CS 5114: Theory of Algorithms

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Blacksburg, Virginia

Spring 2010

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Title page

Tractable Problems

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A problem is said to be **tractable** if an algorithm exists to solve it with polynomial time complexity: $O(p(n))$.

- It is said to be **intractable** if the best known algorithm requires exponential time.

Examples:

- Sorting: $O(n^2)$
- Convex Hull: $O(n^2)$
- Single source shortest path: $O(n^2)$
- All pairs shortest path: $O(n^3)$
- Matrix multiplication: $O(n^3)$

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Log-polynomial is $O(n \log n)$

Like any simple rule of thumb for catagorizing, in some cases the distinction between polynomial and exponential could break down. For example, one can argue that, for practical problems, 1.01^n is preferable to n^{25}. But the reality is that very few polynomial-time algorithms have high degree, and exponential-time algorithms nearly always have a constant of 2 or greater. Nearly all algorithms are either low-degree polynomials or "real" exponentials, with very little in between.

Tractable Problems (cont)

The technique we will use to classify one group of algorithms is based on two concepts:

1. A special kind of reduction.
2. Nondeterminism.

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no notes

Decision Problems

(I, S) such that S(X) is always either "yes" or "no."

- Usually formulated as a question.

Example:

- Instance: A weighted graph $G = (V, E)$, two vertices s and t, and an integer K.
- Question: Is there a path from s to t of length $\leq K$? In this example, the answer is "yes."

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Need a graph here.

Decision Problems (cont)

Can also be formulated as a language recognition problem:

- Let L be the subset of I consisting of instances whose answer is "yes." Can we recognize L?

The class of tractable problems $\mathcal{P}$ is the class of languages or decision problems recognizable in polynomial time.

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Following our graph example: It is possible to translate from a graph to a string representation, and to define a subset of such strings as corresponding to graphs with a path from s to t. This subset defines a language to "recognize."

Polynomial Reducibility

Reduction of one language to another language.

Let $L_1 \subset I_1$ and $L_2 \subset I_2$ be languages. L_1 is **polynomially reducible** to L_2 if there exists a transformation $f : I_1 \rightarrow I_2$, computable in polynomial time, such that $f(x) \in L_2$ if and only if $x \in L_1$.
We write: $L_1 \leq_p L_2$ or $L_1 \leq L_2$.

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Or one decision problem to another.

Specialized case of reduction from Chapter 10.

Examples

- CLIQUE $\leq_p$ INDEPENDENT SET.
- An instance I of CLIQUE is a graph $G = (V, E)$ and an integer K.
- The instance $I' = f(I)$ of INDEPENDENT SET is the graph $G' = (V, E')$ and the integer K, were an edge $(u, v) \in E'$ iff $(u, v) \notin E$.
- f is computable in polynomial time.

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Transformation Example

- G has a clique of size $\geq K$ iff G' has an independent set of size $\geq K$.
- Therefore, CLIQUE $\leq_p$ INDEPENDENT SET.
- **IMPORTANT WARNING:** The reduction does not **solve** either INDEPENDENT SET or CLIQUE, it merely transforms one into the other.

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Need a graph here.

If nodes in G' are independent, then no connections. Thus, in G they all connect.

Nondeterminism

Nondeterminism allows an algorithm to make an arbitrary choice among a finite number of possibilities.

Implemented by the “nd-choice” primitive:
nd-choice(ch_1, ch_2, ..., ch_j)
returns one of the choices ch_1, ch_2, ... **arbitrarily**.

Nondeterministic algorithms can be thought of as “correctly guessing” (choosing nondeterministically) a solution.

