CS 5114: Theory of Algorithms

Clifford A. Shaffer

Department of Computer Science
Virginia Tech
Blacksburg, Virginia

Spring 2010

Title page

CS5114: Theory of Algorithms

- Emphasis: Creation of Algorithms
- Less important:
  - Analysis of algorithms
  - Problem statement
  - Programming
- Central Paradigm: Mathematical Induction
  - Find a way to solve a problem by solving one or more smaller problems

Creation of algorithms comes through exploration, discovery, techniques, intuition: largely by **lots** of examples and **lots** of practice (HW exercises).
We will use Analysis of Algorithms as a tool.
Problem statement (in the software eng. sense) is not important because our problems are easily described, if not easily solved.
Smaller problems may or may not be the same as the original problem.
Divide and conquer is a way of solving a problem by solving one more more smaller problems.
Claim on induction: The processes of constructing proofs and constructing algorithms are similar.

Review of Mathematical Induction

- The paradigm of **Mathematical Induction** can be used to solve an enormous range of problems.
- **Purpose**: To prove a parameterized theorem of the form:
  Theorem: $\forall n \geq c, \mathbf{P}(n)$.
  - Use only positive integers $\geq c$ for n.
- Sample $\mathbf{P}(n)$:
  $n + 1 \leq n^2$

$\mathbf{P}(n)$ is a statement containing n as a variable.

This sample $\mathbf{P}(n)$ is true for $n \geq 2$, but false for $n = 1$.

Principle of Mathematical Induction

- IF the following two statements are true:
  1. $\mathbf{P}(c)$ is true.
  2. For $n > c, \mathbf{P}(n-1)$ is true $\rightarrow \mathbf{P}(n)$ is true.

  ... **THEN** we may conclude: $\forall n \geq c$, $\mathbf{P}(n)$.
- The assumption "$\mathbf{P}(n-1)$ is true" is the **induction hypothesis**.
- Typical induction proof form:
  1. Base case
  2. State induction Hypothesis
  3. Prove the implication (induction step)
- What does this remind you of?

Important: The goal is to prove the **implication**, not the theorem! That is, prove that $\mathbf{P}(n-1) \rightarrow \mathbf{P}(n)$. **NOT** to prove $P(n)$. This is much easier, because we can assume that $\mathbf{P}(n)$ is true.
Consider the truth table for implication to see this. Since $A \rightarrow B$ is (vacuously) true when A is false, we can just assume that A is true since the implication is true anyway A is false. That is, we only need to worry that the implication could be false if A is true.

The power of induction is that the induction hypothesis "comes for free." We often try to make the most of the extra information provided by the induction hypothesis.
This is like recursion! There you have a base case and a recursive call that must make progress toward the base case.

Induction Example 1

Theorem: Let

$$S(n) = \sum_{i=1}^{n} i = 1 + 2 + \cdots + n.$$

Then, $\forall n \geq 1, S(n) = \frac{n(n+1)}{2}$.

Base Case: $\mathbf{P}(n)$ is true since $S(1) = 1 = 1(1+1)/2$.
Induction Hypothesis: $S(i) = \frac{i(i+1)}{2}$ for $i < n$.
Induction Step:

$$\begin{aligned} S(n) &= S(n-1) + n = (n-1)n/2 + n \\ &= \frac{n(n+1)}{2} \end{aligned}$$

Therefore, $\mathbf{P}(n-1) \rightarrow \mathbf{P}(n)$.
By the principle of Mathematical Induction,
$\forall n \geq 1, S(n) = \frac{n(n+1)}{2}$.
MI is often an ideal tool for **verification** of a hypothesis.
Unfortunately it does not help to construct a hypothesis.

Induction Example 2

Theorem: $\forall n \geq 1, \forall$ real x such that $1 + x > 0$,
$(1+x)^n \geq 1 + nx$.

What do we do induction on? Can't be a real number, so must be n.
$\mathbf{P}(n) : (1+x)^n \geq 1 + nx$.

Base Case: $(1+x)^1 = 1 + x \geq 1 + 1x$
Induction Hypothesis: Assume $(1+x)^{n-1} \geq 1 + (n-1)x$
Induction Step:

$$\begin{aligned} (1+x)^n &= (1+x)(1+x)^{n-1} \\ &\geq (1+x)(1+(n-1)x) \\ &= 1 + nx - x + x + nx^2 - x^2 \\ &= 1 + nx + (n-1)x^2 \\ &\geq 1 + nx. \end{aligned}$$

Induction Example 3

Theorem: 2¢ and 5¢ stamps can be used to form any denomination (for denominations ≥ 4).

Base case: $4 = 2 + 2$.

Induction Hypothesis: Assume $\mathbf{P}(k)$ for $4 \leq k < n$.

Induction Step:
Case 1: $n - 1$ is made up of all 2¢ stamps. Then, replace 2 of these with a 5¢ stamp.

Case 2: $n - 1$ includes a 5¢ stamp. Then, replace this with 3 2¢ stamps.

Colorings

4-color problem: For any set of polygons, 4 colors are sufficient to guarentee that no two adjacent polygons share the same color.

Restrict the problem to regions formed by placing (infinite) lines in the plane. How many colors do we need?
Candidates:

- 4: Certainly
- 3: ?
- 2: ?
- 1: No!

Let's try it for 2...

Induction is useful for much more than checking equations!

If we accept the statement about the general 4-color problem, then of course 4 colors is enough for our restricted version.

If 2 is enough, then of course we can do it with 3 or more.

Two-coloring Problem

Given: Regions formed by a collection of (infinite) lines in the plane.
Rule: Two regions that share an edge cannot be the same color.

Theorem: It is possible to two-color the regions formed by n lines.

Picking what to do induction on can be a problem. Lines? Regions? How can we "add a region?" We can't, so try induction on lines.
Base Case: $n = 1$. Any line divides the plane into two regions.
Induction Hypothesis: It is possible to two-color the regions formed by $n-1$ lines.
Induction Step: Introduce the n'th line.
This line cuts some colored regions in two.
Reverse the region colors on one side of the n'th line.
A valid two-coloring results.

- Any boundary surviving the addition still has opposite colors.
- Any new boundary also has opposite colors after the switch.

Strong Induction

IF the following two statements are true:

1. $\mathbf{P}(c)$
2. $\mathbf{P}(i), i = 1, 2, \cdots, n-1 \rightarrow \mathbf{P}(n)$,

... **THEN** we may conclude: $\forall n \geq c$, $\mathbf{P}(n)$.

Advantage: We can use statements other than $\mathbf{P}(n-1)$ in proving $\mathbf{P}(n)$.

The previous examples were all very straightforward – simply add in the n'th item and justify that the IH is maintained.
Now we will see examples where we must do more sophisticated (creative!) maneuvers such as

- go backwards from n.
- prove a stronger IH.

to make the most of the IH.

Graph Problem

An **Independent Set** of vertices is one for which no two vertices are adjacent.

Theorem: Let $G = (V, E)$ be a **directed** graph. Then, G contains some independent set $S(G)$ such that every vertex can be reached from a vertex in $S(G)$ by a path of length at most 2.

Example: a graph with 3 vertices in a cycle. Pick any one vertex as $S(G)$.

It should be obvious that the theorem is true for an undirected graph.
Naive approach: Assume the theorem is true for any graph of $n-1$ vertices. Now add the nth vertex and its edges. But this won't work for the graph $1 \leftarrow 2$. Initially, vertex 1 is the independent set. We can't add 2 to the graph. Nor can we reach it from 1.
Going forward is good for proving existance.
Going backward (from an arbitrary instance into the IH) is usually necessary to prove that a property holds in all instances. This is because going forward requires proving that you reach all of the possible instances.

Graph Problem (cont)

Theorem: Let $G = (V, E)$ be a **directed** graph. Then, G contains some independent set $S(G)$ such that every vertex can be reached from a vertex in $S(G)$ by a path of length at most 2.
Base Case: Easy if $n \leq 3$ because there can be no path of length > 2.
Induction Hypothesis: The theorem is true if $|V| < n$.
Induction Step ($n > 3$):
Pick any $v \in V$.
Define: $N(v) = \{v\} \cup \{w \in V | (v, w) \in E\}$.
$H = G - N(v)$.
Since the number of vertices in H is less than n, there is an independent set $S(H)$ that satisfies the theorem for H.

$N(v)$ is all vertices reachable (directly) from v. That is, the Neighbors of v.
H is the graph induced by $V - N(v)$.

OK, so why remove both v and $N(v)$ from the graph? If we only remove v, we have the same problem as before. If G is $1 \rightarrow 2 \rightarrow 3$, and we remove 1, then the independent set for H must be vertex 2. We can't just add back 1. But if we remove both 1 and 2, then we'll be able to do something...

Graph Proof (cont)

There are two cases:

1. $S(H) \cup \{v\}$ is independent.
   Then $S(G) = S(H) \cup \{v\}$.
2. $S(H) \cup \{v\}$ is not independent.
   Let $w \in S(H)$ such that $(w, v) \in E$.
   Every vertex in $N(v)$ can be reached by w with path of length ≤ 2.
   So, set $S(G) = S(H)$.

By Strong Induction, the theorem holds for all G.

"$S(H) \cup \{v\}$ is not independent" means that there is an edge from something in $S(H)$ to v.
IMPORTANT: There cannot be an edge from v to $S(H)$ because whatever we can reach from v is in $N(v)$ and would have been removed in H.
We need strong induction for this proof because we don't know how many vertices are in $N(v)$.

Fibonacci Numbers

Define Fibonacci numbers inductively as:

$$\begin{aligned} F(1) &= F(2) = 1 \\ F(n) &= F(n-1) + F(n-2), n > 2. \end{aligned}$$

Theorem: $\forall n \geq 1, F(n)^2 + F(n+1)^2 = F(2n+1)$.

Induction Hypothesis:
$F(n-1)^2 + F(n)^2 = F(2n-1)$.

Expand both sides of the theorem, then cancel like terms:
$F(2n+1) = F(2n) + F(2n-1)$ and,

$$\begin{aligned} F(n)^2 + F(n+1)^2 &= F(n)^2 + (F(n) + F(n-1))^2 \\ &= F(n)^2 + F(n)^2 + 2F(n)F(n-1) + F(n-1)^2 \\ &= F(n)^2 + F(n-1)^2 + F(n)^2 + 2F(n)F(n-1) \\ &= F(2n-1) + F(n)^2 + 2F(n)F(n-1). \end{aligned}$$

Want: $F(n)^2 + F(n+1)^2 = F(2n+1) = F(2n) + F(2n-1)$
Steps above gave:
$F(2n) + F(2n-1) = F(2n-1) + F(n)^2 + 2F(n)F(n-1)$
So we need to show that: $F(n)^2 + 2F(n)F(n-1) = F(2n)$
To prove the original theorem, we must prove this. Since we must do it anyway, we should take advantage of this in our IH!

Fibonacci Numbers (cont)

With a stronger theorem comes a stronger IH!

Theorem:
$F(n)^2 + F(n+1)^2 = F(2n+1)$ and
$F(n)^2 + 2F(n)F(n-1) = F(2n)$.

Induction Hypothesis:
$F(n-1)^2 + F(n)^2 = F(2n-1)$ and
$F(n-1)^2 + 2F(n-1)F(n-2) = F(2n-2)$.

$F(n)^2 + 2F(n)F(n-1)$

$$\begin{aligned} &= F(n)^2 + 2(F(n-1) + F(n-2))F(n-1) \\ &= F(n)^2 + F(n-1)^2 + 2F(n-1)F(n-2) + F(n-1)^2 \\ &= F(2n-1) + F(2n-2) \\ &= F(2n). \end{aligned}$$

$$\begin{aligned} F(n)^2 + F(n+1)^2 &= F(n)^2 + [F(n) + F(n-1)]^2 \\ &= F(n)^2 + F(n)^2 + 2F(n)F(n-1) + F(n-1)^2 \\ &= F(n)^2 + F(2n) + F(n-1)^2 \\ &= F(2n-1) + F(2n) \\ &= F(2n+1). \end{aligned}$$

... which proves the theorem. The original result could not have been proved without the stronger induction hypothesis.

Another Example

Theorem: All horses are the same color.

Proof: $\mathbf{P}(n)$: If S is a set of n horses, then all horses in S have the same color.
Base case: $n = 1$ is easy.
Induction Hypothesis: Assume $\mathbf{P}(i), i < n$.
Induction Step:

- Let S be a set of horses, $|S| = n$.
- Let S' be $S - \{h\}$ for some horse h.
- By IH, all horses in S' have the same color.
- Let h' be some horse in S'.
- IH implies $\{h, h'\}$ have all the same color.

Therefore, $\mathbf{P}(n)$ holds.

The problem is that the base case does not give enough strength to give the **particular** instance of $n = 2$ used in the last step.

Algorithm Analysis

- We want to “measure” algorithms.
- What do we measure?
- What factors affect measurement?
- Objective: Measures that are independent of all factors except input.

What do we measure?
Time and space to run; ease of implementation (this changes with language and tools); code size

What affects measurement?
Computer speed and architecture; Programming language and compiler; System load; Programmer skill; Specifics of input (size, arrangement)

If you compare two programs running on the same computer under the same conditions, all the other factors (should) cancel out.
Want to measure the relative efficiency of two algorithms without needing to implement them on a real computer.

Time Complexity

- Time and space are the most important computer resources.
- Function of input: **T**(input)
- Growth of time with size of input:
  - Establish an (integer) **size** n for inputs
  - n numbers in a list
  - n edges in a graph
- Consider time for all inputs of size n:
  - Time varies widely with specific input
  - Best case
  - Average case
  - Worst case
- Time complexity $\mathbf{T}(n)$ counts **steps** in an algorithm.

Sometimes analyze in terms of more than one variable.
Best case usually not of interest.
Average case is usually what we want, but can be hard to measure.
Worst case appropriate for “real-time” applications, often best we can do in terms of measurement.
Examples of “steps:” comparisons, assignments, arithmetic/logical operations. What we choose for “step” depends on the algorithm. Step cost must be “constant” – not dependent on n.

Asymptotic Analysis

- It is undesirable/impossible to count the exact number of steps in most algorithms.
  - Instead, concentrate on main characteristics.
- Solution: Asymptotic analysis
  - Ignore small cases:
    - Consider behavior approaching infinity
  - Ignore constant factors, low order terms:
    - $2n^2$ looks the same as $5n^2 + n$ to us.

Undesirable to count number of machine instructions or steps because issues like processor speed muddy the waters.

O Notation

O notation is a measure for “upper bound” of a growth rate.

- pronounced “Big-oh”

Definition: For $\mathbf{T}(n)$ a non-negatively valued function, $\mathbf{T}(n)$ is in the set $\mathrm{O}(f(n))$ if there exist two positive constants c and n_0 such that $\mathbf{T}(n) \leq cf(n)$ for all $n > n_0$.

Examples:

- $5n + 8 \in \mathrm{O}(n)$
- $2n^2 + n \log n \in \mathrm{O}(n^2) \in \mathrm{O}(n^3 + 5n^2)$
- $2n^2 + n \log n \in \mathrm{O}(n^2) \in \mathrm{O}(n^3 + n^2)$

Remember: The time equation is for some particular set of inputs – best, worst, or average case.

O Notation (cont)

We seek the "simplest" and "strongest" f.

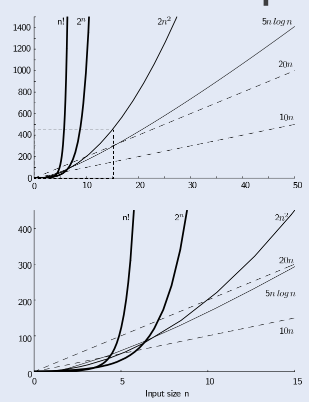
Big-O is somewhat like "$\leq$":
$n^2 \in \mathrm{O}(n^3)$ and $n^2 \log n \in O(n^3)$, but

- $n^2 \neq n^2 \log n$
- $n^2 \in \mathrm{O}(n^2)$ while $n^2 \log n \notin \mathrm{O}(n^2)$

A common misunderstanding:

- "The best case for my algorithm is $n = 1$ because that is the fastest." WRONG!
- Big-oh refers to a growth rate as n grows to ∞.
- Best case is defined for the input of size n that is cheapest among all inputs of size n.

Growth Rate Graph



2^n is an exponential algorithm. $10n$ and $20n$ differ only by a constant.

Speedups

What happens when we buy a computer 10 times faster?

$\mathbf{T}(n)$	n	n'	Change	n'/n
$10n$	$1,000$	$10,000$	$n' = 10n$	10
$20n$	500	$5,000$	$n' = 10n$	10
$5n \log n$	250	$1,842$	$\sqrt{10}n < n' < 10n$	7.37
$2n^2$	70	223	$n' = \sqrt{10}n$	3.16
2^n	13	16	$n' = n + 3$	$--$

n: Size of input that can be processed in one hour (10,000 steps).

n': Size of input that can be processed in one hour on the new machine (100,000 steps).

How much speedup? 10 times. More important: How much increase in problem size for same time? Depends on growth rate.
For n^2, if $n = 1000$, then n' would be 1003.
Compare $\mathbf{T}(n) = n^2$ to $\mathbf{T}(n) = n \log n$. For $n > 58$, it is faster to have the $\Theta(n \log n)$ algorithm than to have a computer that is 10 times faster.

Some Rules for Use

Definition: f is **monotonically growing** if $n_1 \geq n_2$ implies $f(n_1) \geq f(n_2)$.
We typically assume our time complexity function is monotonically growing.

Theorem 3.1: Suppose f is monotonically growing.
$\forall c > 0$ and $\forall a > 1, (f(n))^c \in O(a^{f(n)})$
In other words, an **exponential** function grows faster than a **polynomial** function.

Lemma 3.2: If $f(n) \in O(s(n))$ and $g(n) \in O(r(n))$ then

- $f(n) + g(n) \in O(s(n) + r(n)) \equiv O(\max(s(n), r(n)))$
- $f(n)g(n) \in O(s(n)r(n))$.
- If $s(n) \in O(h(n))$ then $f(n) \in O(h(n))$
- For any constant k, $f(n) \in O(ks(n))$

Assume monitonic growth because larger problems should take longer to solve. However, many real problems have "cyclically growing" behavior.
Is $O(2^{f(n)}) \in O(3^{f(n)})$? Yes, but not vice versa.
$3^n = 1.5^n \times 2^n$ so no constant could ever make 2^n bigger than 3^n for all n.functional composition

Other Asymptotic Notation

$\Omega(f(n))$ – lower bound ($\geq$)
Definition: For $\mathbf{T}(n)$ a non-negatively valued function, $\mathbf{T}(n)$ is in the set $\Omega(g(n))$ if there exist two positive constants c and n_0 such that $\mathbf{T}(n) \geq cg(n)$ for all $n > n_0$.
Ex: $n^2 \log n \in \Omega(n^2)$.

$\Theta(f(n))$ – Exact bound ($=$)
Definition: $g(n) = \Theta(f(n))$ if $g(n) \in O(f(n))$ and $g(n) \in \Omega(f(n))$.
Important!: It is Θ if it is both in big-Oh and in Ω.
Ex: $5n^3 + 4n^2 + 9n + 7 = \Theta(n^3)$

Ω is most userful to discuss cost of problems, not algorithms. Once you have an equation, the bounds have met. So this is more interesting when discussing your level of uncertainty about the difference between the upper and lower bound.

You have Θ when you have the upper and the lower bounds meeting. So Θ means that you know a lot more than just Big-oh, and so is perferred when possible.

A common misunderstanding:

- Confusing worst case with upper bound.
- Upper bound refers to a growth rate.
- Worst case refers to the worst input from among the choices for possible inputs of a given size.

Other Asymptotic Notation (cont)

$o(f(n))$ – little o ($<$)
Definition: $g(n) \in o(f(n))$ if $\lim_{n\to\infty} \frac{g(n)}{f(n)} = 0$
Ex: $n^2 \in o(n^3)$

$\omega(f(n))$ – little omega ($>$)
Definition: $g(n) \in w(f(n))$ if $f(n) \in o(g(n))$.
Ex: $n^5 \in w(n^2)$

$\infty(f(n))$
Definition: $T(n) = \infty(f(n))$ if $T(n) = O(f(n))$ but the constant in the O is so large that the algorithm is impractical.

We won't use these too much.

Aim of Algorithm Analysis

Typically want to find "simple" $f(n)$ such that $T(n) = \Theta(f(n))$.

- Sometimes we settle for $O(f(n))$.

Usually we measure T as "worst case" time complexity.
Sometimes we measure "average case" time complexity.
Approach: Estimate number of "steps"

- Appropriate step depends on the problem.
- Ex: measure key comparisons for sorting

Summation: Since we typically count steps in different parts of an algorithm and sum the counts, techniques for computing sums are important (loops).

Recurrence Relations: Used for counting steps in recursion.

We prefer Θ over Big-oh because Θ means that we understand our bounds and they met. But if we just can't find that the bottom meets the top, then we are stuck with just Big-oh. Lower bounds can be hard. For **problems** we are often interested in Ω

– but this is often hard for non-trivial situations!

