Divide and Conquer Algorithms

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Divide and Conquer Algorithms

- Study three divide and conquer algorithms:
  - Counting inversions.
  - Finding the closest pair of points.
  - Integer multiplication.
- First two problems use clever conquer strategies.
- Third problem uses a clever divide strategy.

Motivation

- Collaborative filtering: match one user's preferences to those of other users.
- Meta-search engines: merge results of multiple search engines to into a better search result.

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- Fundamental question: how do we compare a pair of rankings?
- Suggestion: two rankings are very similar if they have few inversions.
  - Assume one ranking is the ordered list of integers from 1 to n.
  - The other ranking is a permutation $a_1, a_2, \dots, a_n$ of the integers from 1 to n.
  - The second ranking has an *inversion* if there exist i, j such that $i < j$ but $a_i > a_j$.
  - The number of inversions s is a measure of the difference between the rankings.
- Question also arises in statistics: *Kendall's rank correlation* of two lists of numbers is $1 - 2s/\left(n(n-1)\right)$.

Counting Inversions

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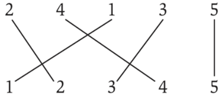

Figure 5.4 Counting the number of inversions in the sequence 2, 4, 1, 3, 5. Each crossing pair of line segments corresponds to one pair that is in the opposite order in the input list and the ascending list—in other words, an inversion.

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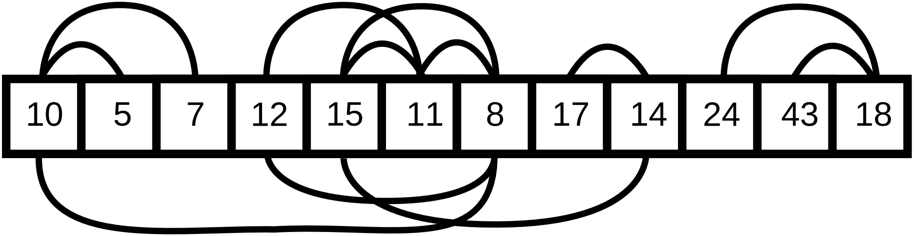

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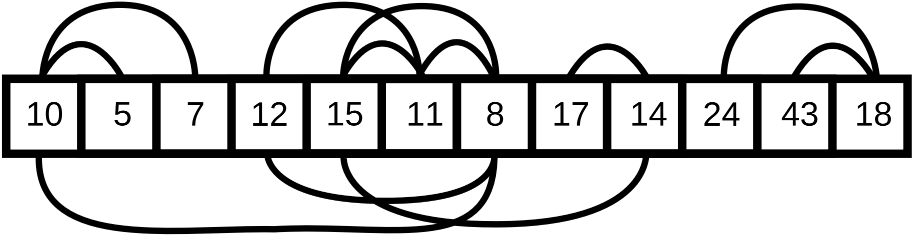

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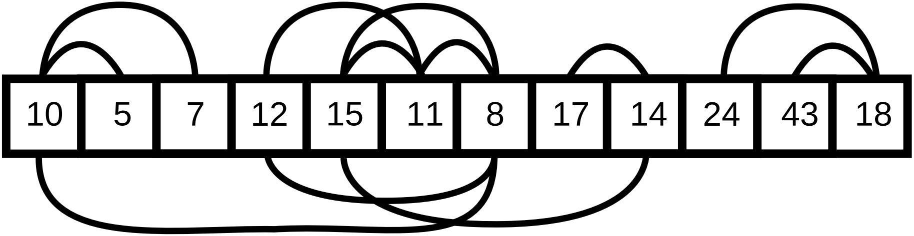

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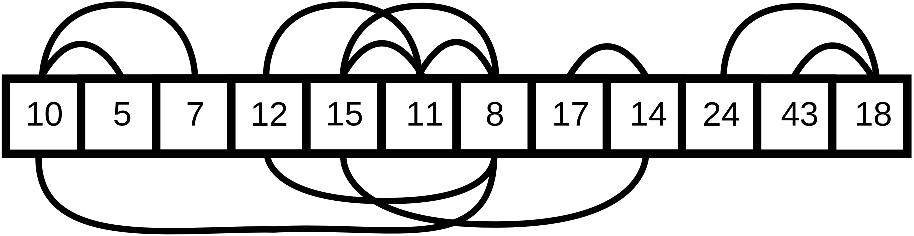