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no notes

Nondeterministic CLIQUE Algorithm

```
procedure nd-CLIQUE(Graph G, int K) {
  VertexSet S = EMPTY;  int size = 0;
  for (v in G.V)
    if (nd-choice(YES, NO) == YES) then {
      S = union(S, v);
      size = size + 1;
    }
  if (size < K) then
    REJECT;      // S is too small
  for (u in S)
    for (v in S)
      if ((u <> v) && ((u, v) not in E))
        REJECT;  // S is missing an edge
  ACCEPT;
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What makes this different than random guessing is that **all** choices happen “in parallel.”

Nondeterministic Acceptance

- (G, K) is in the “language” CLIQUE iff there exists a sequence of nd-choice guesses that causes nd-CLIQUE to accept.
- Definition of acceptance by a nondeterministic algorithm:
  - An instance is accepted iff there exists a sequence of nondeterministic choices that causes the algorithm to accept.
- An unrealistic model of computation.
  - There are an exponential number of possible choices, but only one must accept for the instance to be accepted.
- Nondeterminism is a useful concept
  - It provides insight into the nature of certain hard problems.

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no notes

Class $\mathcal{NP}$

- The class of languages accepted by a nondeterministic algorithm in polynomial time is called $\mathcal{NP}$.
- There are an **exponential** number of different executions of nd-CLIQUE on a single instance, but any one execution requires only **polynomial time** in the size of that instance.
- Time complexity of nondeterministic algorithm is greatest amount of time required by any **one** of its executions.

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Note that Towers of Hanoi is not in $\mathcal{NP}$.

Class $\mathcal{NP}$(cont)

Alternative Interpretation:

- $\mathcal{NP}$ is the class of algorithms that, never mind how we got the answer, can check if the answer is correct in polynomial time.
- If you cannot verify an answer in polynomial time, you cannot hope to find the right answer in polynomial time!

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- If you cannot verify an answer in polynomial time, you cannot hope to find the right answer in polynomial time!

This is worded a bit loosely. Specifically, we assume that we can get the answer fast enough – that is, in polynomial time non-deterministically.

How to Get Famous

Clearly, $\mathcal{P} \subset \mathcal{NP}$.

Extra Credit Problem:

- Prove or disprove: $\mathcal{P} = \mathcal{NP}$.

This is important because there are many natural decision problems in $\mathcal{NP}$ for which no $\mathcal{P}$ (tractable) algorithm is known.

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no notes

$\mathcal{NP}$-completeness

A theory based on identifying problems that are as hard as any problems in $\mathcal{NP}$.

The next best thing to knowing whether $\mathcal{P} = \mathcal{NP}$ or not.

A decision problem A is **$\mathcal{NP}$-hard** if every problem in $\mathcal{NP}$ is polynomially reducible to A, that is, for all

$$B \in \mathcal{NP}, \quad B \leq_p A.$$

A decision problem A is **$\mathcal{NP}$-complete** if $A \in \mathcal{NP}$ and A is $\mathcal{NP}$-hard.

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A decision problem A is $\mathcal{NP}$-complete if $A \in \mathcal{NP}$ and A is $\mathcal{NP}$-hard.

A is not permitted to be harder than $\mathcal{NP}$. For example, Tower of Hanoi is not in $\mathcal{NP}$. It requires exponential time to verify a set of moves.

Satisfiability

Let E be a Boolean expression over variables $x_1, x_2, \cdots, x_n$ in conjunctive normal form (CNF), that is, an AND of ORs.

