CS 5114: Theory of Algorithms

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Spring 2010

Title page

Graph Algorithms

Graphs are useful for representing a variety of concepts:

- Data Structures
- Relationships
- Families
- Communication Networks
- Road Maps

- A **graph** $\mathrm{G} = (\mathrm{V}, \mathrm{E})$ consists of a set of **vertices** V, and a set of **edges** E, such that each edge in E is a connection between a pair of vertices in V.
- Directed vs. Undirected
- Labeled graph, weighted graph
- Labels for edges vs. weights for edges
- Multiple edges, loops
- Cycle, Circuit, path, simple path, tours
- Bipartite, acyclic, connected
- Rooted tree, unrooted tree, free tree

A Tree Proof

- **Definition**: A **free tree** is a connected, undirected graph that has no cycles.
- **Theorem**: If T is a free tree having n vertices, then T has exactly $n - 1$ edges.
- **Proof**: By induction on n.
- **Base Case**: $n = 1$. T consists of 1 vertex and 0 edges.
- **Inductive Hypothesis**: The theorem is true for a tree having $n - 1$ vertices.
- **Inductive Step**:
  - If T has n vertices, then T contains a vertex of degree 1.
  - Remove that vertex and its incident edge to obtain T', a free tree with $n - 1$ vertices.
  - By IH, T' has $n - 2$ edges.
  - Thus, T has $n - 1$ edges.

This is close to a satisfactory definition for free tree. There are several equivalent definitions for free trees, with similar proofs to relate them.

Why do we know that some vertex has degree 1? Because the definition says that the Free Tree has no cycles.

Graph Traversals

Various problems require a way to **traverse** a graph – that is, visit each vertex and edge in a systematic way.

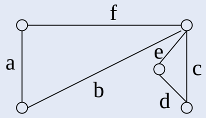
Three common traversals:

1. Eulerian tours
   Traverse each edge exactly once
2. Depth-first search
   Keeps vertices on a stack
3. Breadth-first search
   Keeps vertices on a queue

a vertex may be visited multiple times

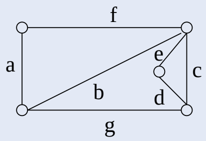
Eulerian Tours

A circuit that contains every edge exactly once.
Example:



Tour: b a f c d e.

Example:



No Eulerian tour. How can you tell for sure?

Why no tour? Because some vertices have odd degree.

All even nodes is a necessary condition. Is it sufficient?

Eulerian Tour Proof

- **Theorem**: A connected, undirected graph with m edges that has no vertices of odd degree has an Eulerian tour.
- **Proof**: By induction on m.
- **Base Case**:
- **Inductive Hypothesis**:
- **Inductive Step**:
  - Start with an arbitrary vertex and follow a path until you return to the vertex.
  - Remove this circuit. What remains are connected components G_1, G_2, ..., G_k each with nodes of even degree and $< m$ edges.
  - By IH, each connected component has an Eulerian tour.
  - Combine the tours to get a tour of the entire graph.

Base case: 0 edges and 1 vertex fits the theorem.
IH: The theorem is true for $< m$ edges.
Always possible to find a circuit starting at any arbitrary vertex, since each vertex has even degree.

Depth First Search

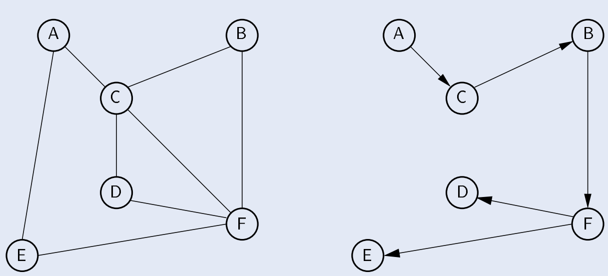
```
void DFS(Graph G, int v) { // Depth first search
  PreVisit(G, v);     // Take appropriate action
  G.setMark(v, VISITED);
  for (Edge w = each neighbor of v)
    if (G.getMark(G.v2(w)) == UNVISITED)
      DFS(G, G.v2(w));
  PostVisit(G, v);    // Take appropriate action
}
```

Initial call: `DFS(G, r)` where r is the **root** of the DFS.

Cost: $\Theta(|V| + |E|)$.

no notes

Depth First Search Example



(a) (b)

The directions are imposed by the traversal. This is the Depth First Search Tree.

DFS Tree

If we number the vertices in the order that they are marked, we get **DFS numbers**.

