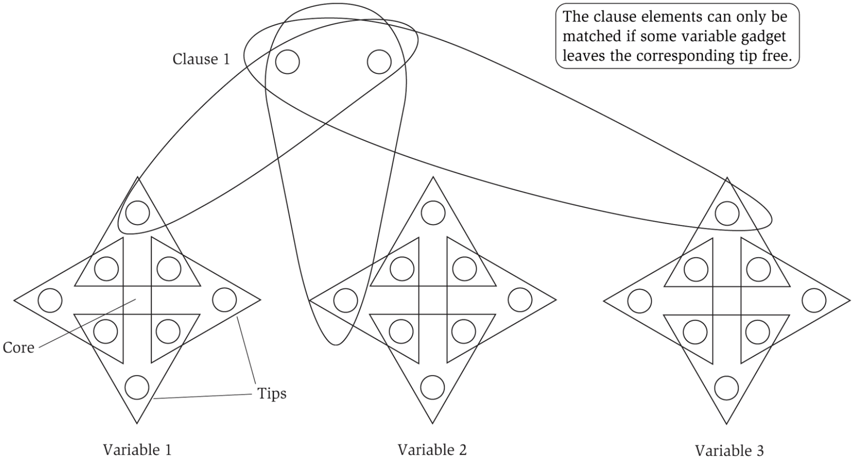

Figure 8.9 The reduction from 3-SAT to 3-Dimensional Matching.

3-SAT $\leq_P$ 3-Dimensional Matching: Proof

- Satisfying assignment $\rightarrow$ matching.

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- Satisfying assignment $\rightarrow$ matching.
  - Make appropriate choices for the core of each variable gadget.
  - At least one free tip available for each clause gadget, allowing core elements of clause gadgets to be covered.

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- Satisfying assignment $\rightarrow$ matching.
  - Make appropriate choices for the core of each variable gadget.
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  - We have not covered all the tips!

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  - We have not covered all the tips!
  - Add $(n-1)k$ *cleanup gadgets* to allow the remaining $(n-1)k$ tips to be covered: cleanup gadget i contains two core elements $Q = \{q_i, q'_i\}$ and triple (q_i, q'_i, b) for every tip b in variable gadget i.

3-SAT $\leq_P$ 3-Dimensional Matching: Proof

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- Matching $\rightarrow$ satisfying assignment.

3-SAT $\leq_P$ 3-Dimensional Matching: Proof

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- Matching $\rightarrow$ satisfying assignment.
  - Matching chooses all even a_{ij} $(x_i = 0)$ or all odd a_{ij} $(x_i = 1)$.

3-SAT $\leq_P$ 3-Dimensional Matching: Proof

- Satisfying assignment $\rightarrow$ matching.
  - Make appropriate choices for the core of each variable gadget.
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  - Matching chooses all even a_{ij} $(x_i = 0)$ or all odd a_{ij} $(x_i = 1)$.
  - Is clause C_j satisfied?

3-SAT $\leq_P$ 3-Dimensional Matching: Proof

- Satisfying assignment $\rightarrow$ matching.
  - Make appropriate choices for the core of each variable gadget.
  - At least one free tip available for each clause gadget, allowing core elements of clause gadgets to be covered.
  - We have not covered all the tips!
  - Add $(n-1)k$ *cleanup gadgets* to allow the remaining $(n-1)k$ tips to be covered: cleanup gadget i contains two core elements $Q = \{q_i, q_i'\}$ and triple (q_i, q_i', b) for every tip b in variable gadget i.
- Matching $\rightarrow$ satisfying assignment.
  - Matching chooses all even a_{ij} $(x_i = 0)$ or all odd a_{ij} $(x_i = 1)$.
  - Is clause C_j satisfied? Core in clause gadget j is covered by some triple $\Rightarrow$ other element in the triple must be a tip element from the correct odd/even set in the three variable gadgets corresponding to a term in C_j.

3-SAT $\leq_P$ 3-Dimensional Matching: Finale

- Did we create an instance of 3-DIMENSIONAL MATCHING?

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- Did we create an instance of 3-DIMENSIONAL MATCHING?
- We need three sets X, Y, and Z of equal size.
- How many elements do we have?
  - $2nk$ a_{ij} elements.
  - $2nk$ b_{ij} elements.
  - k p_j elements.
  - k p'_j elements.
  - $(n-1)k$ q_i elements.
  - $(n-1)k$ q'_i elements.

