Analysis of Algorithms

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- Measure resource requirements: how does the amount of time and space an algorithm uses scale with increasing input size?
- How do we put this notion on a concrete footing?
- What does it mean for one function to grow faster or slower than another?

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- How do we put this notion on a concrete footing?
- What does it mean for one function to grow faster or slower than another?
- Goal: Develop algorithms that provably run quickly and use low amounts of space.

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- Why worst-case? Why not average-case or on random inputs?
- *Input size* = number of elements in the input. Values in the input do not matter.
- Assume all elementary operations take unit time: assignment, arithmetic on a fixed-size number, comparisons, array lookup, following a pointer, etc.

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Definition

An algorithm is *efficient* if it has a polynomial running time.

Upper and Lower Bounds

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Asymptotic upper bound: A function $f(n)$ is $O(g(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ such that for all $n \geq n_0$, we have $f(n) \leq cg(n)$.

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Asymptotic lower bound: A function $f(n)$ is $\Omega(g(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ such that for all $n \geq n_0$, we have $f(n) \geq cg(n)$.

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- In these definitions, c is a constant independent of n.
- Abuse of notation: say $g(n) = O(f(n))$, $g(n) = \Omega(f(n))$, $g(n) = \Theta(f(n))$.

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Transitivity

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- For every $x > 0$, $\log n = O(n^x)$.
- For every $r > 1$ and every $d > 0$, $n^d = O(r^n)$.

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- Finding the minimum, merging two sorted lists.
- Sub-linear time. Binary search in a sorted array of n numbers takes $O(\log n)$ time.

$O(n \log n)$ Time

- Any algorithm where the costliest step is sorting.

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- Given a set of n points in the plane, find the pair that are the closest. Surprising fact: can solve this problem in $O(n \log n)$ time later in the semester.

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- Running time is $O(k^2\binom{n}{k}) = O(n^k)$.

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- Algorithm: For each $1 \leq i \leq n$, check if the graph has an independent size of size i. Output largest independent set found.
- What is the running time?

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- Algorithm: For each $1 \leq i \leq n$, check if the graph has an independent size of size i. Output largest independent set found.
- What is the running time? $O(n^2 2^n)$.