Coping with NP-Completeness

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How Do We Tackle an $\mathcal{NP}$-Complete Problem?

- These problems come up in real life.

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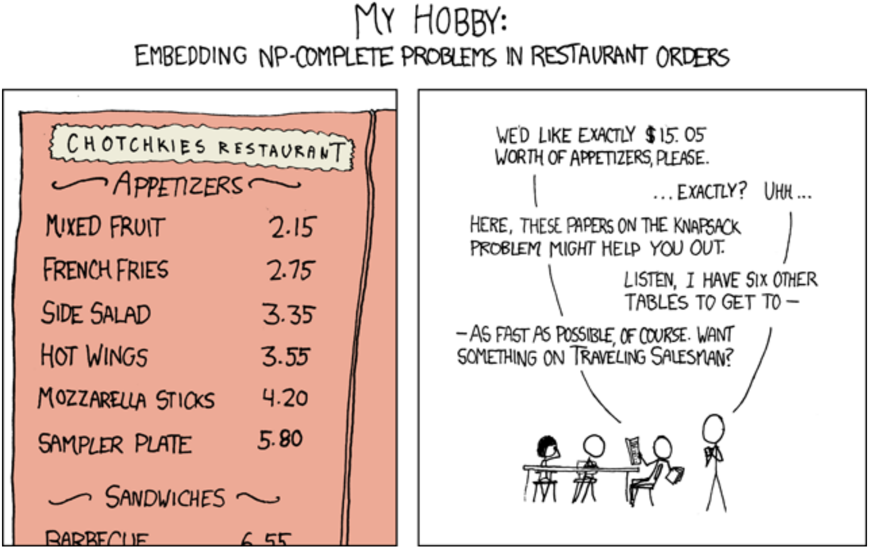

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- These problems come up in real life.
- $\mathcal{NP}$-Complete means that a problem is hard to solve in the *worst case*. Can we come up with better solutions at least in *some* cases?
  - Develop algorithms that are exponential in one parameter in the problem.
  - Consider special cases of the input, e.g., graphs that "look like" trees.
  - Develop algorithms that can provably compute a solution close to the optimal.

Vertex Cover Problem

VERTEX COVER

INSTANCE: Undirected graph G and an integer k

QUESTION: Does G contain a vertex cover of size at most k?

- The problem has two parameters: k and n, the number of nodes in G.
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- What is the running time of a brute-force algorithm?
  $O(kn\binom{n}{k}) = O(kn^{k+1})$.
- Can we devise an algorithm whose running time is exponential in k but polynomial in n, e.g., $O(2^k n)$?

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- Consider an edge (u, v). Either u or v must be in the vertex cover.
- Claim: G has a vertex cover of size at most k iff for any edge (u, v) either $G - \{u\}$ or $G - \{v\}$ has a vertex cover of size at most $k - 1$.

Vertex Cover Algorithm

```
To search for a k-node vertex cover in G:
  If G contains no edges, then the empty set is a vertex cover
  If G contains> k |V| edges, then it has no k-node vertex cover
  Else let e = (u, v) be an edge of G
    Recursively check if either of G−{u} or G−{v}
                has a vertex cover of size k − 1
    If neither of them does, then G has no k-node vertex cover
    Else, one of them (say, G−{u}) has a (k − 1)-node vertex cover T
      In this case, T ∪ {u} is a k-node vertex cover of G
    Endif
  Endif
```

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- Claim: $T(n, k) = O(2^k kn)$.

Solving $\mathcal{NP}$-Hard Problems on Trees

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- "$\mathcal{NP}$-Hard": at least as hard as $\mathcal{NP}$-Complete. We will use $\mathcal{NP}$-Hard to refer to optimisation versions of decision problems.
- Many $\mathcal{NP}$-Hard problems can be solved efficiently on trees.
- Intuition: subtree rooted at any node v of the tree "interacts" with the rest of tree only through v. Therefore, depending on whether we include v in the solution or not, we can decouple solving the problem in v's subtree from the rest of the tree.

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- Claim: If a tree T has a a leaf v, then a maximum-size independent set in T is v and a maximum-size independent set in $T - \{v\}$.