Lemma 7.2: Every edge $e \in E$ is either in the DFS tree T, or connects two vertices of G, one of which is an ancestor of the other in T.

Proof: Consider the first time an edge (v, w) is examined, with v the current vertex.

- If w is unmarked, then (v, w) is in T.
- If w is marked, then w has a smaller DFS number than v AND (v, w) is an unexamined edge of w.
- Thus, w is still on the stack. That is, w is on a path from v.

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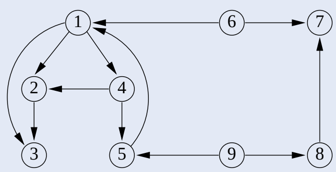
└DFS Tree

Results: No "cross edges." That is, no edges connecting vertices sideways in the tree.

DFS for Directed Graphs

- Main problem: A connected graph may not give a single DFS tree.



- Forward edges: (1, 3)
- Back edges: (5, 1)
- Cross edges: (6, 1), (8, 7), (9, 5), (9, 8), (4, 2)
- **Solution**: Maintain a list of unmarked vertices.
  - Whenever one DFS tree is complete, choose an arbitrary unmarked vertex as the root for a new tree.

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└DFS for Directed Graphs

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Directed Cycles

Lemma 7.4: Let G be a directed graph. G has a directed cycle iff every DFS of G produces a back edge.

Proof:

1. Suppose a DFS produces a back edge (v, w).
   - v and w are in the same DFS tree, w an ancestor of v.
   - (v, w) and the path in the tree from w to v form a directed cycle.
2. Suppose G has a directed cycle C.
   - Do a DFS on G.
   - Let w be the vertex of C with smallest DFS number.
   - Let (v, w) be the edge of C coming into w.
   - v is a descendant of w in a DFS tree.
   - Therefore, (v, w) is a back edge.

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└Directed Cycles

See earlier lemma.

Breadth First Search

- Like DFS, but replace stack with a queue.
- Visit vertex's neighbors before going deeper in tree.

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└Breadth First Search

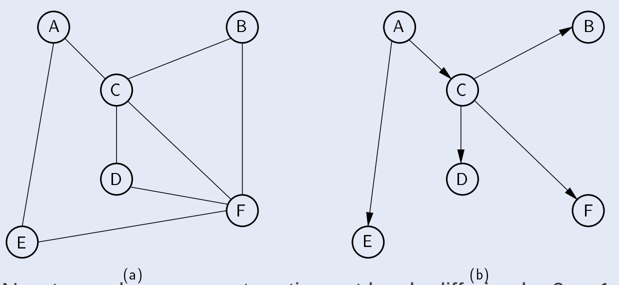
no notes

Breadth First Search Algorithm

```
void BFS(Graph G, int start) {
  Queue Q(G.n());
  Q.enqueue(start);
  G.setMark(start, VISITED);
  while (!Q.isEmpty()) {
    int v = Q.dequeue();
    PreVisit(G, v);   // Take appropriate action
    for (Edge w = each neighbor of v)
      if (G.getMark(G.v2(w)) == UNVISITED) {
        G.setMark(G.v2(w), VISITED);
        Q.enqueue(G.v2(w));
      }
    PostVisit(G, v);  // Take appropriate action
}}
```


no notes

Breadth First Search Example



Non-tree edges connect vertices at levels differing by 0 or 1.

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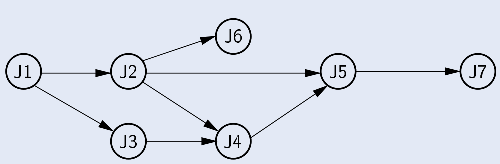
Topological Sort

Problem: Given a set of jobs, courses, etc. with prerequisite constraints, output the jobs in an order that does not violate any of the prerequisites.



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Topological Sort Algorithm

```
void topsort(Graph G) { // Top sort: recursive
  for (int i=0; i<G.n(); i++) // Initialize Mark
    G.setMark(i, UNVISITED);
  for (i=0; i<G.n(); i++)     // Process vertices
    if (G.getMark(i) == UNVISITED)
      tophelp(G, i);          // Call helper
}
void tophelp(Graph G, int v) { // Helper function
  G.setMark(v, VISITED);
  for (Edge w = each neighbor of v)
    if (G.getMark(G.v2(w)) == UNVISITED)
      tophelp(G, G.v2(w));
  printout(v);         // PostVisit for Vertex v
}
```


Prints in reverse order.