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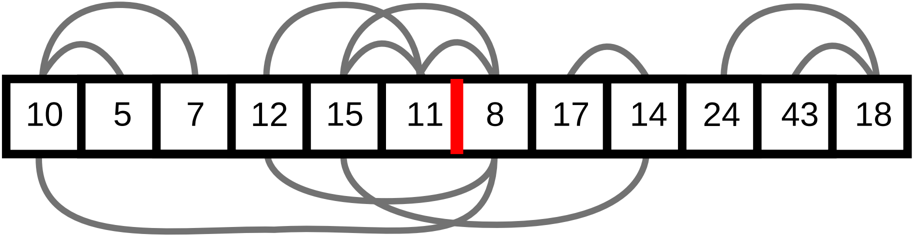

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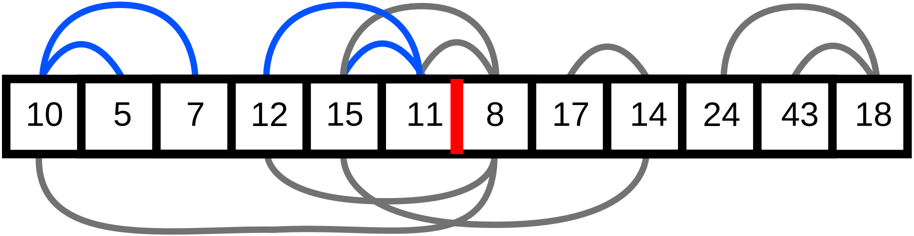

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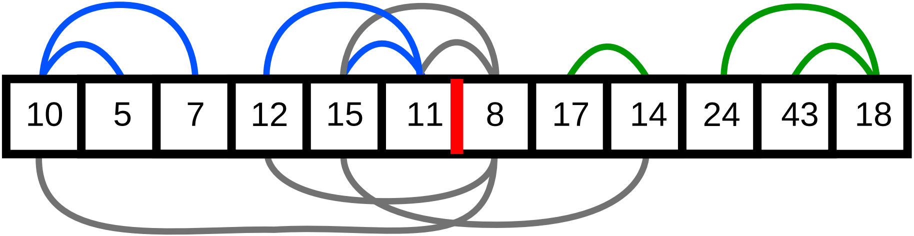

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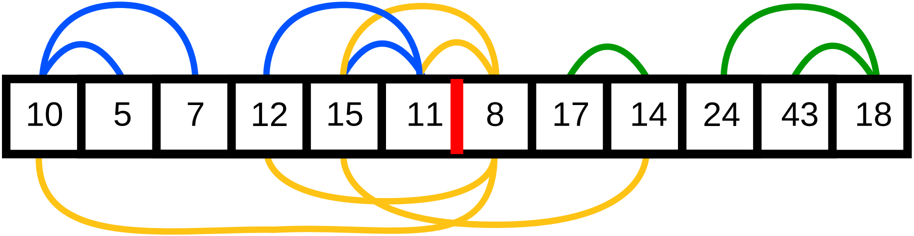

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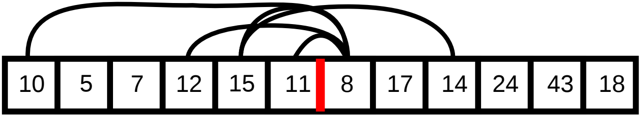

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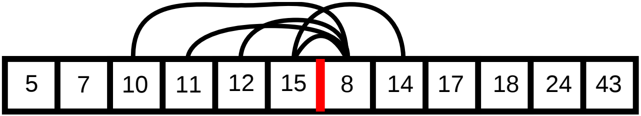