Often prefer average case (except for real-time programming), but worst case is simpler to compute than average case since we need not be concerned with distribution of input.

For the sorting example, key comparisons must be constant-time to be used as a cost measure.

Summation: Guess and Test

Technique 1: Guess the solution and use induction to test.

Technique 1a: Guess the form of the solution, and use simultaneous equations to generate constants. Finally, use induction to test.

no notes

Summation Example

$$S(n) = \sum_{i=0}^{n} i^2.$$

Guess that $S(n)$ is a polynomial $\leq n^3$.
Equivalently, guess that it has the form $S(n) = an^3 + bn^2 + cn + d$.

For $n = 0$ we have $S(n) = 0$ so $d = 0$.
For $n = 1$ we have $a + b + c + 0 = 1$.
For $n = 2$ we have $8a + 4b + 2c = 5$.
For $n = 3$ we have $27a + 9b + 3c = 14$.
Solving these equations yields $a = \frac{1}{3}, b = \frac{1}{2}, c = \frac{1}{6}$

Now, prove the solution with induction.

This is Manber Problem 2.5.

We need to prove by induction since we don't know that the guessed form is correct. All that we **know** without doing the proof is that the form we guessed models some low-order points on the equation properly.

Technique 2: Shifted Sums

Given a sum of many terms, shift and subtract to eliminate intermediate terms.

$$G(n) = \sum_{i=0}^{n} ar^i = a + ar + ar^2 + \cdots + ar^n$$

Shift by multiplying by r.

$$rG(n) = ar + ar^2 + \cdots + ar^n + ar^{n+1}$$

Subtract.

$$G(n) - rG(n) = G(n)(1 - r) = a - ar^{n+1}$$
$$G(n) = \frac{a - ar^{n+1}}{1 - r} \quad r \neq 1$$

We often solve summations in this way – by multiplying by something or subtracting something. The big problem is that it can be a bit like finding a needle in a haystack to decide what "move" to make. We need to do something that gives us a new sum that allows us either to cancel all but a constant number of terms, or else converts all the terms into something that forms an easier summation.

Shift by multiplying by r is a reasonable guess in this example since the terms differ by a factor of r.

Example 3.3

$$G(n) = \sum_{i=1}^{n} i2^i = 1 \times 2 + 2 \times 2^2 + 3 \times 2^3 + \cdots + n \times 2^n$$

Multiply by 2.

$$2G(n) = 1 \times 2^2 + 2 \times 2^3 + 3 \times 2^4 + \cdots + n \times 2^{n+1}$$

Subtract (Note: $\sum_{i=1}^{n} 2^i = 2^{n+1} - 2$)

$$\begin{aligned} 2G(n) - G(n) &= n2^{n+1} - 2^n \cdots 2^2 - 2 \\ G(n) &= n2^{n+1} - 2^{n+1} + 2 \\ &= (n-1)2^{n+1} + 2 \end{aligned}$$

no notes

Recurrence Relations

- A (math) function defined in terms of itself.
- Example: Fibonacci numbers:
  $F(n) = F(n-1) + F(n-2)$ general case
  $F(1) = F(2) = 1$ base cases
- There are always one or more general cases and one or more base cases.
- We will use recurrences for time complexity of recursive (computer) functions.
- General format is $T(n) = E(T, n)$ where $E(T, n)$ is an expression in T and n.
  - $T(n) = 2T(n/2) + n$
- Alternately, an upper bound: $T(n) \leq E(T, n)$.

We won't spend a lot of time on techniques... just enough to be able to use them.

Solving Recurrences

We would like to find a closed form solution for $T(n)$ such that:

$$T(n) = \Theta(f(n))$$

Alternatively, find lower bound

- Not possible for inequalities of form $T(n) \le E(T, n)$.

Methods:

- Guess (and test) a solution
- Expand recurrence
- Theorems

Note that "finding a closed form" means that we have $f(n)$ that doesn't include T.

Can't find lower bound for the inequality because you do not know enough... you don't know *how much bigger* $E(T, n)$ is than $T(n)$, so the result might not be $\Omega(T(n))$.

Guessing is useful for finding an asymptotic solution. Use induction to prove the guess correct.

Guessing

$T(n) = 2T(n/2) + 5n^2 \quad n \ge 2$
$T(1) = 7$
Note that T is defined only for powers of 2.

Guess a solution: $T(n) \le c_1 n^3 = f(n)$
$T(1) = 7$ implies that $c_1 \ge 7$

Inductively, assume $T(n/2) \le f(n/2)$.

$$\begin{aligned} T(n) &= 2T(n/2) + 5n^2 \\ &\le 2c_1(n/2)^3 + 5n^2 \\ &\le c_1(n^3/4) + 5n^2 \\ &\le c_1 n^3 \text{ if } c_1 \ge 20/3. \end{aligned}$$

For Big-oh, not many choices in what to guess.

$7 \times 1^3 = 7$

Because $\frac{20}{4 \cdot 3} n^3 + 5n^2 = \frac{20}{3} n^3$ when $n = 1$, and as n grows, the right side grows even faster.

Guessing (cont)

Therefore, if $c_1 = 7$, a proof by induction yields:
$T(n) \le 7n^3$
$T(n) \in O(n^3)$

Is this the best possible solution?

No - try something tighter.

Guessing (cont)

Guess again.

$$T(n) \le c_2 n^2 = g(n)$$

$T(1) = 7$ implies $c_2 \ge 7$.

Inductively, assume $T(n/2) \le g(n/2)$.

$$\begin{aligned} T(n) &= 2T(n/2) + 5n^2 \\ &\le 2c_2(n/2)^2 + 5n^2 \\ &= c_2(n^2/2) + 5n^2 \\ &\le c_2 n^2 \text{ if } c_2 \ge 10 \end{aligned}$$

Therefore, if $c_2 = 10$, $\quad T(n) \le 10n^2$. $\quad T(n) = O(n^2)$.
Is this the best possible upper bound?

Because $\frac{10}{2} n^2 + 5n^2 = 10n^2$ for $n = 1$, and the right hand side grows faster.

Yes this is best, since $T(n)$ can be as bad as $5n^2$.

Guessing (cont)

Now, reshape the recurrence so that T is defined for all values of n.
$T(n) \le 2T(\lfloor n/2 \rfloor) + 5n^2 \qquad n \ge 2$

For arbitrary n, let $2^{k-1} < n \le 2^k$.

We have already shown that $T(2^k) \le 10(2^k)^2$.

$$\begin{aligned} T(n) &\le T(2^k) \le 10(2^k)^2 \\ &= 10(2^k/n)^2 n^2 \le 10(2)^2 n^2 \\ &\le 40n^2 \end{aligned}$$

Hence, $T(n) = O(n^2)$ for all values of n.