Queue-based Topological Sort

```
void topsort(Graph G) { // Top sort: Queue
  Queue Q(G.n());  int Count[G.n()];
  for (int v=0; v<G.n(); v++) Count[v] = 0;
  for (v=0; v<G.n(); v++) // Process every edge
    for (Edge w each neighbor of v)
       Count[G.v2(w)]++;  // Add to v2's count
  for (v=0; v<G.n(); v++) // Initialize Queue
    if (Count[v] == 0) Q.enqueue(v);
  while (!Q.isEmpty()) {  // Process the vertices
    int v = Q.dequeue();
    printout(v);          // PreVisit for v
    for (Edge w = each neighbor of v) {
      Count[G.v2(w)]--;   // One less prereq
      if (Count[G.v2(w)]==0) Q.enqueue(G.v2(w));
}}}
```


Queue-based Topological Sort

no notes

Shortest Paths Problems

Input: A graph with **weights** or **costs** associated with each edge.

Output: The list of edges forming the shortest path.

Sample problems:

- Find the shortest path between two specified vertices.
- Find the shortest path from vertex S to all other vertices.
- Find the shortest path between all pairs of vertices.

Our algorithms will actually calculate only **distances**.

Shortest Paths Problems

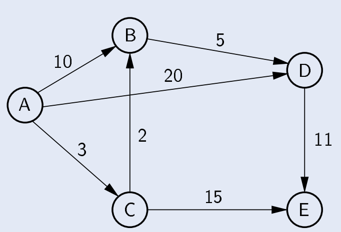
no notes

Shortest Paths Definitions

d(A, B) is the **shortest distance** from vertex A to B.

w(A, B) is the **weight** of the edge connecting A to B.

- If there is no such edge, then w(A, B) $= \infty$.



Shortest Paths Definitions

w(A, D) $=$ 20; d(A, D) $=$ 10 (through ACBD).

Single Source Shortest Paths

Given start vertex s, find the shortest path from s to all other vertices.

Try 1: Visit all vertices in some order, compute shortest paths for all vertices seen so far, then add the shortest path to next vertex x.

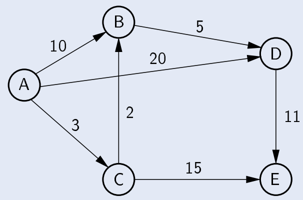
Problem: Shortest path to a vertex already processed might go through x.
Solution: Process vertices in order of distance from s.

Single Source Shortest Paths

no notes

Dijkstra's Algorithm Example

	A	B	C	D	E
Initial	0	∞	∞	∞	∞
Process A	0	10	3	20	∞
Process C	0	5	3	20	18
Process B	0	5	3	10	18
Process D	0	5	3	10	18
Process E	0	5	3	10	18



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└─Dijkstra's Algorithm Example

no notes

Dijkstra's Algorithm: Array (1)

```
void Dijkstra(Graph G, int s) { // Use array
  int D[G.n()];
  for (int i=0; i<G.n(); i++)   // Initialize
    D[i] = INFINITY;
  D[s] = 0;
  for (i=0; i<G.n(); i++) {  // Process vertices
    int v = minVertex(G, D);
    if (D[v] == INFINITY) return; // Unreachable
    G.setMark(v, VISITED);
    for (Edge w = each neighbor of v)
      if (D[G.v2(w)] > (D[v] + G.weight(w)))
        D[G.v2(w)] = D[v] + G.weight(w);
  }
}
```


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└─Dijkstra's Algorithm: Array (1)

no notes

Dijkstra's Algorithm: Array (2)

```
// Get mincost vertex
int minVertex(Graph G, int* D) {
  int v;  // Initialize v to an unvisited vertex;
  for (int i=0; i<G.n(); i++)
    if (G.getMark(i) == UNVISITED)
      { v = i; break; }
  for (i++; i<G.n(); i++) // Find smallest D val
    if ((G.getMark(i)==UNVISITED) && (D[i]<D[v]))
      v = i;
  return v;
}
```

Approach 1: Scan the table on each pass for closest vertex.
Total cost: $\Theta(|V|^2 + |E|) = \Theta(|V|^2)$.