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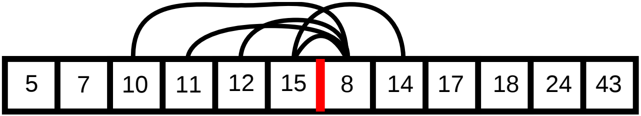

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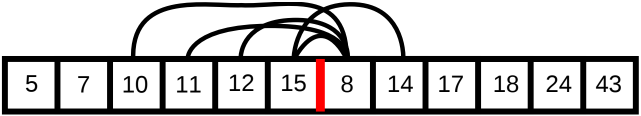

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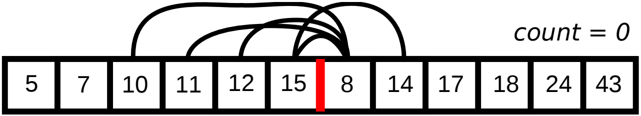

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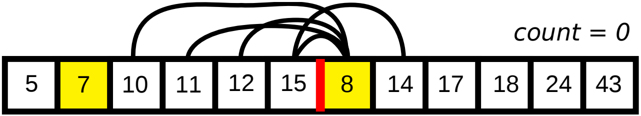

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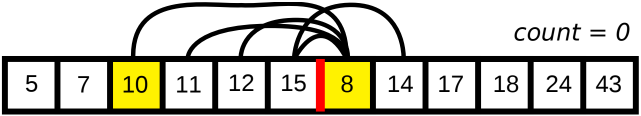

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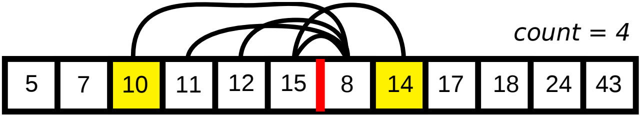

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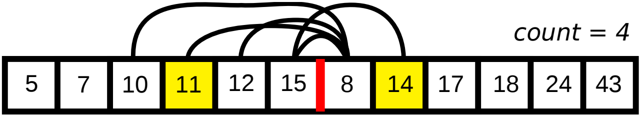

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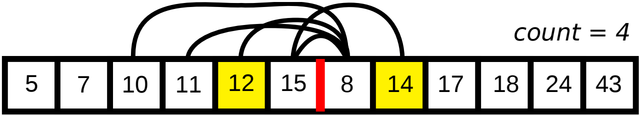

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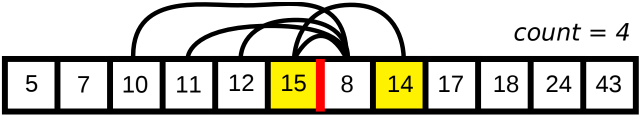

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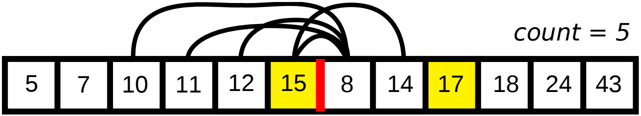

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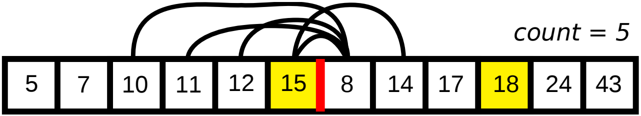

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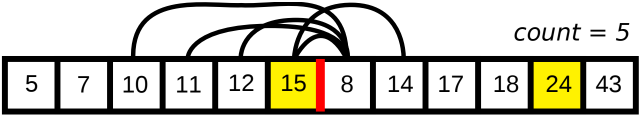

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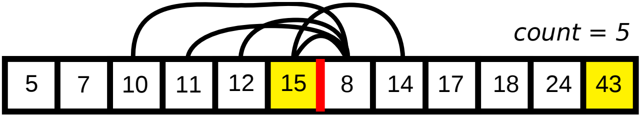