Typically, the bound for powers of two generalizes to all n.

no notes

Expanding Recurrences

Usually, start with equality version of recurrence.

$$\begin{aligned} T(n) &= 2T(n/2) + 5n^2 \\ T(1) &= 7 \end{aligned}$$

Assume n is a power of 2; $n = 2^k$.

no notes

Expanding Recurrences (cont)

$$\begin{aligned} T(n) &= 2T(n/2) + 5n^2 \\ &= 2(2T(n/4) + 5(n/2)^2) + 5n^2 \\ &= 2(2(2T(n/8) + 5(n/4)^2) + 5(n/2)^2) + 5n^2 \\ &= 2^k T(1) + 2^{k-1} \cdot 5(n/2^{k-1})^2 + 2^{k-2} \cdot 5(n/2^{k-2})^2 \\ &\qquad + \cdots + 2 \cdot 5(n/2)^2 + 5n^2 \\ &= 7n + 5\sum_{i=0}^{k-1} n^2/2^i = 7n + 5n^2 \sum_{i=0}^{k-1} 1/2^i \\ &= 7n + 5n^2(2 - 1/2^{k-1}) \\ &= 7n + 5n^2(2 - 2/n). \end{aligned}$$

This it the **exact** solution for powers of 2. $T(n) = \Theta(n^2)$.

no notes

Divide and Conquer Recurrences

These have the form:

$$\begin{aligned} T(n) &= aT(n/b) + cn^k \\ T(1) &= c \end{aligned}$$

... where a, b, c, k are constants.

A problem of size n is divided into a subproblems of size n/b, while cn^k is the amount of work needed to combine the solutions.

no notes

Divide and Conquer Recurrences (cont)

Expand the sum; $n = b^m$.

$$\begin{aligned} T(n) &= a(aT(n/b^2) + c(n/b)^k) + cn^k \\ &= a^m T(1) + a^{m-1}c(n/b^{m-1})^k + \cdots + ac(n/b)^k + cn^k \\ &= ca^m \sum_{i=0}^{m} (b^k/a)^i \end{aligned}$$

$a^m = a^{\log_b n} = n^{\log_b a}$
The summation is a geometric series whose sum depends on the ratio

$$r = b^k/a.$$

There are 3 cases.

$n = b^m \Rightarrow m = log_b n$.

Set $a = b^{\log_b a}$. Switch order of logs, giving $(b^{\log_b n})^{\log_b a} = n^{\log_b a}$.

D & C Recurrences (cont)

(1) $r < 1$.

$$\sum_{i=0}^{m} r^i < 1/(1-r), \qquad \text{a constant.}$$

$$T(n) = \Theta(a^m) = \Theta(n^{\log_b a}).$$

(2) $r = 1$.

$$\sum_{i=0}^{m} r^i = m + 1 = \log_b n + 1$$

$$T(n) = \Theta(n^{\log_b a} \log n) = \Theta(n^k \log n)$$

When $r = 1$, since $r = b^k/a = 1$, we get $a = b^k$.
Recall that $k = log_b a$.

D & C Recurrences (Case 3)

(3) $r > 1$.

$$\sum_{i=0}^{m} r^i = \frac{r^{m+1} - 1}{r - 1} = \Theta(r^m)$$

So, from $T(n) = ca^m \sum r^i$,

$$\begin{aligned} T(n) &= \Theta(a^m r^m) \\ &= \Theta(a^m (b^k/a)^m) \\ &= \Theta(b^{km}) \\ &= \Theta(n^k) \end{aligned}$$

no notes

Summary

Theorem 3.4:

$$T(n) = \begin{cases} \Theta(n^{\log_b a}) & \text{if } a > b^k \\ \Theta(n^k \log n) & \text{if } a = b^k \\ \Theta(n^k) & \text{if } a < b^k \end{cases}$$

Apply the theorem:
$T(n) = 3T(n/5) + 8n^2$.
$a = 3, b = 5, c = 8, k = 2$.
$b^k/a = 25/3$.

Case (3) holds: $T(n) = \Theta(n^2)$.

We simplify by approximating summations.

Examples

- Mergesort: $T(n) = 2T(n/2) + n$.
  $2^1/2 = 1$, so $T(n) = \Theta(n \log n)$.
- Binary search: $T(n) = T(n/2) + 2$.
  $2^0/1 = 1$, so $T(n) = \Theta(\log n)$.
- Insertion sort: $T(n) = T(n-1) + n$.
  Can't apply the theorem. Sorry!
- Standard Matrix Multiply (recursively):
  $T(n) = 8T(n/2) + n^2$.
  $2^2/8 = 1/2$ so $T(n) = \Theta(n^{\log_2 8}) = \Theta(n^3)$.

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

In the straightforward implementation, 2×2 case is:

$$\begin{aligned} c_{11} &= a_{11}b_{11} + a_{12}b_{21} \\ c_{12} &= a_{11}b_{12} + a_{12}b_{22} \\ c_{21} &= a_{21}b_{11} + a_{22}b_{21} \\ c_{22} &= a_{21}b_{12} + a_{22}b_{22} \end{aligned}$$

So the recursion is 8 calls of half size, and the additions take $\Theta(n^2)$ work.

Useful log Notation

- If you want to take the log of $(\log n)$, it is written $\log \log n$.
- $(\log n)^2$ can be written $\log^2 n$.
- Don't get these confused!
- $\log^* n$ means "the number of times that the log of n must be taken before $n \leq 1$.
  - For example, $65536 = 2^{16}$ so $\log^* 65536 = 4$ since $\log 65536 = 16$, $\log 16 = 4$, $\log 4 = 2$, $\log 2 = 1$.

no notes

Amortized Analysis

Consider this variation on STACK:

```
void init(STACK S);
element examineTop(STACK S);
void push(element x, STACK S);
void pop(int k, STACK S);
```

... where pop removes k entries from the stack.

"Local" worst case analysis for pop:
$O(n)$ for n elements on the stack.

Given m_1 calls to push, m_2 calls to pop:
Naive worst case: $m_1 + m_2 \cdot n = m_1 + m_2 \cdot m_1$.

no notes

Alternate Analysis

Use amortized analysis on multiple calls to push, pop:

Cannot pop more elements than get pushed onto the stack.

After many pushes, a single pop has high **potential**.

Once that potential has been expended, it is not available for future pop operations.

The cost for m_1 pushes and m_2 pops:

$$m_1 + (m_2 + m_1) = O(m_1 + m_2)$$

Actual number of (constant time) push calls + (Actual number of pop calls + Total potential for the pops)

CLR has an entire chapter on this – we won't go into this much, but we use Amortized Analysis implicitly sometimes.

Creative Design of Algorithms by Induction

Analogy: Induction $\leftrightarrow$ Algorithms

Begin with a problem:

- "Find a solution to problem Q."

Think of Q as a set containing an infinite number of **problem instances**.

Example: Sorting

- Q contains all finite sequences of integers.

Now that we have completed the tool review, we will do two things:

1. Survey algorithms in application areas
2. Try to understand how to create efficient algorithms

This chapter is about the second. The remaining chapters do the second in the context of the first.

I $\leftarrow$ A is reasonably obvious – we often use induction to prove that an algorithm is correct. The intellectual claim of Manber is that I $\rightarrow$ A gives insight into problem solving.

Solving Q

First step:

- Parameterize problem by size: $Q(n)$

Example: Sorting

- $Q(n)$ contains all sequences of n integers.

Q is now an infinite sequence of problems:

- $Q(1), Q(2), ..., Q(n)$

Algorithm: Solve for an instance in $Q(n)$ by solving instances in $Q(i), i < n$ and combining as necessary.