Greedy Algorithm for Independent Set

- A *forest* is a graph where every connected component is a tree.

```
To find a maximum-size independent set in a forest F:
  Let S be the independent set to be constructed (initially empty)
  While F has at least one edge
     Let e = (u, v) be an edge of F such that v is a leaf
     Add v to S
     Delete from F nodes u and v, and all edges incident to them
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- Running time of the algorithm is $O(n)$.
- The algorithm works correctly on any graph for which we can repeatedly find a leaf.

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- Consider the INDEPENDENT SET problem but with a weight w_v on every node v.
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- But there are still only two possibilities: either include u in the independent set or include *all* neighbours of u that are leaves.
- Suggests dynamic programming algorithm.

Designing Dynamic Programming Algorithm for Maximum Weight Independent Set

- Dynamic programming algorithm needs a set of sub-problems, recursion to combine sub-problems, and order over sub-problems.
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  - Sub-problems are T_u: subtree induced by u and all its descendants.

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  - Sub-problems are T_u: subtree induced by u and all its descendants.
- Ordering the sub-problems: start at leaves and work our way up to the root.

Recursion for Dynamic Programming Algorithm for Maximum Weight Independent Set

- Either we include u in an optimal solution or exclude u.
  - $OPT_{in}(u)$: maximum weight of an independent set in T_u that includes u.
  - $OPT_{out}(u)$: maximum weight of an independent set in T_u that excludes u.

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- Base cases: For a leaf u, $\text{OPT}_{\text{in}}(u) = w_u$ and $\text{OPT}_{\text{out}}(u) = 0$.
- Recurrence:
  1. If we include u, all children must be excluded.
  2. If we exclude u, a child may or may not be excluded.

Dynamic Programming Algorithm for Maximum Weight Independent Set

```
To find a maximum-weight independent set of a tree T:
    Root the tree at a node r
    For all nodes u of T in post-order
        If u is a leaf then set the values:
```

$$M_{out}[u] = 0$$

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$$M_{out}[u] = \sum_{v \in children(u)} \max(M_{out}[u],\ M_{in}[u])$$

$$M_{in}[u] = w_u + \sum_{v \in children(u)} M_{out}[u].$$

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Aren't Trees Too Restrictive?

- Trees are only a very specific sub-class of graphs. What use are algorithms for $\mathcal{NP}$-Hard problems that work well on trees?

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- Trees are only a very specific sub-class of graphs. What use are algorithms for $\mathcal{NP}$-Hard problems that work well on trees?
- These ideas can be generalised to graphs that "look like" trees: graphs with bounded treewidth.

Example of Tree Decomposition

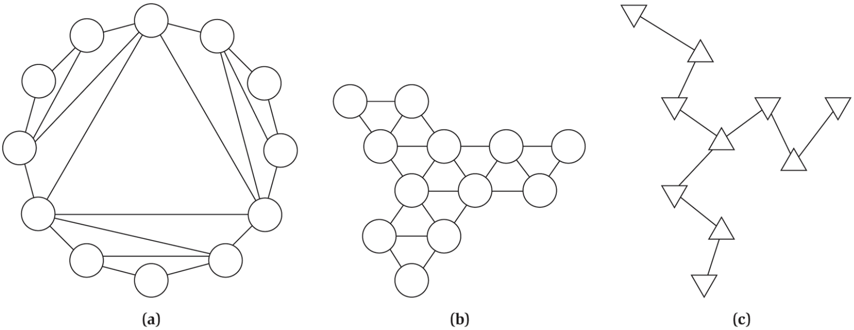

Figure 10.5 Parts (a) and (b) depict the same graph drawn in different ways. The drawing in (b) emphasizes the way in which it is composed of ten interlocking triangles. Part (c) illustrates schematically how these ten triangles "fit together."

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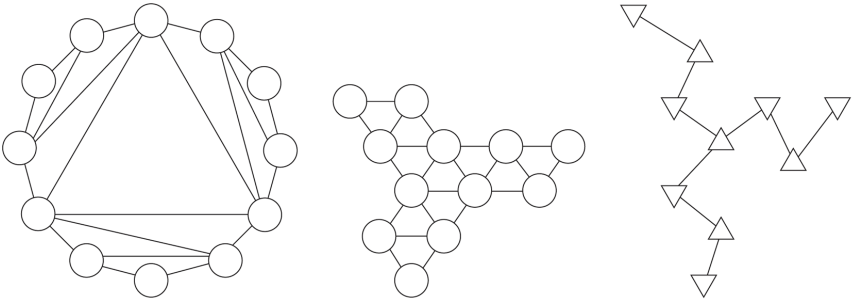

(a) (b) (c)

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- Definition of "tree-like" should capture graphs that we can decompose into disconnected pieces by removing a small number of nodes.
- Definition should make precise the notion of "tree-like" structures in the figure.

Tree Decompositions

A *Tree decomposition* of a graph $G(V, E)$ consists of

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(Coherence): Let t_1, t_2, and t_3 be three nodes in T such that t_2 lies on the path from t_1 to t_3. Then, if a node v in G belongs to V_{t_1} and V_{t_3}, it also belongs to V_{t_2}.

Properties of Tree Decompositions

- Trees have two nice separation properties:
  1. If we delete an edge from a tree, the tree splits into two connected components.
  2. If we delete a node and all incident edges from a tree, the tree splits into a number of connected components equal to the degree of the node.
- Tree decompositions have analogous properties.

Node Separation in a Tree Decomposition

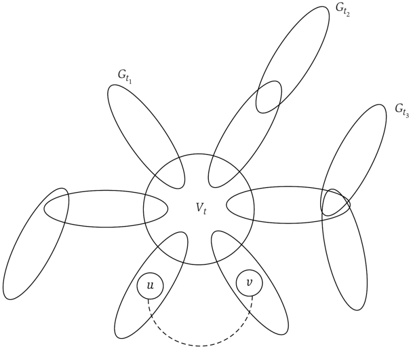

Figure 10.6 Separations of the tree T translate to separations of the graph G.

- If T' is a subgraph of T, let $G_{T'}$ denote the subgraph of G induced by the nodes $\cup_{t \in T'} V_t$.

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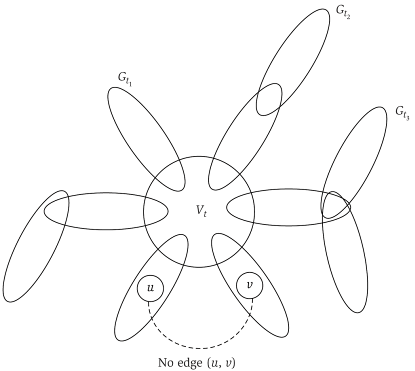

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- If T' is a subgraph of T, let $G_{T'}$ denote the subgraph of G induced by the nodes $\cup_{t \in T'} V_t$.
- Claim: Suppose $T - \{t\}$ has the components $T_1, T_2, \ldots T_d$. Then the subgraphs

$$G_{T_1} - V_t, G_{T_t} - V_t, \ldots, G_{T_d} - V_t$$

have no nodes in common and there are no edges between nodes in different subgraphs.

Edge Separation in a Tree Decomposition

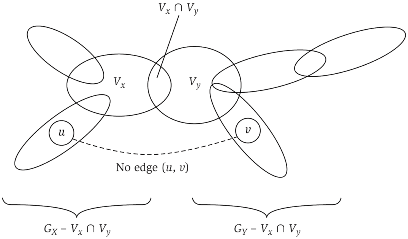

Figure 10.7 Deleting an edge of the tree T translates to separation of the graph G.

- Claim: Let X and Y be the two components of T after the deletion of the edge (x, y). Then deleting the set $V_X \cap V_Y$ from G disconnects G into the two subgraphs $G_X - (V_X \cap V_Y)$ and $G_Y - (V_X \cap V_Y)$

Uses of Tree Decompositions

- *Width* of a tree decomposition is the size of the largest piece.
- *Treewidth* of a graph is the smallest width of a tree decomposition of the graph.
- If we have a tree decomposition of small width, we can perform dynamic programming over the decomposition.
- Cost of the algorithm is exponential in the width of the decomposition.

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- Cost of the algorithm is exponential in the width of the decomposition.
- Does a graph a tree decomposition with width at most w? $\mathcal{NP}$-Complete!
- (Chapter 10.5): Given a graph and a parameter w, there is an algorithm that runs in $O(f(w)mn)$ time and either
  1. produces a tree decomposition of width at most $4w$ or
  2. reports correctly that G does not have a tree decomposition with width less than w.

Approximation Algorithms

- Methods for optimisation versions of $\mathcal{NP}$-Complete problems.
- Run in polynomial time.
- Solution returned is guaranteed to be within a small factor of the optimal solution

Load Balancing Problem

- Given set of m machines $M_1, M_2, \ldots M_n$.
- Given a set of m jobs: job j has processing time t_j.
- Assign each job to one machine so that the total time spent is minimised.

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- Let $A(i)$ be the set of jobs assigned to machine M_i.
- $T_i = \sum_{k \in A(i)} t_k$.
- Minimise *makespan* $T = \max_i T_i$.

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- Minimising makespan is $\mathcal{NP}$-Complete.

Greedy-Balance Algorithm

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Greedy-Balance:
Start with no jobs assigned
Set T_i = 0 and A(i) = Ø for all machines M_i
For j = 1, ..., n
  Let M_i be a machine that achieves the minimum min_k T_k
  Assign job j to machine M_i
  Set A(i) ← A(i) ∪ {j}
  Set T_i ← T_i + t_j
EndFor
```