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└─Dijkstra's Algorithm: Array (2)

no notes

Dijkstra's Algorithm: Priority Queue (1)

```
class Elem { public: int vertex, dist; };
int key(Elem x) { return x.dist; }
void Dijkstra(Graph G, int s) { // priority queue
  int v;   Elem temp;
  int D[G.n()];  Elem E[G.e()];
  temp.dist = 0; temp.vertex = s; E[0] = temp;
  heap H(E, 1, G.e());         // Create the heap
  for (int i=0; i<G.n(); i++) D[i] = INFINITY;
  D[s] = 0;
  for (i=0; i<G.n(); i++) {     // Get distances
    do { temp = H.removemin(); v = temp.vertex; }
      while (G.getMark(v) == VISITED);
    G.setMark(v, VISITED);
    if (D[v] == INFINITY) return; // Unreachable
```


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└─Dijkstra's Algorithm: Priority Queue (1)

no notes

Dijkstra's Algorithm: Priority Queue (2)

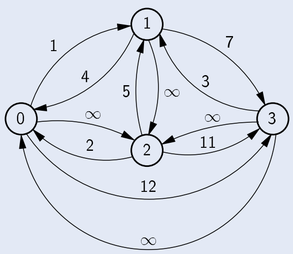
```
    for (Edge w = each neighbor of v)
      if (D[G.v2(w)] > (D[v] + G.weight(w))) {
        D[G.v2(w)] = D[v] + G.weight(w);
        temp.dist = D[G.v2(w)];
        temp.vertex = G.v2(w);
        H.insert(temp);  // Insert new distance
}}}
```

- Approach 2: Store unprocessed vertices using a min-heap to implement a priority queue ordered by D value. Must update priority queue for each edge.
- Total cost: $\Theta((|V| + |E|) \log |V|)$.

no notes

All Pairs Shortest Paths

- For every vertex $u, v \in V$, calculate d(u, v).
- Could run Dijkstra's Algorithm |V| times.
- Better is **Floyd's Algorithm**.
- Define a **k-path** from u to v to be any path whose intermediate vertices all have indices less than k.



Multiple runs of Dijkstra's algorithm Cost: $|V||E| \log |V| = |V|^3 \log |V|$ for dense graph.

The issue driving the concept of "k paths" is how to efficiently check all the paths without computing any path more than once.

0,3 is a 0-path. 2,0,3 is a 1-path. 0,2,3 is a 3-path, but not a 2 or 1 path. Everything is a 4 path.

Floyd's Algorithm

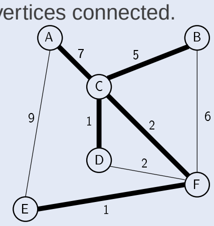
```
void Floyd(Graph G) { // All-pairs shortest paths
  int D[G.n()][G.n()]; // Store distances
  for (int i=0; i<G.n(); i++) // Initialize D
    for (int j=0; j<G.n(); j++)
      D[i][j] = G.weight(i, j);
  for (int k=0; k<G.n(); k++) // Compute k paths
    for (int i=0; i<G.n(); i++)
      for (int j=0; j<G.n(); j++)
        if (D[i][j] > (D[i][k] + D[k][j]))
          D[i][j] = D[i][k] + D[k][j];
}
```


no notes

Minimum Cost Spanning Trees

Minimum Cost Spanning Tree (MST) Problem:

- Input: An undirected, connected graph G.
- Output: The subgraph of G that
  1. has minimum total cost as measured by summing the values for all of the edges in the subset, and
  2. keeps the vertices connected.



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Key Theorem for MST

Let V_1, V_2 be an arbitrary, non-trivial partition of V. Let (v_1, v_2), $v_1 \in V_1, v_2 \in V_2$, be the cheapest edge between V_1 and V_2. Then (v_1, v_2) is in some MST of G.

Proof:

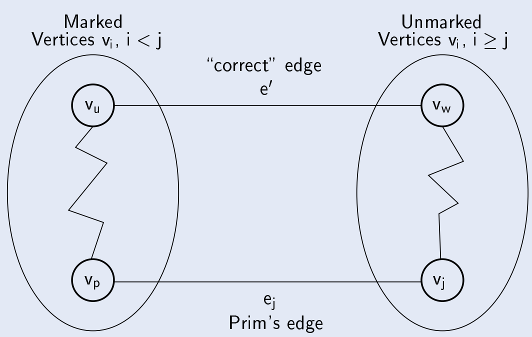
- Let T be an arbitrary MST of G.
- If (v_1, v_2) is in T, then we are done.
- Otherwise, adding (v_1, v_2) to T creates a cycle C.
- At least one edge (u_1, u_2) of C other than (v_1, v_2) must be between V_1 and V_2.
- $c(u_1, u_2) \geq c(v_1, v_2)$.
- Let $T' = T \cup \{(v_1, v_2)\} - \{(u_1, u_2)\}$.
- Then, T' is a spanning tree of G and $c(T') \leq c(T)$.
- But $c(T)$ is minimum cost.