This is a "meta" algorithm – An algorithm for finding algorithms!

Induction

Goal: Prove that we can solve for an instance in $Q(n)$ by assuming we can solve instances in $Q(i), i < n$.

Don't forget the base cases!

Theorem: $\forall n \geq 1$, we can solve instances in $Q(n)$.

- This theorem embodies the **correctness** of the algorithm.

Since an induction proof is mechanistic, this should lead directly to an algorithm (recursive or iterative).

Just one (new) catch:

- Different inductive proofs are possible.
- We want the most **efficient** algorithm!

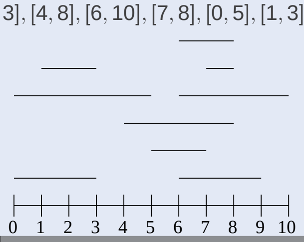
The goal is using Strong Induction.
Correctness is proved by induction.
Example: Sorting

- Sort $n-1$ items, add nth item (insertion sort)
- Sort 2 sets of $n/2$, merge together (mergesort)
- Sort values $< x$ and $> x$ (quicksort)

Interval Containment

Start with a list of non-empty intervals with integer endpoints.

Example:
$[6, 9], [5, 7], [0, 3], [4, 8], [6, 10], [7, 8], [0, 5], [1, 3], [6, 8]$



no notes

Interval Containment (cont)

Problem: Identify and mark all intervals that are contained in some other interval.

Example:

- Mark $[6, 9]$ since $[6, 9] \subseteq [6, 10]$

$[5, 7] \subseteq [4, 8]$
$[0, 3] \subseteq [0, 5]$
$[7, 8] \subseteq [6, 10]$
$[1, 3] \subseteq [0, 5]$
$[6, 8] \subseteq [6, 10]$
$[6, 9] \subseteq [6, 10]$

Interval Containment (cont)

- $Q(n)$: Instances of n intervals
- **Base case**: $Q(1)$ is easy.
- **Inductive Hypothesis**: For $n > 1$, we know how to solve an instance in $Q(n-1)$.
- **Induction step**: Solve for $Q(n)$.
  - Solve for first $n-1$ intervals, applying inductive hypothesis.
  - Check the nth interval against intervals $i = 1, 2, \cdots$
  - If interval i contains interval n, mark interval n. (stop)
  - If interval n contains interval i, mark interval i.
- **Analysis**:
  $T(n) = T(n-1) + cn$
  $T(n) = \Theta(n^2)$

Base case: Nothing is contained

"Creative" Algorithm

Idea: Choose a special interval as the nth interval.

Choose the nth interval to have rightmost left endpoint, and if there are ties, leftmost right endpoint.
(1) No need to check whether nth interval contains other intervals.

(2) nth interval should be marked iff the rightmost endpoint of the first $n-1$ intervals exceeds or equals the right endpoint of the nth interval.

Solution: Sort as above.

In the example, the nth interval is $[7, 8]$.
Every other interval has left endpoint to left, or right endpoint to right.
We must keep track of the current right-most endpont.

"Creative" Solution Induction

Induction Hypothesis: Can solve for $Q(n-1)$ AND interval n is the "rightmost" interval AND we know R (the rightmost endpoint encountered so far) for the first $n-1$ segments.

Induction Step: (to solve $Q(n)$)

- Solve for first $n-1$ intervals recursively, and remember R.
- If the rightmost endpoint of nth interval is $\leq$ R, then mark the nth interval.
- Else R $\leftarrow$ right endpoint of nth interval.

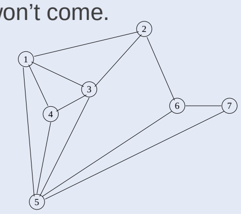
Analysis: $\Theta(n \log n) + \Theta(n)$.

Lesson: Preprocessing, often sorting, can help sometimes.

We strengthened the induction hypothesis. In algorithms, this does cost something.
We must sort.
Analysis: Time for sort + constant time per interval.

Maximal Induced Subgraph

Problem: Given a graph $G = (V, E)$ and an integer k, find a maximal induced subgraph $H = (U, F)$ such that all vertices in H have degree $\geq k$.
Example: Scientists interacting at a conference. Each one will come only if k colleagues come, and they know in advance if somebody won't come.
Example: For $k = 3$.



Solution:

Induced subgraph: U is a subset of V, F is a subset of E such that both ends of $e \in E$ are members of U.
Solution is: $U = \{1, 3, 4, 5\}$

Max Induced Subgraph Solution

$Q(s, k)$: Instances where $|V| = s$ and k is a fixed integer.

Theorem: $\forall s, k > 0$, we can solve an instance in $Q(s, k)$.

Analysis: Should be able to implement algorithm in time $\Theta(|V| + |E|)$.

Base Case: $s = 1$ H is the empty graph.
Induction Hypothesis: Assume $s > 1$. we can solve instances of $Q(s - 1, k)$.
Induction Step: Show that we can solve an instance of $G(V, E)$ in $Q(s, k)$. Two cases:

(1) Every vertex in G has degree $\geq k$. $H = G$ is the only solution.
(2) Otherwise, let $v \in V$ have degree $< k$. $G - v$ is an instance of $Q(s - 1, k)$ which we know how to solve.

By induction, the theorem follows.
Visit all edges to generate degree counts for the vertices. Any vertex with degree below k goes on a queue. Pull the vertices off the queue one by one, and reduce the degree of their neighbors. Add the neighbor to the queue if it drops below k.

Celebrity Problem

In a group of n people, a **celebrity** is somebody whom everybody knows, but who knows no one else.

Problem: If we can ask questions of the form "does person i know person j?" how many questions do we need to find a celebrity, if one exists?

How should we structure the information?

no notes

Celebrity Problem (cont)

Formulate as an $n \times n$ boolean matrix M.
$M_{ij} = 1$ iff i knows j.

Example: $\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

A celebrity has all 0's in his row and all 1's in his column.

There can be at most one celebrity.

Clearly, $O(n^2)$ questions suffice. Can we do better?

The celebrity in this example is 4.

Efficient Celebrity Algorithm

Appeal to induction:

- If we have an $n \times n$ matrix, how can we reduce it to an $(n-1) \times (n-1)$ matrix?

What are ways to select the n'th person?

This induction implies that we go backwards. Natural thing to try: pick arbitrary n'th person.
Assume that we can solve for $n-1$. What happens when we add nth person?

- Celebrity candidate in $n-1$ – just ask two questions.
- Celebrity is n – must check $2(n-1)$ positions. $O(n^2)$.
- No celebrity. Again, $O(n^2)$.

So we will have to look for something special. Who can we eliminate? There are only two choices: A celebrity or a non-celebrity. It doesn't make sense to eliminate a celebrity. Is there an easy way to guarentee that we eliminate a non-celeberity?

Efficient Celebrity Algorithm (cont)

Eliminate one person if he is a non-celebrity.

- Strike one row and one column.