3-SAT $\leq_P$ 3-Dimensional Matching: Finale

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- We need three sets X, Y, and Z of equal size.
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  - $2nk$ a_{ij} elements.
  - $2nk$ b_{ij} elements.
  - k p_j elements.
  - k p'_j elements.
  - $(n-1)k$ q_i elements.
  - $(n-1)k$ q'_i elements.
- X is the union of a_{ij} with even j, the set of all p_j and the set of all q_i.
- Y is the union of a_{ij} with odd j, the set if all p'_j and the set of all q'_i.
- Z is the set of all b_{ij}.

3-SAT $\leq_P$ 3-Dimensional Matching: Finale

- Did we create an instance of 3-DIMENSIONAL MATCHING?
- We need three sets X, Y, and Z of equal size.
- How many elements do we have?
  - $2nk$ a_{ij} elements.
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  - k p_j elements.
  - k p'_j elements.
  - $(n-1)k$ q_i elements.
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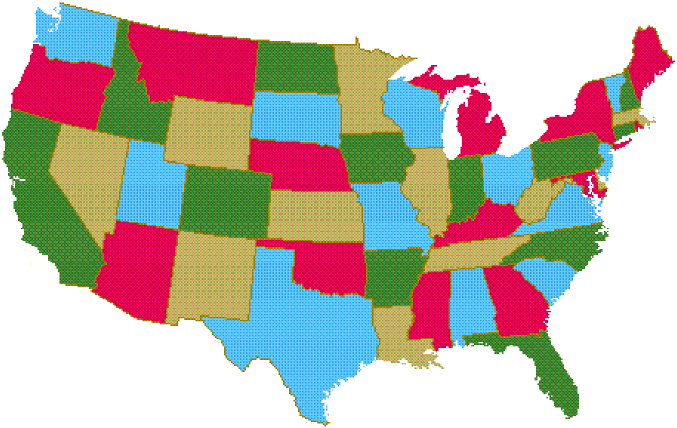
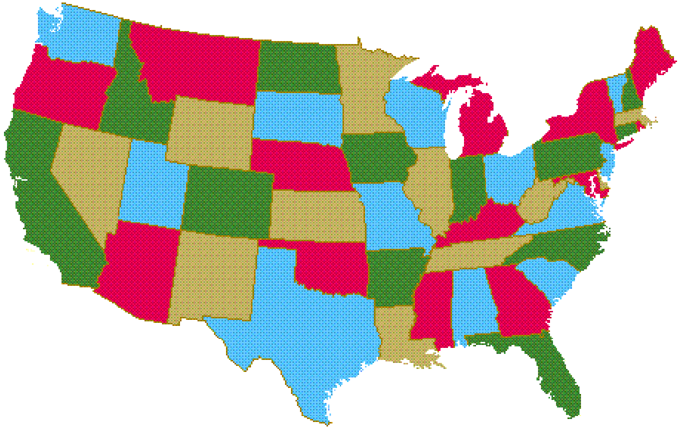
- X is the union of a_{ij} with even j, the set of all p_j and the set of all q_i.
- Y is the union of a_{ij} with odd j, the set if all p'_j and the set of all q'_i.
- Z is the set of all b_{ij}.
- Each triple contains exactly one element from X, Y, and Z.

Colouring maps

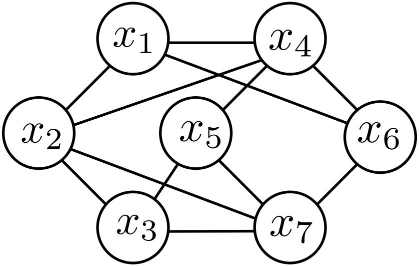


Colouring maps



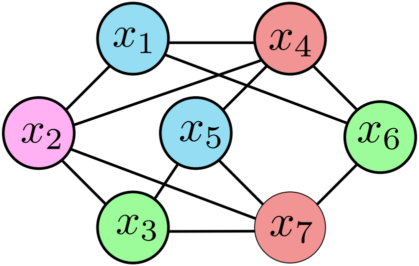
- Any map can be coloured with four colours (Appel and Hakken, 1976).

Graph Colouring



- Given an undirected graph $G(V, E)$, a *k-colouring* of G is a function $f : V \rightarrow \{1, 2, \ldots k\}$ such that for every edge $(u, v) \in E$, $f(u) \neq f(v)$.