$$E = (x_5 + x_7 + \overline{x_8} + x_{10}) \cdot (\overline{x_2} + x_3) \cdot (x_1 + \overline{x_3} + x_6).$$

A variable or its negation is called a **literal**.
Each sum is called a **clause**.

SATISFIABILITY (SAT):

- Instance: A Boolean expression E over variables $x_1, x_2, \cdots, x_n$ in CNF.
- Question: Is E satisfiable?

Cook's Theorem: SAT is $\mathcal{NP}$-complete.

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Cook's Theorem: SAT is $\mathcal{NP}$-complete.

Is there a truth assignment for the variables that makes E true?Cook won a Turing award for this work.

Proof Sketch

SAT $\in \mathcal{NP}$:

- A non-deterministic algorithm **guesses** a truth assignment for $x_1, x_2, \cdots, x_n$ and **checks** whether E is true in polynomial time.
- It accepts iff there is a satisfying assignment for E.

SAT is $\mathcal{NP}$-hard:

- Start with an arbitrary problem $B \in \mathcal{NP}$.
- We know there is a polynomial-time, nondeterministic algorithm to accept B.
- Cook showed how to transform an instance X of B into a Boolean expression E that is satisfiable if the algorithm for B accepts X.

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The proof of this last step is usually several pages long. One approach is to develop a nondeterministic Turing Machine program to solve an arbitrary problem B in $\mathcal{NP}$.

Implications

(1) Since SAT is $\mathcal{NP}$-complete, we have not defined an empty concept.

(2) If SAT $\in \mathcal{P}$, then $\mathcal{P} = \mathcal{NP}$.

(3) If $\mathcal{P} = \mathcal{NP}$, then SAT $\in \mathcal{P}$.

(4) If $A \in \mathcal{NP}$ and B is $\mathcal{NP}$-complete, then $B \leq_p A$ implies A is $\mathcal{NP}$-complete.
Proof:

- Let $C \in \mathcal{NP}$.
- Then $C \leq_p B$ since B is $\mathcal{NP}$-complete.
- Since $B \leq_p A$ and $\leq_p$ is transitive, $C \leq_p A$.
- Therefore, A is $\mathcal{NP}$-hard and, finally, $\mathcal{NP}$-complete.

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Implications (cont)

(5) This gives a simple two-part strategy for showing a decision problem A is $\mathcal{NP}$-complete.

(a) Show $A \in \mathcal{NP}$.

(b) Pick an $\mathcal{NP}$-complete problem B and show $B \leq_p A$.

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Proving $A \in \mathcal{NP}$ is usually easy.

Don't get the reduction backwards!

$\mathcal{NP}$-completeness Proof Paradigm

To show that decision problem B is $\mathcal{NP}$-complete:

1. $B \in \mathcal{NP}$
   - Give a polynomial time, non-deterministic algorithm that accepts B.
     1. Given an instance X of B, **guess** evidence Y.
     2. **Check** whether Y is evidence that $X \in B$. If so, accept X.
2. B is $\mathcal{NP}$-hard.
   - Choose a known $\mathcal{NP}$-complete problem, A.
   - Describe a polynomial-time transformation T of an **arbitrary** instance of A to a [not necessarily arbitrary] instance of B.
   - Show that $X \in A$ if and only if $T(X) \in B$.

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$B \in \mathcal{NP}$ is usually the easy part.

The first two steps of the $\mathcal{NP}$-hard proof are usually the hardest.

3-SATISFIABILITY (3SAT)

Instance: A Boolean expression E in CNF such that each clause contains exactly 3 literals.

Question: Is there a satisfying assignment for E?

A special case of SAT.

One might hope that 3SAT is easier than SAT.

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What about 2SAT? This is in $\mathcal{P}$.

Effectively a 2-coloring graph problem. Join 2 vertices if they are in same clause, also join x_i and $\overline{x_i}$. Then, try to 2-color the graph with a DFS.

How to solve 1SAT? Answer is "yes" iff x_i and $\overline{x_i}$ are not both in list for any i.

3SAT is $\mathcal{NP}$-complete

(1) 3SAT $\in \mathcal{NP}$.