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Does 1 know 3? No. 3 is a non-celebrity.
Does 2 know 5? Yes. 2 is a non-celebrity.
Observation: Each question eliminates one non-celebrity.

no notes

Celebrity Algorithm

Algorithm:

1. Ask $n-1$ questions to eliminate $n-1$ non-celebrities. This leaves one candidate who might be a celebrity.
2. Ask $2(n-1)$ questions to check candidate.

Analysis:

- $\Theta(n)$ questions are asked.

Example:

- Does 1 know 2? No. Eliminate 2
- Does 1 know 3? No. Eliminate 3
- Does 1 know 4? Yes. Eliminate 1
- Does 4 know 5? No. Eliminate 5

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

4 remains as candidate.

no notes

Maximum Consecutive Subsequence

Given a sequence of integers, find a contiguous subsequence whose sum is maximum.

The sum of an empty subsequence is 0.

- It follows that the maximum subsequence of a sequence of all negative numbers is the empty subsequence.

Example:
2, 11, -9, 3, 4, -6, -7, 7, -3, 5, 6, -2

Maximum subsequence:
7, -3, 5, 6 Sum: 15

no notes

Finding an Algorithm

Induction Hypothesis: We can find the maximum subsequence sum for a sequence of $< n$ numbers.

Note: We have changed the problem.

- First, figure out how to compute the sum.
- Then, figure out how to get the subsequence that computes that sum.

no notes

Finding an Algorithm (cont)

Induction Hypothesis: We can find the maximum subsequence sum for a sequence of $< n$ numbers.
Let $S = x_1, x_2, \cdots, x_n$ be the sequence.
Base case: $n = 1$
Either $x_1 < 0 \Rightarrow \mathrm{sum} = 0$
Or sum $= x_1$.
Induction Step:

- We know the maximum subsequence SUM(n-1) for $x_1, x_2, \cdots, x_{n-1}$.
- Where does x_n fit in?
  - Either it is not in the maximum subsequence or it ends the maximum subsequence.
- If x_n ends the maximum subsequence, it is appended to trailing maximum subsequence of $x_1, \cdots, x_{n-1}$.

That is, of the numbers seen **so far**.

Finding an Algorithm (cont)

Need: TRAILINGSUM(n-1) which is the maximum sum of a subsequence that ends $x_1, \cdots, x_{n-1}$.

To get this, we need a stronger induction hypothesis.

no notes

Maximum Subsequence Solution

New Induction Hypothesis: We can find SUM(n-1) and TRAILINGSUM(n-1) for any sequence of $n - 1$ integers.

Base case:
SUM(1) $=$ TRAILINGSUM(1) $=$ Max(0, x_1).

Induction step:
SUM(n) $=$ Max(SUM(n-1), TRAILINGSUM(n-1) $+x_n$).
TRAILINGSUM(n) = Max(0, TRAILINGSUM(n-1) $+x_n$).

no notes

Maximum Subsequence Solution (cont)

Analysis:
Important Lesson: If we calculate and remember some additional values as we go along, we are often able to obtain a more efficient algorithm.

This corresponds to strengthening the induction hypothesis so that we compute more than the original problem (appears to) require.

How do we find sequence as opposed to sum?

$\mathrm{O}(n)$. $T(n) = T(n-1) + 2$.
Remember position information as well.

The Knapsack Problem

Problem:

- Given an integer capacity K and n items such that item i has an integer size k_i, find a subset of the n items whose sizes exactly sum to K, if possible.
- That is, find $S \subseteq \{1, 2, \cdots, n\}$ such that

$$\sum_{i \in S} k_i = K.$$

Example:
Knapsack capacity $K = 163$.
10 items with sizes

$$4, 9, 15, 19, 27, 44, 54, 68, 73, 101$$

This version of Knapsack is one of several variations.
Think about solving this for 163. An answer is:

$$S = \{9, 27, 54, 73\}$$

Now, try solving for $K = 164$. An answer is:

$$S = \{19, 44, 101\}.$$

There is no relationship between these solutions!

Knapsack Algorithm Approach

Instead of parameterizing the problem just by the number of items n, we parameterize by both n and by K.

$P(n, K)$ is the problem with n items and capacity K.

First consider the decision problem: Is there a subset S?

Induction Hypothesis:
We know how to solve $P(n-1, K)$.

Is there a subset S such that $\sum S_i = K$?

Knapsack Induction

Induction Hypothesis:
We know how to solve $P(n-1, K)$.

Solving $P(n, K)$:

- If $P(n-1, K)$ has a solution, then it is also a solution for $P(n, K)$.
- Otherwise, $P(n, K)$ has a solution iff $P(n-1, K-k_n)$ has a solution.

So what should the induction hypothesis really be?

But... I don't know how to solve $P(n-1, K-k_n)$ since it is not in my induction hypothesis! So, we must strengthen the induction hypothesis.

New Induction Hypothesis:
We know how to solve $P(n-1, k), 0 \leq k \leq K$.

Knapsack: New Induction

- **New Induction Hypothesis**:
  We know how to solve $P(n-1, k), 0 \leq k \leq K$.
- To solve $P(n, K)$:
  If $P(n-1, K)$ has a solution,
  Then $P(n, K)$ has a solution.
  Else If $P(n-1, K-k_n)$ has a solution,
  Then $P(n, K)$ has a solution.
  Else $P(n, K)$ has no solution.

Need to solve two subproblems: $P(n-1, k)$ and $P(n-1, k-k_n)$.

Algorithm Complexity

- Resulting algorithm complexity:
  $T(n) = 2T(n-1) + c \qquad n \geq 2$
  $T(n) = \Theta(2^n) \qquad$ by expanding sum.
- Alternate: change variable from n to $m = 2^n$.
  $2T(m/2) + c_1 n^0$.
  From Theorem 3.4, we get $\Theta(m^{\log_2 2}) = \Theta(2^n)$.
- But, there are only $n(K+1)$ problems defined.
  - It must be that problems are being re-solved many times by this algorithm. Don't do that.

Problem: Can't use Theorem 3.4 in this form.
This form uses n^0 because we also need an exponent of n to fit the form of the theorem.

Efficient Algorithm Implementation

The key is to avoid re-computing subproblems.

Implementation:

- Store an $n \times (K+1)$ matrix to contain solutions for all the $P(i, k)$.
- Fill in the table row by row.
- Alternately, fill in table using logic above.

Analysis:
$T(n) = \Theta(nK)$.
Space needed is also $\Theta(nK)$.

To solve $P(i, k)$ look at entry in the table.
If it is marked, then OK.
Otherwise solve recursively.
Initially, fill in all $P(i, 0)$.

Example

$K = 10$, with 5 items having size 9, 2, 7, 4, 1.

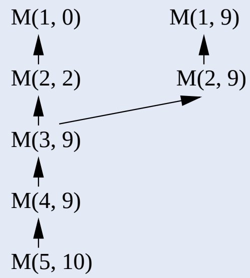
	0	1	2	3	4	5	6	7	8	9	10
$k_1 = 9$	O	–	–	–	–	–	–	–	–	I	–
$k_2 = 2$	O	–	I	–	–	–	–	–	–	O	–
$k_3 = 7$	O	–	O	–	–	–	–	I	–	I/O	–
$k_4 = 4$	O	–	O	–	I	–	I	O	–	O	–
$k_5 = 1$	O	I	O	I	O	I	O	I/O	I	O	I

Key:
– No solution for $P(i, k)$
O Solution(s) for $P(i, k)$ with i omitted.
I Solution(s) for $P(i, k)$ with i included.
I/O Solutions for $P(i, k)$ both with i included and with i omitted.