Lower Bounds on the Optimal Makespan

- We need a lower bound on the optimum makespan T^*.

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- The two bounds below will suffice:

$$T^* \geq \frac{1}{m} \sum_j t_j$$

$$T^* \geq \max_j t_j$$

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- Let T be the computed makespan.
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- Let T be the computed makespan.
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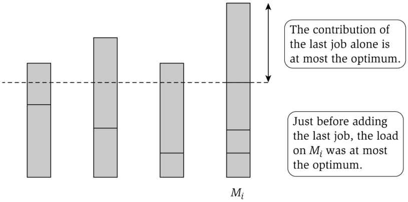

Figure 11.2 Accounting for the load on machine M_i in two parts: the last job to be added, and all the others.

- M_i had the smallest load and its load was $T - t_j$.
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- Therefore, $T - t_j \leq 1/m \sum_k T_k \leq T^*$.
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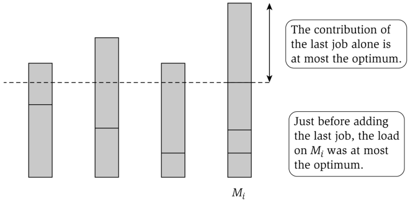

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- $T \leq 2T^*$

Improving the Bound

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Improving the Bound

- It is easy to construct an example for which the greedy algorithm produces a solution close to a factor of 2 away from optimal.
- How can we improve the algorithm?
- What if we process the jobs in decreasing order of processing time?