- Given lists $A = a_1, a_2, \ldots, a_m$ and $B = b_1, b_2, \ldots b_m$, compute the number of pairs a_i and b_j such $a_i > b_j$.
- Key idea: problem is much easier if A and B are sorted!
- MERGE-AND-COUNT procedure:
  1. Maintain a *current* pointer for each list.
  2. Maintain a variable *count* initialised to 0.
  3. Initialise each pointer to the front of the list.
  4. While both lists are nonempty:
     - 4.1 Let a_i and b_j be the elements pointed to by the *current* pointers.
     - 4.2 Append the smaller of the two to the output list.
     - 4.3 If $b_j < a_i$, increment *count* by the number of elements remaining in A.
     - 4.4 Advance *current* in the list containing the smaller element.
  5. Append the rest of the non-empty list to the output.
  6. Return *count* and the merged list.
- Running time of this algorithm is $O(m)$.

Counting Inversions: Final Algorithm

```
Sort-and-Count(L)
  If the list has one element then
      there are no inversions
  Else
      Divide the list into two halves:
         A contains the first ⌈n/2⌉ elements
         B contains the remaining ⌊n/2⌋ elements
      (r_A, A) = Sort-and-Count(A)
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      (r, L) = Merge-and-Count(A, B)
   Endif
   Return r = r_A + r_B + r, and the sorted list L
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- Running time $T(n)$ of the algorithm is $O(n \log n)$ because $T(n) \leq 2T(n/2) + O(n)$.

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- Prove by induction. Strategy: every inversion in the data is counted exactly once.

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- Inductive step: Pick an arbitrary k and l such that $k < l$ but $x_k > x_l$. When is this inversion counted?
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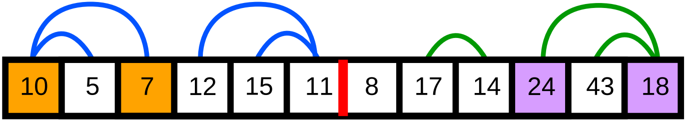

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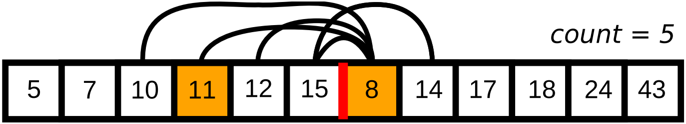

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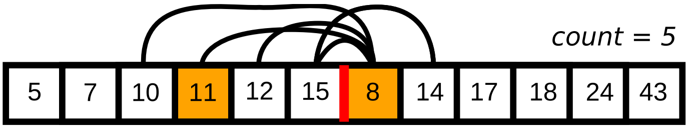

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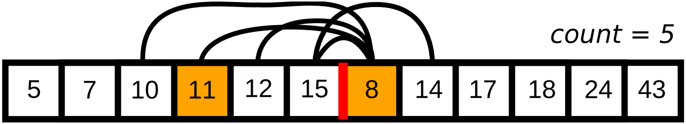

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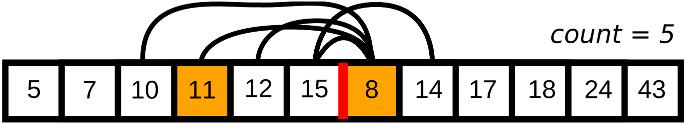

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Figure 5.8 The elementary-school algorithm for multiplying two integers, in (a) decimal and (b) binary representation.

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$$\begin{aligned} T(n) &\leq 3T(n/2) + cn \\ &\leq O(n^{\log_2 3}) = O(n^{1.59}) \end{aligned}$$

Final Algorithm