Graph Colouring



- Given an undirected graph $G(V, E)$, a *k-colouring* of G is a function $f : V \rightarrow \{1, 2, \ldots k\}$ such that for every edge $(u, v) \in E$, $f(u) \neq f(v)$.

GRAPH COLOURING (k-COLOURING)

INSTANCE: An undirected graph $G(V, E)$ and an integer $k > 0$.

QUESTION: Does G have a k-colouring?

Applications of Graph Colouring

1. Job scheduling: assign jobs to n processors under constraints that certain pairs of jobs cannot be scheduled at the same time.
2. Compiler design: assign variables to k registers but two variables being used at the same time cannot be assigned to the same register.
3. Wavelength assignment: assign one of k transmitting wavelengths to each of n wireless devices. If two devices are close to each other, they must get different wavelengths.

2-Colouring

- How hard is 2-COLOURING?

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- Claim: A graph is 2-colourable if and only if it is bipartite.

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- Testing 2-colourability is possible in $O(|V| + |E|)$ time.

2-Colouring

- How hard is 2-COLOURING?
- Claim: A graph is 2-colourable if and only if it is bipartite.
- Testing 2-colourability is possible in $O(|V| + |E|)$ time.
- What about 3-COLOURING? Is it easy to exhibit a certificate that a graph *cannot* be coloured with three colours?

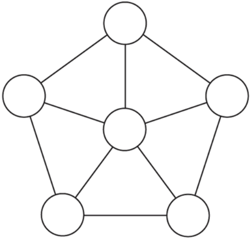

Figure 8.10 A graph that is not 3-colorable.

3-Colouring is $\mathcal{NP}$-Complete

- Why is 3-Colouring in $\mathcal{NP}$?

3-Colouring is $\mathcal{NP}$-Complete

- Why is 3-Colouring in $\mathcal{NP}$?
- 3-SAT $\leq_P$ 3-Colouring.

3-SAT $\leq_P$ 3-Colouring: Encoding Variables

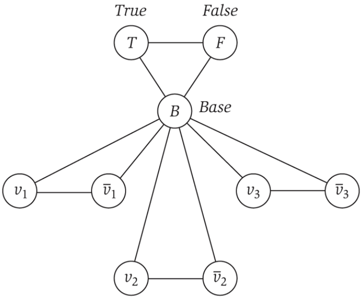

Figure 8.11 The beginning of the reduction for 3-Coloring.

- x_i corresponds to node v_i and $\overline{x_i}$ corresponds to node $\overline{v_i}$.

3-SAT $\leq_P$ 3-Colouring: Encoding Variables

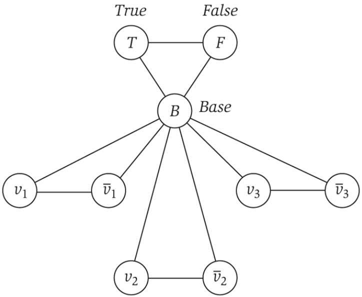

Figure 8.11 The beginning of the reduction for 3-Coloring.

- x_i corresponds to node v_i and $\overline{x_i}$ corresponds to node $\overline{v_i}$.
- In any 3-Colouring, nodes v_i and $\overline{v_i}$ get a colour different from *Base*.
- *True colour*: colour assigned to the *True* node; *False colour*: colour assigned to the *False* node.
- Set x_i to 1 iff v_i gets the *True* colour.

3-SAT $\leq_P$ 3-Colouring: Encoding Clauses

- Consider the clause $C_1 = x_1 \vee \overline{x_2} \vee x_3$.

3-SAT $\leq_P$ 3-Colouring: Encoding Clauses

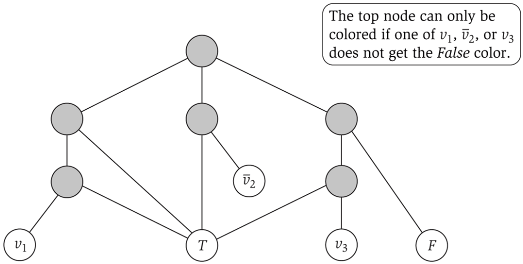

Figure 8.12 Attaching a subgraph to represent the clause $x_1 \vee \overline{x}_2 \vee x_3$.

- Consider the clause $C_1 = x_1 \vee \overline{x_2} \vee x_3$.
- Attach a six-node subgraph for this clause to the rest of the graph.