Sorted-Balance Algorithm

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Sorted-Balance:
Start with no jobs assigned
Set T_i = 0 and A(i) = Ø for all machines M_i
Sort jobs in decreasing order of processing times t_j
Assume that t_1 ≥ t_2 ≥ ... ≥ t_n
For j = 1, ..., n
  Let M_i be the machine that achieves the minimum min_k T_k
  Assign job j to machine M_i
  Set A(i) ← A(i) ∪ {j}
  Set T_i ← T_i + t_j
EndFor
```

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- Using same proof as before, $T = T_i \leq 3T^*/2$.

Set Cover

SET COVER

INSTANCE: A set U of n elements, a collection $S_1, S_2, \ldots, S_m$ of subsets of U, each with an associated weight w.

SOLUTION: A collection $\mathcal{C}$ of sets in the collection such that $\sum_{S_i \in C} w_i$ is minimised.

Greedy-Set-Cover

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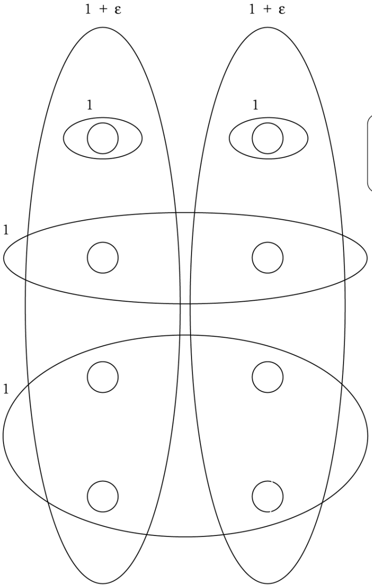
- The algorithm computes a set cover whose weight is at most $O(\log n)$ times the optimal weight (Johnson 1974, Lovász 1975, Chvatal 1979).

Example of Greedy-Set-Cover



Two sets can be used to cover everything, but the greedy algorithm doesn't find them.

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$$\sum_{S_i \in \mathcal{C}} w_i = \sum_{S_i \in \mathcal{C}} \left(\sum_{s \in S_i} c_s \right) = \sum_{s \in U} c_s.$$

Upper Bounding Cost-by-Weight Ratio

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- Claim: For every set S_k, the sum $\sum_{s \in S_k} \leq H(|S_K|)w_k$.

Why is the Bound Useful?

- Let us assume $\sum_{s \in S_k} c_s \leq H(|S_K|) w_k$.
- Let d^* be the size of the largest set in the collection.
- Let $\mathcal{C}^*$ denote the optimal set cover: $w^* = \sum_{S_i \in \mathcal{C}^*} w_i$.

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- We have proven that GREEDY-SET-COVER computes a set cover whose weight is at most $H(d^*)$ times the optimal weight.

Proving $\sum_{s \in S_k} c_s \leq H(|S_K|)w_k$

- Renumber elements in U so that elements in S_k are the first $d = |S_k|$ elements of U, i.e., $S_k = \{s_1, s_2, \ldots, s_d\}$.
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- We are done!

$$\sum_{s \in S_k} c_s = \sum_{i=1}^{d} c_{s_j} \leq \sum_{i=1}^{d} \frac{w_k}{d - j + 1} = H(d)w_k.$$

How Badly Can Greedy-Set-Cover Perform?

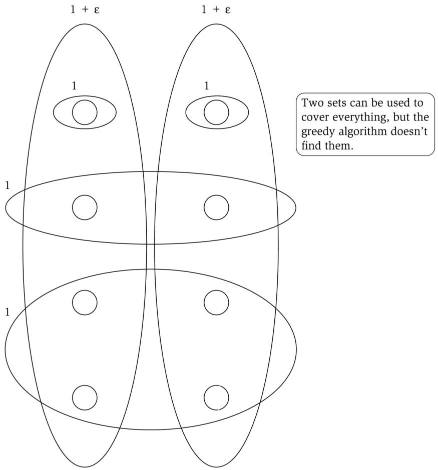

Figure 11.6 An instance of the Set Cover Problem where the weights of sets are either 1 or $1+\varepsilon$ for some small $\varepsilon > 0$. The greedy algorithm chooses sets of total weight 4, rather than the optimal solution of weight $2+2\varepsilon$.

- Generalise this example to show that algorithm produces a set cover of weight $\Omega(\log n)$ even though optimal weight is $2+\varepsilon$.
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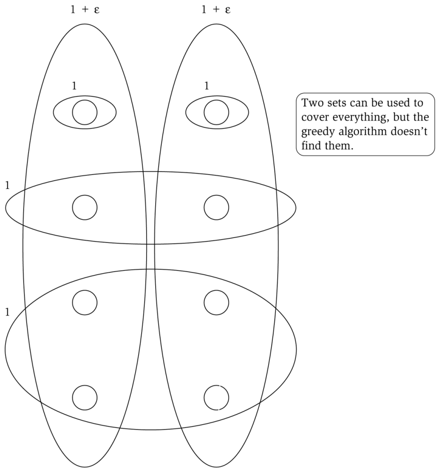

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- More complex constructions show greedy algorithm incurs a weight close to $H(n)$ times the optimal weight.
- No polynomial time algorithm can achieve an approximation bound better than $H(n)$ times optimal unless $\mathcal{P} = \mathcal{NP}$ (Lund and Yannakakis, 1994).