Therefore, $c(T') = c(T)$ and T' is a MST containing (v_1, v_2).

Key Theorem for MST

There can only be multiple MSTs when there are edges with equal cost.

Key Theorem Figure



Key Theorem Figure

no notes

Prim's MST Algorithm (1)

```
void Prim(Graph G, int s) {     // Prim's MST alg
  int D[G.n()];  int V[G.n()];  // Distances
  for (int i=0; i<G.n(); i++)   // Initialize
    D[i] = INFINITY;
  D[s] = 0;
  for (i=0; i<G.n(); i++) {   // Process vertices
    int v = minVertex(G, D);
    G.setMark(v, VISITED);
    if (v != s) AddEdgetoMST(V[v], v);
    if (D[v] == INFINITY) return; //v unreachable
    for (Edge w = each neighbor of v)
      if (D[G.v2(w)] > G.weight(w)) {
        D[G.v2(w)] = G.weight(w); // Update dist
        V[G.v2(w)] = v;           // who came from
}}}
```


Prim's MST Algorithm (1)

no notes

Prim's MST Algorithm (2)

```
int minVertex(Graph G, int* D) {
  int v;  // Initialize v to any unvisited vertex
  for (int i=0; i<G.n(); i++)
    if (G.getMark(i) == UNVISITED)
      { v = i; break; }
  for (i=0; i<G.n(); i++) // Find smallest value
    if ((G.getMark(i)==UNVISITED) && (D[i]<D[v]))
      v = i;
  return v;
}
```

This is an example of a **greedy** algorithm.

Prim's MST Algorithm (2)

no notes

Alternative Prim's Implementation (1)

Like Dijkstra's algorithm, can implement with priority queue.

```
void Prim(Graph G, int s) {
  int v;                 // The current vertex
  int D[G.n()];          // Distance array
  int V[G.n()];          // Who's closest
  Elem temp;
  Elem E[G.e()];         // Heap array
  temp.distance = 0; temp.vertex = s;
  E[0] = temp;           // Initialize heap array
  heap H(E, 1, G.e());   // Create the heap
  for (int i=0; i<G.n(); i++) D[i] = INFINITY;
  D[s] = 0;
```


└─Alternative Prim's Implementation (1)

no notes

Alternative Prim's Implementation (2)

```
  for (i=0; i<G.n(); i++) { // Now build MST
    do { temp = H.removemin(); v = temp.vertex; }
      while (G.getMark(v) == VISITED);
    G.setMark(v, VISITED);
    if (v != s) AddEdgetoMST(V[v], v);
    if (D[v] == INFINITY) return; // Unreachable
    for (Edge w = each neighbor of v)
      if (D[G.v2(w)] > G.weight(w)) { // Update D
        D[G.v2(w)] = G.weight(w);
        V[G.v2(w)] = v;          // Who came from
        temp.distance = D[G.v2(w)];
        temp.vertex = G.v2(w);
        H.insert(temp);    // Insert dist in heap
      }
}}
```


└─Alternative Prim's Implementation (2)

no notes

Kruskal's MST Algorithm (1)

```
Kruskel(Graph G) { // Kruskal's MST algorithm
  Gentree A(G.n()); // Equivalence class array
  Elem E[G.e()];   // Array of edges for min-heap
  int edgecnt = 0;
  for (int i=0; i<G.n(); i++) // Put edges into E
    for (Edge w = G.first(i);
         G.isEdge(w); w = G.next(w)) {
      E[edgecnt].weight = G.weight(w);
      E[edgecnt++].edge = w;
    }
  heap H(E, edgecnt, edgecnt); // Heapify edges
  int numMST = G.n();  // Init w/ n equiv classes
```


└─Kruskal's MST Algorithm (1)

no notes

Kruskal's MST Algorithm (2)

```
  for (i=0; numMST>1; i++) { // Combine
    Elem temp = H.removemin(); // Next cheap edge
    Edge w = temp.edge;
    int v = G.v1(w);  int u = G.v2(w);
    if (A.differ(v, u)) { // If different
      A.UNION(v, u);      //   Combine
      AddEdgetoMST(G.v1(w), G.v2(w));  // Add
      numMST--;           // Now one less MST
    }
  }
}
```

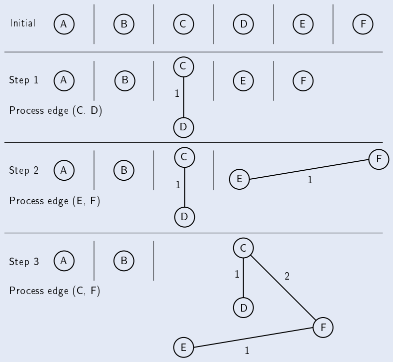
How do we compute function `MSTof(v)`?
Solution: UNION-FIND algorithm (Section 4.3).

└─Kruskal's MST Algorithm (2)

no notes

Kruskal's Algorithm Example

Total cost: $\Theta(|V| + |E| \log |E|)$.

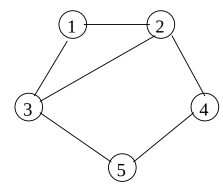


Cost is dominated by the edge sort.
Alternative: Use a min heap, quit when only one set left.
"Kth-smallest" implementation.

Matching

- Suppose there are n workers that we want to work in teams of two. Only certain pairs of workers are willing to work together.
- **Problem**: Form as many compatible non-overlapping teams as possible.
- Model using G, an undirected graph.
  - Join vertices if the workers will work together.
- A **matching** is a set of edges in G with no vertex in more than one edge (the edges are independent).
  - A **maximal matching** has no free pairs of vertices that can extend the matching.
  - A **maximum matching** has the greatest possible number of edges.
  - A **perfect matching** includes every vertex.

An example:
(1-3) is a matching.
(1-3) (5, 4) is both maximal and maximum.
Take away the edge (5-4). Then (3, 2) would be maximal but not a maximum matching.



Very Dense Graphs (1)

Theorem: Let $G = (V, E)$ be an undirected graph with $|V| = 2n$ and every vertex having degree $\geq n$. Then G contains a perfect matching.

Proof: Suppose that G does not contain a perfect matching.

- Let $M \subseteq E$ be a max matching. $|M| < n$.
- There must be two unmatched vertices v_1, v_2 that are not adjacent.
- Every vertex adjacent to v_1 or to v_2 is matched.
- Let $M' \subseteq M$ be the set of edges involved in matching the neighbors of v_1 and v_2.
- There are $\geq 2n$ edges from v_1 and v_2 to vertices covered by M', but $|M'| < n$.

There must be two unmatched vertices not adjacent: Otherwise it would either be perfect (if there are no 2 free vertices) or we could just match v_1 and v_2 (because they are adjacent).

Every adjacent vertex is matched, otherwise the matching would not be maximal.

See Manber Figure 3.76.

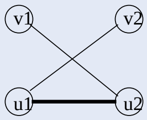
Very Dense Graphs (2)

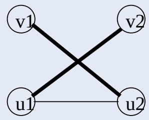
Proof: (continued)

- Thus, some edge of M' is adjacent to 3 edges from v_1 and v_2.
- Let (u_1, u_2) be such an edge.
- Replacing (u_1, u_2) with (v_1, u_2) and (v_2, u_1) results in a larger matching.
- Theorem proven by contradiction.

Pigeonhole Principle

Generalizing the Insight





- v_1, u_2, u_1, v_2 is a path from an unmatched vertex to an unmatched vertex such that alternate edges are unmatched and matched.
- In one step, switch unmatched and matched edges.
- Let $G = (V, E)$ be an undirected graph and $M \subseteq E$ a matching.
- An **alternating path** P goes from v to u, consists of alternately matched and unmatched edges, and both v and u are not in the match.

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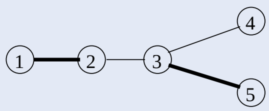
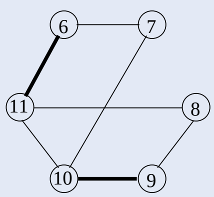
Matching Example





1, 2, 3, 5 is NOT an alternating path (it does not start with an unmatch vertex).

7, 6, 11, 10, 9, 8 is an alternating path with respect to the given matching.

Observation: If a matching has an alternating path, then the size of the matching can be increased by one by switching matched and unmatched edges along the alternating path.

The Alternating Path Theorem (1)

Theorem: A matching is maximum iff it has no alternating paths.

Proof:

- Clearly, if a matching has alternating paths, then it is not maximum.
- Suppose M is a non-maximum matching.
- Let M' be any maximum matching. Then, $|M'| > |M|$.
- Let $M \oplus M'$ be the symmetric difference of M and M'.

$$M \oplus M' = M \cup M' - (M \cap M').$$

- $G' = (V, M \oplus M')$ is a subgraph of G having maximum degree ≤ 2.

The first point is the obvious part of the iff. If there is an alternating path, simply switch the match and umatched edges to augment the match.

Symmetric difference: Those in either, but not both.

A vertex matches one different vertex in M and M'.

The Alternating Path Theorem (2)

Proof: (continued)

- Therefore, the connected components of G' are either even-length cycles or a path with alternating edges.
- Since $|M'| > |M|$, there must be a component of G' that is an alternating path having more M' edges than M edges.

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Bipartite Matching

- A **bipartite graph** $G = (U, V, E)$ consists of two disjoint sets of vertices U and V together with edges E such that every edge has an endpoint in U and an endpoint in V.
- Bipartite matching naturally models a number of assignment problems, such as assignment of workers to jobs.
- Alternating paths will work to find a maximum bipartite matching. An alternating path always has one end in U and the other in V.
- If we direct unmatched edges from U to V and matched edges from V to U, then a directed path from an unmatched vertex in U to an unmatched vertex in V is an alternating path.

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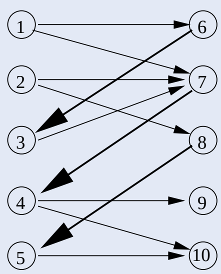
Bipartite Matching Example



2, 8, 5, 10 is an alternating path.

1, 6, 3, 7, 4, 9 and 2, 8, 5, 10 are **disjoint** alternating paths that we can augment **independently**.

Naive algorithm: Find a maximal matching (greedy algorithm).

For each vertex:
  Do a DFS or other search until an alternating path is found.
  Use the alternating path to improve the match.

$|V|(|V| + |E|)$

Algorithm for Maximum Bipartite Matching

Construct BFS subgraph from the set of unmatched vertices in U until a level with unmatched vertices in V is found.

Greedily select a maximal set of disjoint alternating paths.

Augment along each path independently.

Repeat until no alternating paths remain.

Time complexity $O((|V| + |E|)\sqrt{|V|})$.

Order doesn't matter. Find a path, remove its vertices, then repeat.Augment along the paths independently since they are disjoint.

Network Flows

Models distribution of utilities in networks such as oil pipelines, waters systems, etc. Also, highway traffic flow.

Simplest version:

A **network** is a directed graph $G = (V, E)$ having a distinguished source vertex s and a distinguished sink vertex t. Every edge (u, v) of G has a **capacity** $c(u, v) \geq 0$. If $(u, v) \notin E$, then $c(u, v) = 0$.

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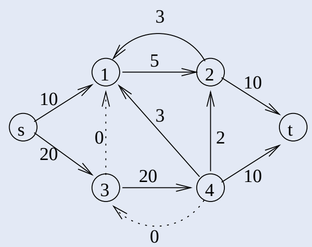
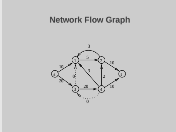
Network Flow Graph





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Network Flow Definitions

A **flow** in a network is a function $f : V \times V \rightarrow R$ with the following properties.

(i) **Skew Symmetry**:

$$\forall v, w \in V, \quad f(v, w) = -f(w, v).$$

(ii) **Capacity Constraint**:

$$\forall v, w, \in V, \quad f(v, w) \leq c(v, w).$$

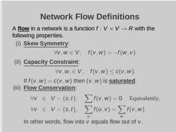
If $f(v, w) = c(v, w)$ then (v, w) is **saturated**.

(iii) **Flow Conservation**:

$$\forall v \in V - \{s, t\}, \quad \sum f(v, w) = 0. \quad \text{Equivalently,}$$

$$\forall v \in V - \{s, t\}, \quad \sum_u f(u, v) = \sum_w f(v, w).$$

In other words, flow into v equals flow out of v.



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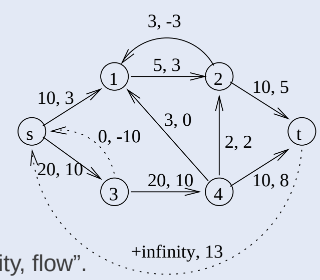
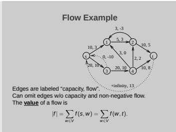
Flow Example



Edges are labeled “capacity, flow”.
Can omit edges w/o capacity and non-negative flow.
The **value** of a flow is

$$|f| = \sum_{w \in V} f(s, w) = \sum_{w \in V} f(w, t).$$



3, -3 is an illustration of “negative flow” returning. Every node can be thought of as having negative flow. We will make use of this later – augmenting paths.