Example: M(3, 9) contains O because $P(2, 9)$ has a solution.
It contains I because $P(2, 2) = P(2, 9-7)$ has a solution.
How can we find a solution to $P(5, 10)$ from M?
How can we find **all** solutions for $P(5, 10)$?

Solution Graph

Find all solutions for $P(5, 10)$.



The result is an n-level DAG.

Alternative approach:
Do not precompute matrix. Instead, solve subproblems as necessary, marking in the array during backtracking.
To avoid storing the large array, use hashing for storing (and retrieving) subproblem solutions.

Dynamic Programming

This approach of storing solutions to subproblems in a table is called **dynamic programming**.

It is useful when the number of distinct subproblems is not too large, but subproblems are executed repeatedly.

Implementation: Nested `for` loops with logic to fill in a single entry.

Most useful for **optimization problems**.

no notes

Fibonacci Sequence

```
int Fibr(int n) {
  if (n <= 1) return 1;         // Base case
  return Fibr(n-1) + Fibr(n-2); // Recursion
}
```

- Cost is Exponential. Why?
- If we could eliminate redundancy, cost would be greatly reduced.

Essentially, we are making as many function calls as the value of the Fibonacci sequence itself. It is roughly (though not quite) two function calls of size $n - 1$ each.

Fibonacci Sequence (cont)

- Keep a table

```
int Fibrt(int n, int* Values) {
  // Assume Values has at least n slots, and
  // all slots are initialized to 0
  if (n <= 1) return 1;  // Base case
  if (Values[n] == 0)    // Compute and store
    Values[n] = Fibrt(n-1, Values) +
                Fibrt(n-2, Values);
  return Values[n];
}
```

- Cost?
- We don't need table, only last 2 values.
  - Key is working bottom up.

no notes

Chained Matrix Multiplication

Problem: Compute the product of n matrices

$$M = M_1 \times M_2 \times \cdots \times M_n$$

as efficiently as possible.

If A is $r \times s$ and B is $s \times t$, then
COST$(A \times B) =$
SIZE$(A \times B) =$

If C is $t \times u$ then
COST$((A \times B) \times C) =$
COST$((A \times (B \times C))) =$

$A \times B$:
rst
$r \times t$

$rst + (r \times t)(t \times u) = rst + rtu$.
$(r \times s)[(s \times t)(t \times u)] = (r \times s)(s \times u)$.
$rsu + stu$.

Order Matters

Example:

$$A = 2 \times 8; B = 8 \times 5; C = 5 \times 20$$

COST$((A \times B) \times C) =$
COST$(A \times (B \times C)) =$

View as binary trees:

$2 \cdot 8 \cdot 5 + 2 \cdot 5 \cdot 20 = 280$.
$8 \cdot 5 \cdot 20 + 2 \cdot 8 \cdot 20 = 1120$.

Tree for $((A \times B) \times C) =: \cdot \cdot ABC$
Tree for $(A \times (B \times C) =: \cdot A \cdot BC$

We would like to find the optimal order for computation before actually doing the matrix multiplications.

Chained Matrix Induction

Induction Hypothesis: We can find the optimal evaluation tree for the multiplication of $\leq n - 1$ matrices.

Induction Step: Suppose that we start with the tree for:

$$M_1 \times M_2 \times \cdots \times M_{n-1}$$

and try to add M_n.

Two obvious choices:
1. Multiply $M_{n-1} \times M_n$ and replace M_{n-1} in the tree with a subtree.
2. Multiply M_n by the result of $P(n - 1)$: make a new root.

Visually, adding M_n may radically order the (optimal) tree.

Problem: There is no reason to believe that either of these yields the optimal ordering.

Alternate Induction

Induction Step: Pick some multiplication as the root, then recursively process each subtree.
- Which one? Try them all!
- Choose the cheapest one as the answer.
- How many choices?

Observation: If we know the ith multiplication is the root, then the left subtree is the optimal tree for the first $i - 1$ multiplications and the right subtree is the optimal tree for the last $n - i - 1$ multiplications.

Notation: for $1 \leq i \leq j \leq n$,
$c[i, j] =$ minimum cost to multiply $M_i \times M_{i+1} \times \cdots \times M_j$.

$$\text{So,} c[1, n] = \min_{1 \leq i \leq n-1} r_0 r_i r_n + c[1, i] + c[i + 1, n].$$

$n - 1$ choices for root.

Analysis

Base Cases: For $1 \le k \le n$, $c[k,k] = 0$.
More generally:

$$c[i,j] = \min_{1 \le k \le j-1} r_{i-1} r_k r_j + c[i,k] + c[k+1,j]$$

Solving $c[i,j]$ requires $2(j-i)$ recursive calls.
Analysis:

$$\begin{aligned} T(n) &= \sum_{k=1}^{n-1} (T(k) + T(n-k)) = 2 \sum_{k=1}^{n-1} T(k) \\ T(1) &= 1 \\ T(n+1) &= T(n) + 2T(n) = 3T(n) \\ T(n) &= \Theta(3^n) \quad \text{Ugh!} \end{aligned}$$

But there are only $\Theta(n^2)$ values $c[i,j]$ to be calculated!

2 calls for each root choice, with $(j-i)$ choices for root. But, these don't all have equal cost.

Actually, since $j > i$, only about half that needs to be done.

Dynamic Programming

Make an $n \times n$ table with entry $(i,j) = c[i,j]$.

$c[1,1]$	$c[1,2]$	$\cdots$	$c[1,n]$
	$c[2,2]$	$\cdots$	$c[2,n]$
		$\cdots$ $\cdots$	$\cdots$ $\cdots$
			$c[n,n]$

Only upper triangle is used.
Fill in table diagonal by diagonal.
$c[i,i] = 0$.
For $1 \le i < j \le n$,

$$c[i,j] = \min_{i \le k \le j-1} r_{i-1} r_k r_j + c[i,k] + c[k+1,j].$$

The array is processed starting with the middle diagonal (all zeros), diagonal by diagonal toward the upper left corner.

Dynamic Programming Analysis

- The time to calculate $c[i,j]$ is proportional to $j-i$.
- There are $\Theta(n^2)$ entries to fill.
- $T(n) = O(n^3)$.
- Also, $T(n) = \Omega(n^3)$.
- How do we actually **find** the best evaluation order?

For middle diagonal of size $n/2$, each costs $n/2$.

For each $c[i,j]$, remember the k (the root of the tree) that minimizes the expression.
So, store in the table the next place to go.

Summary

- Dynamic programming can often be added to an inductive proof to make the resulting algorithm as efficient as possible.
- Can be useful when divide and conquer **fails** to be efficient.
- Usually applies to optimization problems.
- Requirements for dynamic programming:
  1. Small number of subproblems, small amount of information to store for each subproblem.
  2. Base case easy to solve.
  3. Easy to solve one subproblem given solutions to smaller subproblems.

no notes