```
procedure nd-3SAT(E) {
  for (i = 1 to n)
    x[i] = nd-choice(TRUE, FALSE);
  Evaluate E for the guessed truth assignment.
  if (E evaluates to TRUE)
    ACCEPT;
  else
    REJECT;
}
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nd-3SAT is a polynomial-time nondeterministic algorithm that accepts 3SAT.

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Proving 3SAT $\mathcal{NP}$-hard

1. Choose SAT to be the known $\mathcal{NP}$-complete problem.
   - We need to show that SAT $\leq_p$ 3SAT.
2. Let $E = C_1 \cdot C_2 \cdots C_k$ be any instance of SAT.

Strategy: Replace any clause C_i that does not have exactly 3 literals with two or more clauses having exactly 3 literals.

Let $C_i = y_1 + y_2 + \cdots + y_j$ where $y_1, \cdots, y_j$ are literals.
(a) $j = 1$

- Replace (y_1) with

$$(y_1 + v + w) \cdot (y_1 + \overline{v} + w) \cdot (y_1 + v + \overline{w}) \cdot (y_1 + \overline{v} + \overline{w})$$

where v and w are new variables.

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where v and w are new variables.

SAT is the only choice that we have so far!

Replacing (y_1) with $(y_1 + y_1 + y_1)$ seems like a reasonable alternative. But some of the theory behind the definitions rejects clauses with duplicated literals.

Proving 3SAT $\mathcal{NP}$-hard (cont)

(b) $j = 2$

- Replace $(y_1 + y_2)$ with $(y_1 + y_2 + z) \cdot (y_1 + y_2 + \overline{z})$ where z is a new variable.

(c) $j > 3$

- Relace $(y_1 + y_2 + \cdots + y_j)$ with

$$(y_1 + y_2 + z_1) \cdot (y_3 + \overline{z_1} + z_2) \cdot (y_4 + \overline{z_2} + z_3) \cdots$$
$$(y_{j-2} + \overline{z_{j-4}} + z_{j-3}) \cdot (y_{j-1} + y_j + \overline{z_{j-3}})$$

where $z_1, z_2, \cdots, z_{j-3}$ are new variables.

- After replacements made for each C_i, a Boolean expression E' results that is an instance of 3SAT.
- The replacement clearly can be done by a polynomial-time deterministic algorithm.

Proving 3SAT $\mathcal{NP}$-hard (cont)

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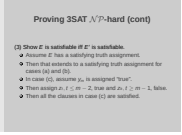
no notes

Proving 3SAT $\mathcal{NP}$-hard (cont)

(3) Show E is satisfiable iff E' is satisfiable.

- Assume E has a satisfying truth assignment.
- Then that extends to a satisfying truth assignment for cases (a) and (b).
- In case (c), assume y_m is assigned "true".
- Then assign $z_t, t \leq m-2$, true and $z_k, t \geq m-1$, false.
- Then all the clauses in case (c) are satisfied.

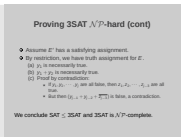


no notes

Proving 3SAT $\mathcal{NP}$-hard (cont)

- Assume E' has a satisfying assignment.
- By restriction, we have truth assignment for E.
  (a) y_1 is necessarily true.
  (b) $y_1 + y_2$ is necessarily true.
  (c) Proof by contradiction:
    - ⋆ If $y_1, y_2, \cdots, y_j$ are all false, then $z_1, z_2, \cdots, z_{j-3}$ are all true.
    - ⋆ But then $(y_{j-1} + y_{j-2} + \overline{z_{j-3}})$ is false, a contradiction.

We conclude SAT $\leq$ 3SAT and 3SAT is $\mathcal{NP}$-complete.



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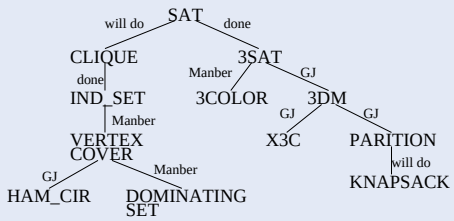
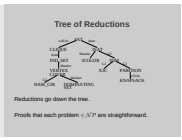
Tree of Reductions



Reductions go down the tree.

Proofs that each problem $\in \mathcal{NP}$ are straightforward.



Refer to handout of $\mathcal{NP}$-complete problems

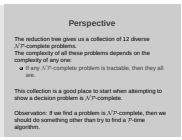
Perspective

The reduction tree gives us a collection of 12 diverse $\mathcal{NP}$-complete problems.
The complexity of all these problems depends on the complexity of any one:

- If any $\mathcal{NP}$-complete problem is tractable, then they all are.

This collection is a good place to start when attempting to show a decision problem is $\mathcal{NP}$-complete.

Observation: If we find a problem is $\mathcal{NP}$-complete, then we should do something other than try to find a $\mathcal{P}$-time algorithm.



Hundreds of problems, from many fields, have been shown to be $\mathcal{NP}$-complete.

More on this observation later.

SAT $\leq_p$ CLIQUE

(1) Easy to show CLIQUE in $\mathcal{NP}$.
(2) An instance of SAT is a Boolean expression

$$B = C_1 \cdot C_2 \cdots C_m,$$

where

$$C_i = y[i,1] + y[i,2] + \cdots + y[i,k_i].$$

Transform this to an instance of CLIQUE $G = (V, E)$ and K.

$$V = \{v[i,j] | 1 \leq i \leq m, 1 \leq j \leq k_i\}$$

Two vertices $v[i_1,j_1]$ and $v[i_2,j_2]$ are adjacent in G if $i_1 \neq i_2$ AND EITHER $y[i_1,j_1]$ and $y[i_2,j_2]$ are the same literal OR $y[i_1,j_1]$ and $y[i_2,j_2]$ have different underlying variables.
$K = m$.

One vertex for each literal in B.

No join if one is the negation of the other

SAT $\leq_p$ CLIQUE (cont)

Example: $B = (x_1 + x_2) \cdot (\overline{x_1} + x_2 + x_3)$.
K = 2.

(3) B is satisfiable iff G has clique of size $\geq K$.

- B is satisfiable implies there is a truth assignment such that $y[i,j_i]$ is true for each i.
- But then $v[i,j_i]$ must be in a clique of size $K = m$.
- If G has a clique of size $\geq K$, then the clique must have size exactly K and there is one vertex $v[i,j_i]$ in the clique for each i.
- There is a truth assignment making each $y[i,j_i]$ true. That truth assignment satisfies B.

We conclude that CLIQUE is $\mathcal{NP}$-hard, therefore $\mathcal{NP}$-complete.

Need figure here. Another example is shown in Manber Figure 11.3.

It must connect to the other $m-1$ literals that are also true.

No clique can have more than one member from the same clause, since there are no links between members of a clause.

Co-$\mathcal{NP}$

- Note the asymmetry in the definition of $\mathcal{NP}$.
  - The non-determinism can identify a clique, and you can verify it.
  - But what if the correct answer is "NO"? How do you verify that?
- Co-$\mathcal{NP}$: The complements of problems in $\mathcal{NP}$.
  - Is a boolean expression **always** false?
  - Is there no clique of size k?
- It seems unlikely that $\mathcal{NP} =$ co-$\mathcal{NP}$.

no notes

Is $\mathcal{NP}$-complete $= \mathcal{NP}$?

- It has been proved that if $\mathcal{P} \neq \mathcal{NP}$, then $\mathcal{NP}$-complete $\neq \mathcal{NP}$.
- The following problems are not known to be in $\mathcal{P}$ or $\mathcal{NP}$, but seem to be of a type that makes them unlikely to be in $\mathcal{NP}$.
  - GRAPH ISOMORPHISM: Are two graphs isomorphic?
  - COMPOSITE NUMBERS: For positive integer K, are there integers $m, n > 1$ such that $K = mn$?
  - LINEAR PROGRAMMING

no notes

PARTITION $\leq_p$ KNAPSACK

PARTITION is a special case of KNAPSACK in which

$$K = \frac{1}{2}\sum_{a \in A} s(a)$$

assuming $\sum s(a)$ is even.

Assuming PARTITION is $\mathcal{NP}$-complete, KNAPSACK is $\mathcal{NP}$-complete.

The assumption about PARITION is true, though we do not prove it.

The "transformation" is simply to pass the input of PARTITION to KNAPSACK.

"Practical" Exponential Problems

- What about our $O(KN)$ dynamic prog algorithm?
- Input size for KNAPSACK is $O(N \log K)$
  - Thus $O(KN)$ is exponential in $N \log K$.
- The dynamic programming algorithm counts through numbers $1, \cdots, K$. Takes exponential time when measured by number of bits to represent K.
- If K is "small" ($K = O(p(N))$), then algorithm has complexity polynomial in N and is truly polynomial in input size.
- An algorithm that is polynomial-time if the numbers IN the input are "small" (as opposed to number OF inputs) is called a **pseudo-polynomial** time algorithm.

This is an important point, about the input size. It has to do with the "size" of a number (a value). We represent the value n with $\log n$ bits, or more precisely, $\log N$ bits where N is the maximum value. In the case of KNAPSACK, K (the knapsack size) is effectively the maximum number. We will use this observation frequently when we analyze numeric algorithms.