3-SAT $\leq_P$ 3-Colouring: Encoding Clauses

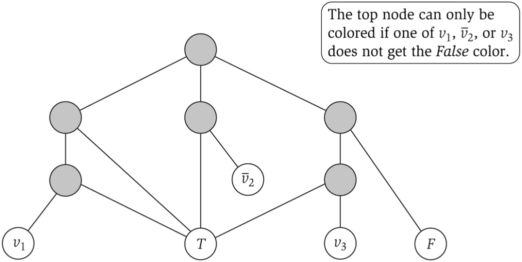

Figure 8.12 Attaching a subgraph to represent the clause $x_1 \vee \overline{x}_2 \vee x_3$.

- Consider the clause $C_1 = x_1 \vee \overline{x_2} \vee x_3$.
- Attach a six-node subgraph for this clause to the rest of the graph.
- Claim: Top node in the subgraph can be coloured in a 3-colouring iff one of v_1, $\overline{v_2}$, or v_3 does not get the *False* colour.

3-SAT $\leq_P$ 3-Colouring: Encoding Clauses

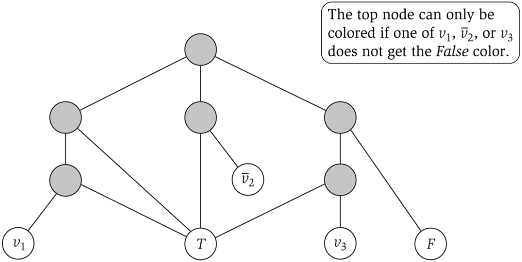

Figure 8.12 Attaching a subgraph to represent the clause $x_1 \vee \bar{x}_2 \vee x_3$.

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- Claim: Top node in the subgraph can be coloured in a 3-colouring iff one of v_1, $\overline{v_2}$, or v_3 does not get the *False* colour.
- Claim: Graph is 3-colourable iff instance of 3-SAT is satisfiable.

Subset Sum

SUBSET SUM

INSTANCE: A set of n natural numbers $w_1, w_2, \ldots, w_n$ and a target W.

QUESTION: Is there a subset of $\{w_1, w_2, \ldots, w_n\}$ whose sum is W?

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- There is a dynamic programming algorithm for SUBSET SUM that runs in $O(nW)$ time.

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- SUBSET SUM is a special case of the KNAPSACK PROBLEM (see Chapter 6.4 of the textbook).
- There is a dynamic programming algorithm for SUBSET SUM that runs in $O(nW)$ time. This algorithm's running time is exponential in the size of the input.

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- Subset Sum is a special case of the Knapsack Problem (see Chapter 6.4 of the textbook).
- There is a dynamic programming algorithm for Subset Sum that runs in $O(nW)$ time. This algorithm's running time is exponential in the size of the input.
- Claim: Subset Sum is $\mathcal{NP}$-Complete, 3-Dimensional Matching $\leq_P$ Subset Sum.

Subset Sum

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- SUBSET SUM is a special case of the KNAPSACK PROBLEM (see Chapter 6.4 of the textbook).
- There is a dynamic programming algorithm for SUBSET SUM that runs in $O(nW)$ time. This algorithm's running time is exponential in the size of the input.
- Claim: SUBSET SUM is $\mathcal{NP}$-Complete, 3-DIMENSIONAL MATCHING $\leq_P$ SUBSET SUM.
- Caveat: Special case of SUBSET SUM in which W is bounded by a polynomial function of n is not $\mathcal{NP}$-Complete (read pages 494–495 of your textbook).

Examples of Hard Computational Problems

(*taken from Adam D. Smith's slides at Penn State University*)

- Aerospace engineering: optimal mesh partitioning for finite elements.
- Biology: protein folding.
- Chemical engineering: heat exchanger network synthesis.
- Civil engineering: equilibrium of urban traffic flow.
- Economics: computation of arbitrage in financial markets with friction.
- Electrical engineering: VLSI layout.
- Environmental engineering: optimal placement of contaminant sensors.
- Financial engineering: find minimum risk portfolio of given return.
- Game theory: find Nash equilibrium that maximizes social welfare.
- Genomics: phylogeny reconstruction.
- Mechanical engineering: structure of turbulence in sheared flows.
- Medicine: reconstructing 3-D shape from biplane angiocardiogram.
- Operations research: optimal resource allocation.
- Physics: partition function of 3-D Ising model in statistical mechanics.
- Politics: Shapley-Shubik voting power.
- Pop culture: Minesweeper consistency.
- Statistics: optimal experimental design.