Divide and Conquer Algorithms

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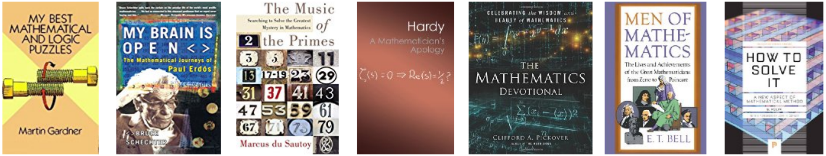
Divide and Conquer Algorithms

- Study three divide and conquer algorithms:
  - Counting inversions.
  - Finding the closest pair of points.
  - Integer multiplication.
- First two problems use clever conquer strategies.
- Third problem uses a clever divide strategy.

Motivation

Inspired by your shopping trends

More top picks for you



- Collaborative filtering: match one user's preferences to those of other users, e.g., purchases, books, music.
- Meta-search engines: merge results of multiple search engines into a better search result.

Fundamental Question

- How do we compare a pair of rankings?
  - My ranking of songs: ordered list of integers from 1 to n.
  - Your ranking of songs: $a_1, a_2, \ldots, a_n$, a permutation of the integers from 1 to n.

1	2	3	4	5	6	7	8	9	10	11	12

4	1	2	6	8	5	3	9	7	11	12	10

Comparing Rankings

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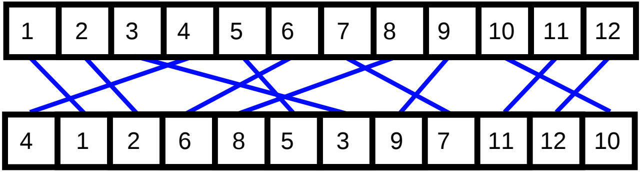
- Suggestion: two rankings of songs are very similar if they have few inversions.

Comparing Rankings



- Suggestion: two rankings of songs are very similar if they have few inversions.
  - The second ranking has an *inversion* if there exist i, j such that $i < j$ but $a_i > a_j$.
  - The number of inversions s is a measure of the difference between the rankings.
- Question also arises in statistics: *Kendall's rank correlation* of two lists of numbers is $1 - 2s/\left(n(n-1)\right)$.

Counting Inversions

COUNT INVERSIONS

INSTANCE: A list $L = x_1, x_2, \ldots, x_n$ of distinct integers between 1 and n.

SOLUTION: The number of pairs $(i, j), 1 \leq i < j \leq n$ such $x_i > x_j$.

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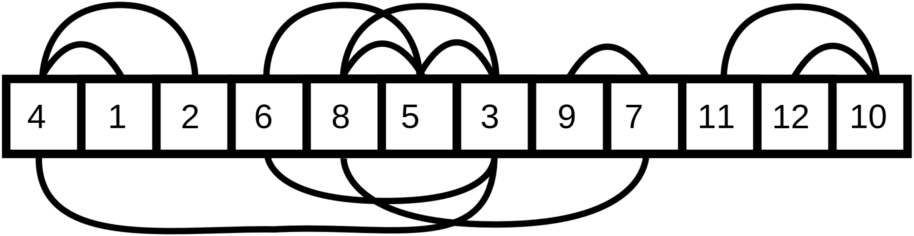

Counting Inversions: Algorithm

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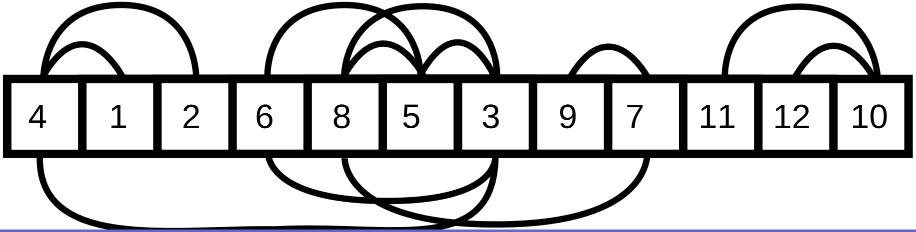

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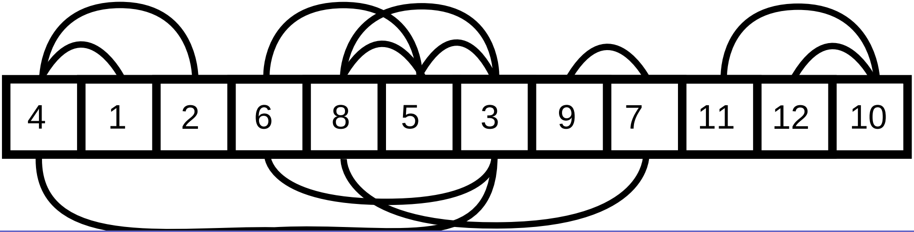

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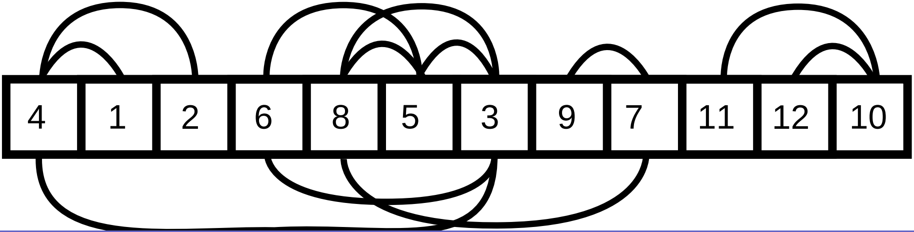

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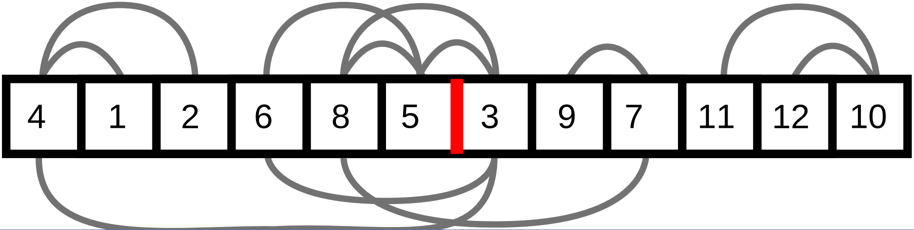

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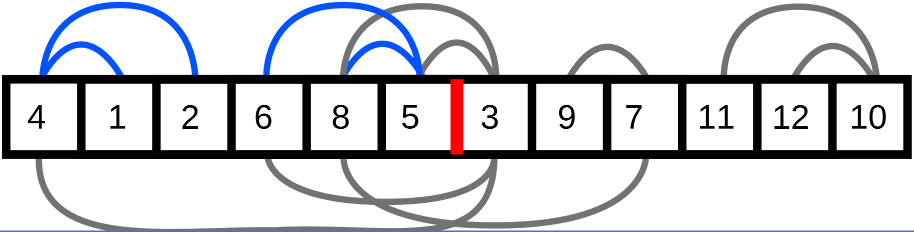

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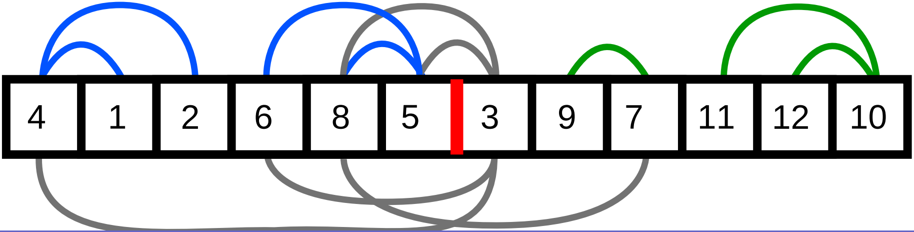

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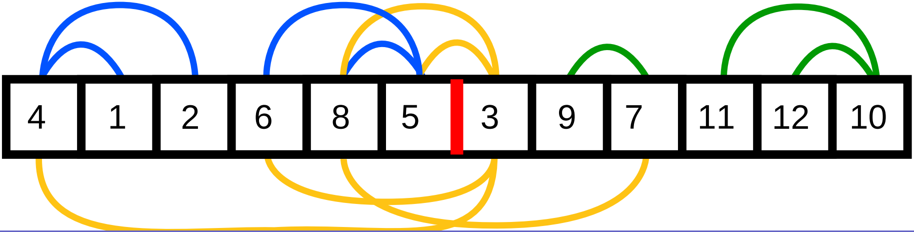

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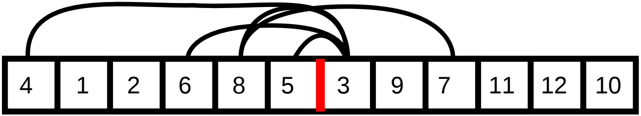

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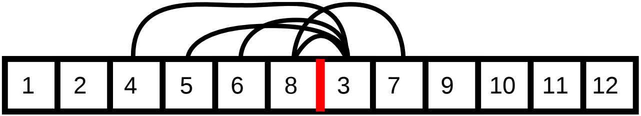

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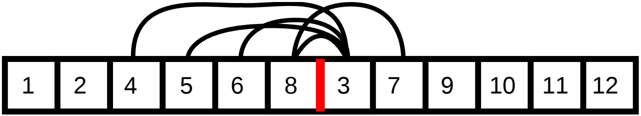

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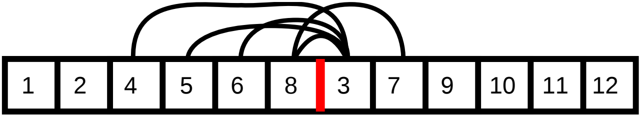

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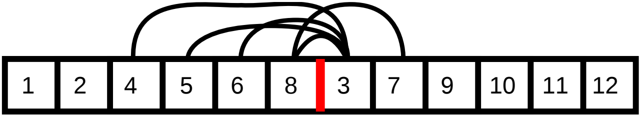

