Chapter 1: Sets, Relations, and Languages

A. Some basic tools from mathematics.

- Implications.

- Set theory.
- Relations and functions.
  - Ordered pair, n-tuple, Cartesian product, relation.
  - A **function** $f : A \rightarrow B$ is a binary relation on A and B such that

  - More terminology: one-to-one, onto, bijection, inverse, isomorphism, composition.
  - Graphical representation of a binary relation.
  - Equivalence relation:
  - Partial order:
- Closure.
  - With respect to a relation, with respect to a function.
  - The reflexive closure of $R \subseteq A \times A$:
  - The transitive closure of $R \subseteq A \times A$:
  - R^* is the reflexive, transitive closure.
- Countable vs. uncountable sets.
  - A is countably infinite if there exists a bijection from A to $\mathcal{N}$.
  - Some famous countability results:

- Proof techniques.

B. Languages

- One answer to the question "What is computation?":
- Hence, an important ingredient in many models of computation is a **formal language**.
- **Definitions**:
  An **alphabet** Σ is

  A **string** is

  Σ^* is

  A **language** L is

  The **closure** or **Kleene star** of a language L:
- **Theorem**. Σ^* is countable, but 2^{Σ^*} is uncountable.
- Representation of languages:
- How many languages can we represent?
  - A finite representation is itself a finite sequence of symbols over some alphabet.
  - Hence, there are countably many finite representations.
  - But there are an uncountable number of languages over a given alphabet.
- **Definition**. The set of **regular expressions** over an alphabet Σ is defined as follows:
- Every regular expression α represents a language $L(\alpha)$:

- Examples.

- **Definition.** Let $B = \{\emptyset\} \cup \{\{a\} \mid a \in \Sigma\}$. Then the set of **regular languages** over Σ is the closure of B
- **Theorem**. The set of regular languages over Σ is exactly the set of languages defined by some regular expression over Σ.
- Remarks: