Finite state machine exercises from Gill

Gill, problem 2.3(c), page 44.

Let R_i be the number of $s_{\nu+1}$ subtables for (i,p,q) machines with no isolated states. Then $R_2 = 2^{2p} - 1$, and, in general,

$$R_n = n^{np} - \left[\binom{n}{1}R_{n-1} + \binom{n}{2}R_{n-2} + \cdots + \binom{n}{n-2}R_2 + 1\right].$$

The number of (n,p,q) machines with no isolated states is thus $q^{np}R_n$.

Gill, problem 2.3(d), page 44.

For a fixed k output symbols, let NE_k be the number of z_r subtables using *exactly* k symbols. $NE_1 = 1$, $NE_2 = 2^{np} - \binom{2}{1}NE_1$, $NE_3 = 3^{np} - \binom{3}{1}NE_1 - \binom{3}{2}NE_2$, ...,

$$NE_q = q^{np} - \sum_{r=1}^{q-1}\binom{q}{r}NE_r.$$

The total count is thus

$$\left[q^{np} - \sum_{r=1}^{q-1}\binom{q}{r}NE_r\right] n^{np}.$$

For $(n,p,q) = (2,2,3) : 576$, $(2,2,4) : 384$.

Gill, problem 2.4, page 45.

It suffices to show that the transition table of each member of the permutation family uniquely determines the permutation, since then there is a 1–1 correspondence between distinct tables and permutations. Consider a table produced by the permutation ρ, i.e., $1,\ldots,6$ have been replaced by $\rho(1),\ldots,\rho(6)$. Examine the $s_{\nu+1}$ subtable. There is only one state σ for which $f_s(\alpha,\sigma) = f_s(\beta,\sigma)$, so σ must be $\rho(1)$. Now $\rho(1)$ occurs exactly twice in the β column, so this second occurrence of $\rho(1)$ determines $\rho(2)$ from the α column. $\rho(2)$ also occurs exactly twice in the β column, and the corresponding α entries must be $\rho(3)$ and $\rho(4)$. $\rho(1)$ and $\rho(2)$ have already been determined, so the remaining state in the β column must be $\rho(3)$. Finding $\rho(3)$ also determines $\rho(4)$ from the α column, as mentioned above. Now only $\rho(5)$ and $\rho(6)$ remain, but they are uniquely determined by the z_r subtable, $\rho(5)$ corresponding to 1 0 and $\rho(6)$ to 1 1.

Gill, problem 2.16, page 47.

$$\text{Number of skeleton matrices} = \left[\sum_{i=1}^{p}\binom{n}{i}\right]^n.$$

$$\text{Number of modified skeleton matrices} = \binom{n+p-1}{p}^n.$$

Gill, problem 2.19, page 47.

The i, k entry of $[M_a][M_b]^t$ is

$$\left([M_a][M_b]^t\right)_{ik} = \sum_{j=1}^{m} a_{ij} b_{kj}.$$

From the definition of $[M_a]$ and $[M_b]$,

$$a_{ij} b_{kj} = \begin{cases} 1, & \beta_j \text{ is a branch from } \sigma_i \text{ to } \sigma_k, \\ 0, & \text{otherwise.} \end{cases}$$

Since at most one of the branches $\beta_1, \ldots, \beta_m$ goes from σ_i to σ_k,

$$\sum_{j=1}^{m} a_{ij} b_{kj} = \begin{cases} 1, & \text{there is a branch from } \sigma_i \text{ to } \sigma_k, \\ 0, & \text{otherwise.} \end{cases}$$

Thus the i, k entry of $[M_a][M_b]^t$ is exactly the same as that of $[\check{M}]$, and since i, k were arbitrary, $[M_a][M_b]^t = [\check{M}]$.

Gill, problem 3.17, page 85.

Let $\Sigma_{i_1}, \ldots, \Sigma_{i_r}$ be all the equivalence classes of M which contain a state of M'. Let M'' be the submachine of $\check{M}$ consisting of states $\sigma_{i_1}{}', \ldots, \sigma_{i_r}{}'$. By the construction of the minimal form $\check{M}$, $M'' = M'$. Since the states in M'' are distinguishable, M'' is a minimal machine equivalent to M', hence M'' is the minimal form of M'. Since M' is persistent or isolated , there are no branches out of the states in $\Sigma_{i_1} \bigcup \cdots \bigcup \Sigma_{i_r}$, and consequently no branches out of the states $\sigma_{i_1}{}', \ldots, \sigma_{i_r}{}'$ in $\check{M}$. Therefore M'' is either isolated, persistent, or all of $\check{M}$.

$A(p, n)$ = number of ways to distribute n classes of objects among p distinct cells, with each class represented, one object per cell and no cell empty

$$= \sum_{\substack{i_1 + \cdots + i_n = p \\ 1 \leq i_j \leq p}} \frac{p!}{i_1! i_2! \cdots i_n!}.$$

Number of (n, p, q) machines, $p \geq n$, such that every state has at least one branch to all n states, is $q^{np}\big(A(p, n)\big)^n$.