"Practical" Problems (cont)

- Lesson: While KNAPSACK is $\mathcal{NP}$-complete, it is often not that hard.
- Many $\mathcal{NP}$-complete problems have no pseudo-polynomial time algorithm unless $\mathcal{P} = \mathcal{NP}$.

The issue is what size input is practical. The problems we want to solve for Traveling Salesman are not practical.

Coping with $\mathcal{NP}$-completeness

(1) Find subproblems of the original problem that have polynomial-time algorithms.

(2) Approximation algorithms.

(3) Randomized Algorithms.

(4) Backtracking; Branch and Bound.

(5) Heuristics.
- Greedy.
- Simulated Annealing.
- Genetic Algorithms.

The subproblems need to be "significant" special cases.

Approximation works for optimization problems (and there are a LOT of those).

Randomized Algorithms typically work well for problems with a lot of solutions.

(4) gives ways to (relatively efficiently) implement nd-choice.

Subproblems

Restrict attention to special classes of inputs.
Examples:

- VERTEX COVER, INDEPENDENT SET, and CLIQUE, when restricted to bipartite graphs, all have polynomial-time algorithms (for VERTEX COVER, by reduction to NETWORK FLOW).
- 2-SATISFIABILITY, 2-DIMENSIONAL MATCHING and EXACT COVER BY 2-SETS all have polynomial time algorithms.
- PARTITION and KNAPSACK have polynomial time algorithms if the numbers in an instance are all $O(p(n))$.
- However, HAMILTONIAN CIRCUIT and 3-COLORABILITY remain $\mathcal{NP}$-complete even for a planar graph.

Backtracking

We may view a nondeterministic algorithm executing on a particular instance as a tree:

1. Each edge represents a particular nondeterministic choice.
2. The checking occurs at the leaves.

Example:

Each leaf represents a different set S. Checking that S is a clique of size $\geq K$ can be done in polynomial time.

Backtracking (cont)

Backtracking can be viewed as an in-order traversal of this tree with two criteria for stopping.

1. A leaf that accepts is found.
2. A partial solution that could not possibly lead to acceptance is reached.

Example:

There cannot possibly be a set S of cardinality ≥ 2 under this node, so backtrack.

Since $(1, 2) \notin E$, no S under this node can be a clique, so backtrack.

Branch and Bound

- For optimization problems.
  More sophisticated kind of backtracking.
- Use the best solution found so far as a **bound** that controls backtracking.
- Example Problem: Given a graph G, find a minimum vertex cover of G.
- Computation tree for nondeterministic algorithm is similar to CLIQUE.
  - Every leaf represents a different subset S of the vertices.
- Whenever a leaf is reached and it contains a vertex cover of size B, B is an upper bound on the size of the minimum vertex cover.
  - Use B to prune any future tree nodes having size $\geq B$.
- Whenever a smaller vertex cover is found, update B.

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Assuming the subclass covers the inputs you are interested in!

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Example for k-CLIQUE

Need a figure here. Manber Figure 11.7 has a similar example.

Backtracking (cont)
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When the corresponding decision problem is $\mathcal{NP}$-complete.

Branch and Bound (cont)

- Improvement:
  - Use a fast, greedy algorithm to get a minimal (not minimum) vertex cover.
  - Use this as the initial bound B.
- While Branch and Bound is better than a brute-force exhaustive search, it is usually exponential time, hence impractical for all but the smallest instances.
  - ... if we insist on an optimal solution.
- Branch and Bound often practical as an approximation algorithm where the search terminates when a "good enough" solution is obtained.

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no notes

Approximation Algorithms

Seek algorithms for optimization problems with a guaranteed bound on the quality of the solution.

VERTEX COVER: Given a graph $G = (V, E)$, find a vertex cover of minimum size.

Let M be a maximal (not necessarily maximum) matching in G and let V' be the set of matched vertices.
If OPT is the size of a minimum vertex cover, then

$$|V'| \leq 2\mathrm{OPT}$$

because at least one endpoint of every matched edge must be in **any** vertex cover.