```
Recursive-Multiply(x,y):
```

Write $x = x_1 \cdot 2^{n/2} + x_0$

$y = y_1 \cdot 2^{n/2} + y_0$

Compute $x_1 + x_0$ and $y_1 + y_0$

p = Recursive-Multiply$(x_1 + x_0,\ \ y_1 + y_0)$

$x_1 y_1$ = Recursive-Multiply(x_1, y_1)

$x_0 y_0$ = Recursive-Multiply(x_0, y_0)

Return $x_1 y_1 \cdot 2^n + (p - x_1 y_1 - x_0 y_0) \cdot 2^{n/2} + x_0 y_0$

Computational Geometry

- Algorithms for geometric objects: points, lines, segments, triangles, spheres, polyhedra, ldots.
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INSTANCE: A set P of n points in the plane

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SOLUTION: The pair of points in P that are the closest to each other.

- At first glance, it seems any algorithm must take $\Omega(n^2)$ time.
- Shamos and Hoey figured out an ingenious $O(n \log n)$ divide and conquer algorithm.

Closest Pair: Set-up

- Let $P = \{p_1, p_2, \ldots, p_n\}$ with $p_i = (x_i, y_i)$.
- Use $d(p_i, p_j)$ to denote the Euclidean distance between p_i and p_j. For a specific pair of points, can compute $d(p_i, p_j)$ in $O(1)$ time.
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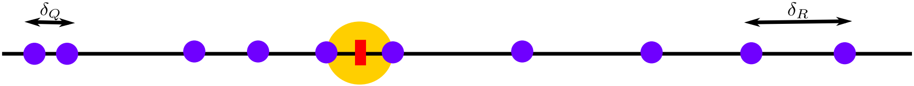

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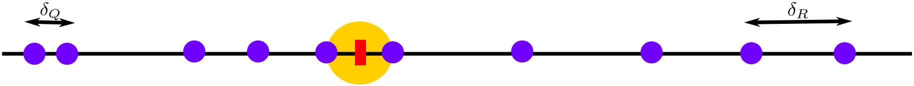
- Generalize the second idea to 2D.



Closest Pair: Algorithm Skeleton

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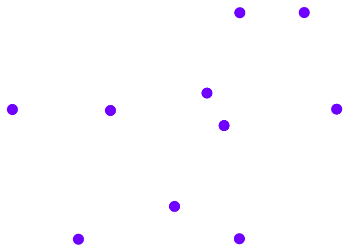

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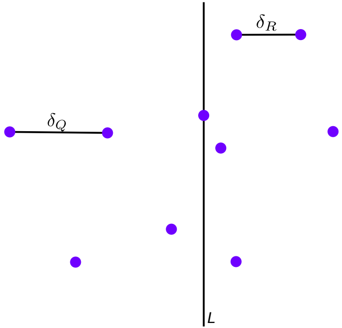

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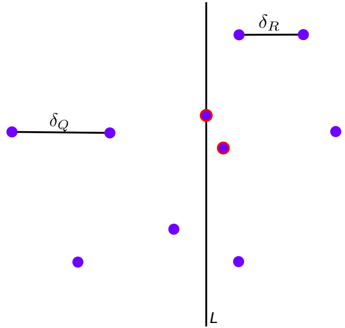
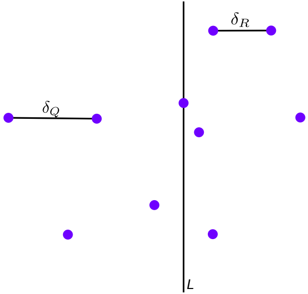
3. Let δ_Q be the distance computed for Q, δ_R be the distance computed for R, and $\delta = \min(\delta_Q, \delta_R)$.
4. Compute pair (q, r) of points such that $q \in Q$, $r \in R$, $d(q, r) < \delta$ and $d(q, r)$ is the smallest possible.



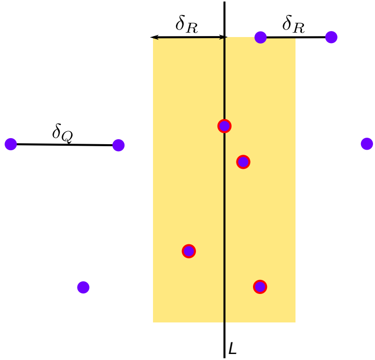
Closest Pair: Proof Sketch

- Prove by induction: Let (s, t) be the closest pair.
  (i) both are in Q: computed correctly by recursive call.
  (ii) both are in R: computed correctly by recursive call.
  (iii) one is in Q and the other is in R: computed correctly in $O(n)$ time by the procedure we will discuss.
- Strategy: Pairs of points for which we do not compute the distance between cannot be the closest pair.