Max Flow Problem

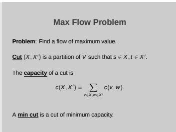
Problem: Find a flow of maximum value.

Cut (X, X') is a partition of V such that $s \in X, t \in X'$.

The **capacity** of a cut is

$$c(X, X') = \sum_{v \in X, w \in X'} c(v, w).$$

A **min cut** is a cut of minimum capacity.



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Cut Flows

For any flow f, the **flow across a cut** is:

$$f(X, X') = \sum_{v \in X, w \in X'} f(v, w).$$

Lemma: For all flows f and all cuts (X, X'), $f(X, X') = |f|$.

- Clearly, the flow out of $s = |f|$ = the flow into t.
- It can be proved that the flow across every other cut is also $|f|$.

Corollary: The value of any flow is less than or equal to the capacity of a min cut.

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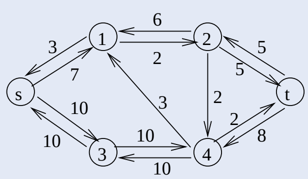
Residual Graph

Given any flow f, the **residual capacity** of the edge is

$$res(v, w) = c(v, w) - f(v, w) \geq 0.$$

Residual graph is a network $R = (V, E_R)$ where E_R contains edges of non-zero residual capacity.



R is the network after f has been subtracted.
Saturated edges do not appear.
Some edges have larger capacity than in G.

Observations

1. Any flow in R can be added to F to obtain a larger flow in G.
2. In fact, a max flow f' in R plus the flow f (written $f + f'$) is a max flow in G.
3. Any path from s to t in R can carry a flow equal to the smallest capacity of any edge on it.
   - Such a path is called an **augmenting path**.
   - For example, the path

     $$s, 1, 2, t$$

     can carry a flow of 2 units $= c(1, 2)$.

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Max-flow Min-cut Theorem

The following are equivalent:

(i) f is a max flow.
(ii) f has no augmenting path in R.
(iii) $|f| = c(X, X')$ for some min cut (X, X').

Proof:
(i) $\Rightarrow$ (ii):

- If f has an augmenting path, then f is not a max flow.

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Max-flow Min-cut Theorem (2)

(ii) $\Rightarrow$ (iii):

- Suppose f has no augmenting path in R.
- Let X be the subset of V reachable from s and $X' = V - X$.
- Then $s \in X, t \in X'$, so (X, X') is a cut.
- $\forall v \in X, w \in X'$, $res(v, w) = c(v, w) - f(v, w) = 0$.
- $f(X, X') = \sum_{v \in X, w \in X'} f(v, w) = \sum_{v \in X, w \in X'} c(v, w) = c(X, X')$.
- By Lemma, $|f| = c(X, X')$ and (X, X') is a min cut.

Line 4: Because no augmenting path.
Line 5: Because we know the residuals are all 0.

In other words, look at the capacity of G at the cut separating s from t in the residual graph. This must be a min cut (for G) with capacity $|f|$.

Max-flow Min-cut Theorem (3)

(iii) $\Rightarrow$ (i)

- Let f be a flow such that $|f| = c(X, X')$ for some (min) cut (X, X').
- By Lemma, all flows f' satisfy $|f'| \leq c(X, X') = |f|$.

Thus, f is a max flow.

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Max-flow Min-cut Corollary

Corollary: The value of a max flow equals the capacity of a min cut.
This suggests a strategy for finding a max flow.

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R = G; f = 0;
repeat
  find a path from s to t in R;
  augment along path to get a larger flow f;
  update R for new flow;
until R has no path s to t.
```

This is the Ford-Fulkerson algorithm.

If capacities are all rational, then it always terminates with f equal to max flow.

Problem with Ford-Fulkerson:
Draw graph with nodes nodes s, t, a, and b. Flow from S to a and b is M, flow from a and b to t is M, flow from a to b is 1.

Now, pick s-a-b-t.
Then s-b-a-t. (reverse 1 unit of flow).
Repeat M times.
M is unrelated to the size of V, E, so this is potentially exponential.

Edmonds-Karp Algorithm

For integral capacities.

Select an augmenting path in R with minimum number of edges.

Performance: $O(|V|^3)$.

There are numerous other approaches to finding augmenting paths, giving a variety of different algorithms.

Network flow remains an active research area.

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