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If OPT is the size of a minimum vertex cover, then
$|V'| \leq 2\mathrm{OPT}$
because at least one endpoint of every matched edge must be in **any** vertex cover.

Vertex cover: A set of vertices such that every edge is incident on at least one vertex in the set.

Then every edge will be have at least one matched vertex (i.e., vertex in the set). Thus the matching qualifies as a vertex cover.

Since a vertex of M cannot cover more than one edge of M.
In fact, we always know how far we are from a **perfect** cover (though we don't always know the size of OPT).

Bin Packing

We have numbers $x_1, x_2, \cdots, x_n$ between 0 and 1 as well as an unlimited supply of bins of size 1.

Problem: Put the numbers into as few bins as possible so that the sum of the numbers in any one bin does not exceed 1.

Example: Numbers 3/4, 1/3, 1/2, 1/8, 2/3, 1/2, 1/4.

Optimal solution: [3/4, 1/8], [1/2, 1/3], [1/2, 1/4], [2/3].

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Example: Numbers 3/4, 1/3, 1/2, 1/8, 2/3, 1/2, 1/4.
Optimal solution: [3/4, 1/8], [1/2, 1/3], [1/2, 1/4], [2/3].

Optimal in that the sum is 3 1/8, and we packed into 4 bins. There is another optimal solution with the first 3 bins packed, but this is more than we need to solve the problem.

First Fit Algorithm

Place x_1 into the first bin.

For each $i, 2 \leq i \leq n$, place x_i in the first bin that will contain it.

No more than 1 bin can be left less than half full.
The number of bins used is no more than twice the sum of the numbers.

The sum of the numbers is a lower bound on the number of bins in the optimal solution.

Therefore, first fit is no more than twice the optimal number of bins.

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Therefore, first fit is no more than twice the optimal number of bins.

Otherwise, the items in the second half-full bin would be put into the first!

First Fit Does Poorly

Let ϵ be very small, e.g., $\epsilon = .00001$.
Numbers (in this order):

- 6 of $(1/7 + \epsilon)$.
- 6 of $(1/3 + \epsilon)$.
- 6 of $(1/2 + \epsilon)$.

First fit returns:

- 1 bin of [6 of $1/7 + \epsilon$]
- 3 bins of [2 of $1/3 + \epsilon$]
- 6 bins of [$1/2 + \epsilon$]

Optimal solution is 6 bins of $[1/7 + \epsilon, 1/3 + \epsilon, 1/2 + \epsilon]$.

First fit is 5/3 larger than optimal.

2010-04-12 CS 5114 └First Fit Does Poorly

no notes

Decreasing First Fit

It can be proved that the worst-case performance of first-fit is 17/10 times optimal.

Use the following heuristic:

- Sort the numbers in decreasing order.
- Apply first fit.
- This is called **decreasing first fit**.

The worst case performance of decreasing first fit is close to 11/9 times optimal.

2010-04-12 CS 5114 └Decreasing First Fit

no notes

Summary

- The theory of $\mathcal{NP}$-completeness gives us a technique for separating tractable from (probably) intractable problems.
- When faced with a new problem requiring algorithmic solution, our thought process might resemble this scheme:

Is it $\mathcal{NP}$-complete? $\rightleftharpoons$ Is it in $\mathcal{P}$?

- Alternately think about each question. Lack of progress on either question might give insights into the answer to the other question.
- Once an affirmative answer is obtained to one of these questions, one of two strategies is followed.

2010-04-12 CS 5114 └Summary

no notes

Strategies

(1) The problem is in $\mathcal{P}$.

- This means there are polynomial-time algorithms for the problem, and presumably we know at least one.
- So, apply the techniques learned in this course to analyze the algorithms and improve them to find the lowest time complexity we can.

(2) The problem is $\mathcal{NP}$-complete.

- Apply the strategies for coping with $\mathcal{NP}$-completeness.
- Especially, find subproblems that are in $\mathcal{P}$, or find approximation algorithms.

2010-04-12 CS 5114 └Strategies

That is the only way we could have proved it is in $\mathcal{P}$.