- Overall running time is $O(n \log n)$.



Closest Pair: Conquer Step

- Line L passes through right-most point in Q.
- Let S be the set of points within distance δ of L. (In image, $\delta = \delta_R$.)
- Claim: There exist $q \in Q$, $r \in R$ such that $d(q, r) < \delta$ if and only if $q, r \in S$.
- Corollary: If $t \in Q - S$ and $u \in R - S$, then (t, u) cannot be the closest pair.



Closest Pair: Packing Argument

- Intuition: "too many" points in S that are closer than δ to each other $\Rightarrow$ there must be a pair in Q or in R that are less than δ apart.

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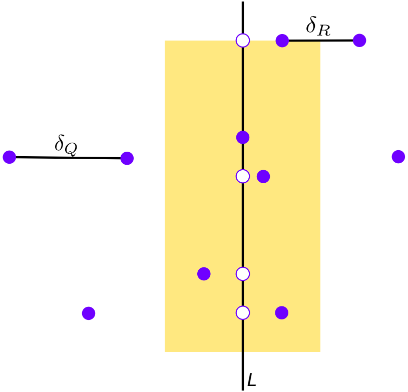

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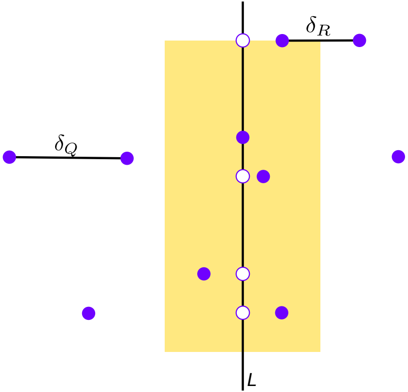
- Let S_y denote the set of points in S sorted by increasing y-coordinate and let s_y denote the y-coordinate of a point $s \in S$.
- Claim: If there exist $s, s' \in S$ such that $d(s, s') < \delta$ then s and s' are at most 15 indices apart in S_y.



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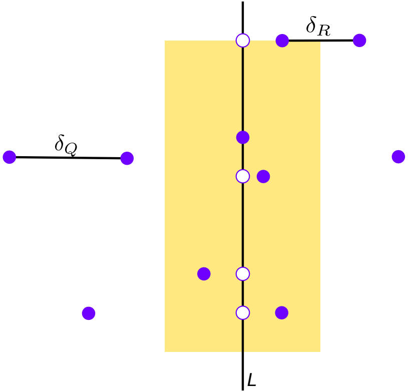

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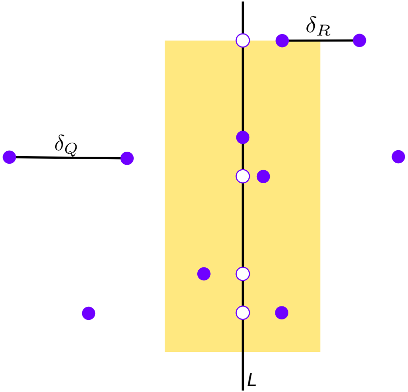
- Use the claim in an algorithm: For every point $s \in S_y$, compute distances only to the next 15 points in S_y.



Closest Pair: Proof of Packing Argument

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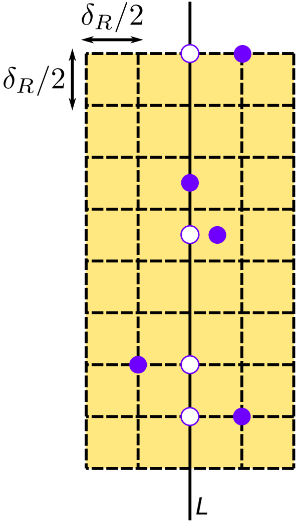
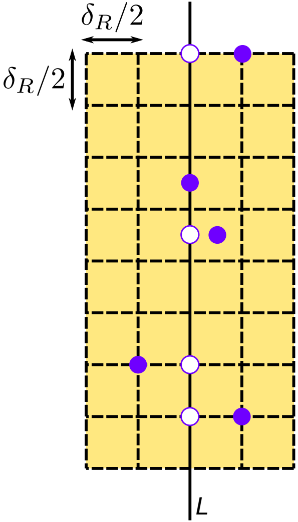


Closest Pair: Proof of Packing Argument

- Claim: If there exist $s, s' \in S$ such that s' appears 16 or more indices after s in S_y, then $s'_y - s_y \geq \delta$.
- Pack the plane with squares of side $\delta/2$.



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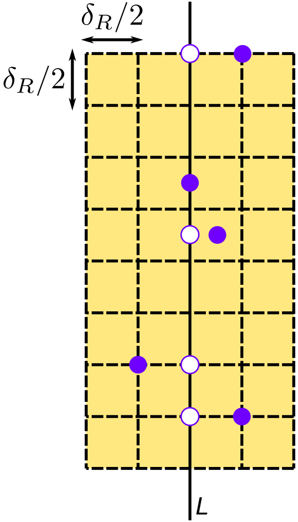
- Each square contains at most one point.



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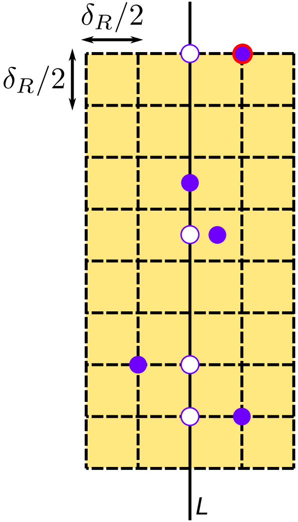
- Let s lie in one of the squares in the first row.



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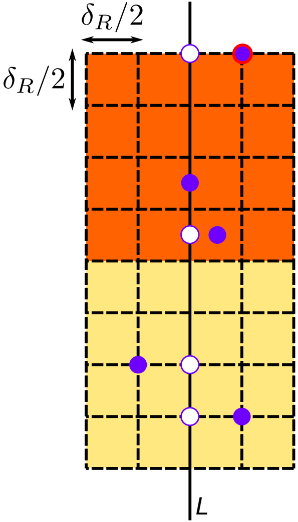
- Each square contains at most one point.
- Let s lie in one of the squares in the first row.
- Any point in the fourth row has a y-coordinate at least δ more than s_y.



Closest Pair: Final Algorithm

```
Closest-Pair(P)
  Construct P_x and P_y  (O(n log n) time)
  (p*_0, p*_1) = Closest-Pair-Rec(P_x,P_y)

Closest-Pair-Rec(P_x, P_y)
  If |P| ≤ 3 then
    find closest pair by measuring all pairwise distances
  Endif

  Construct Q_x, Q_y, R_x, R_y (O(n) time)
  (q*_0,q*_1) = Closest-Pair-Rec(Q_x, Q_y)
  (r*_0,r*_1) = Closest-Pair-Rec(R_x, R_y)

  δ = min(d(q*_0,q*_1), d(r*_0,r*_1))
  x* = maximum x-coordinate of a point in set Q
  L = {(x,y) : x = x*}
  S = points in P within distance δ of L.

  Construct S_y (O(n) time)
  For each point s ∈ S_y, compute distance from s
     to each of next 15 points in S_y
     Let s, s' be pair achieving minimum of these distances
     (O(n) time)

  If d(s,s') < δ then
     Return (s,s')
  Else if d(q*_0,q*_1) < d(r*_0,r*_1) then
     Return (q*_0,q*_1)
  Else
     Return (r*_0,r*_1)
  Endif
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   Let s, s' be pair achieving minimum of these distances
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If d(s,s') < δ then
   Return (s,s')
Else if d(q0*,q1*) < d(r0*,r1*) then
   Return (q0*,q1*)
Else
   Return (r0*,r1*)
Endif
```