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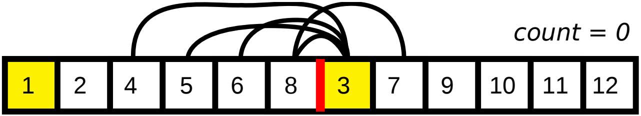

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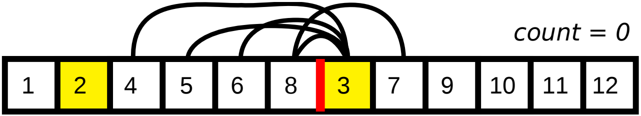

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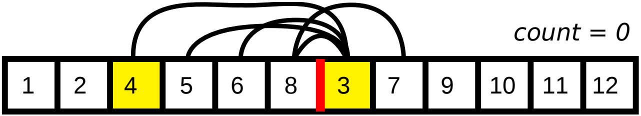

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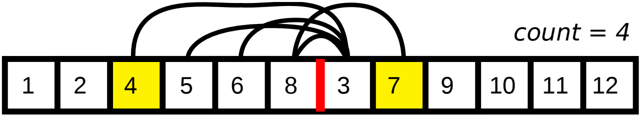

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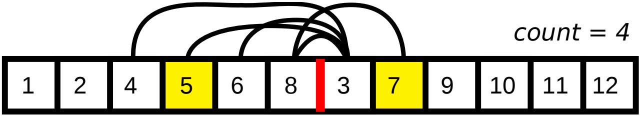

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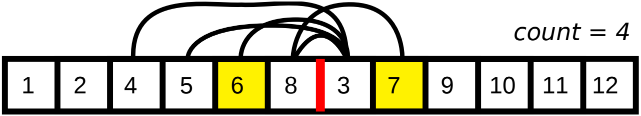

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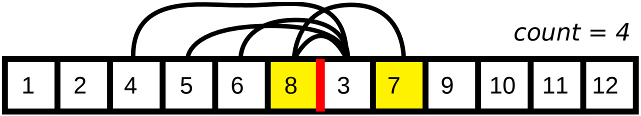

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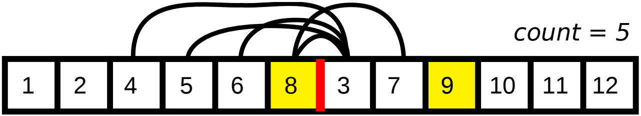

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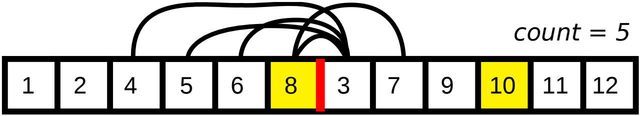

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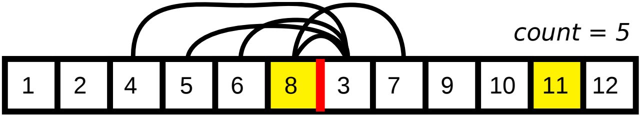

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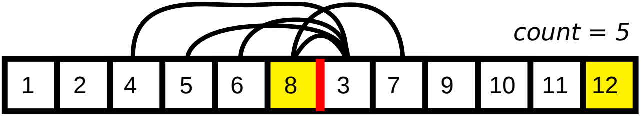

- Given lists $A = a_1, a_2, \ldots, a_m$ and $B = b_1, b_2, \ldots b_m$, compute the number of pairs a_i and b_j such $a_i > b_j$.
- Key idea: problem is much easier if A and B are sorted!
- MERGE-AND-COUNT procedure:
  1. Maintain a *current* pointer for each list.
  2. Maintain a variable *count* initialised to 0.
  3. Initialise each pointer to the front of the list.
  4. While both lists are nonempty:
     1. Let a_i and b_j be the elements pointed to by the *current* pointers.
     2. Append the smaller of the two to the output list.
     3. If $b_j < a_i$, increment *count* by the number of elements remaining in A.
     4. Advance *current* in the list containing the smaller element.
  5. Append the rest of the non-empty list to the output.
  6. Return *count* and the merged list.
- Running time of this algorithm is $O(m)$.

Counting Inversions: Final Algorithm

```
Sort-and-Count(L)
  If the list has one element then
      there are no inversions
  Else
      Divide the list into two halves:
         A contains the first ⌈n/2⌉  elements
         B contains the remaining ⌊n/2⌋ elements
      (r_A, A) = Sort-and-Count(A)
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      (r, L) = Merge-and-Count(A, B)
   Endif
   Return r = r_A + r_B + r, and the sorted list L
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- Running time $T(n)$ of the algorithm is $O(n \log n)$ because $T(n) \leq 2T(n/2) + O(n)$.

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- Prove by induction. Strategy: every inversion in the data is counted exactly once.

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- Inductive step: Pick an arbitrary k and l such that $k < l$ but $x_k > x_l$. When is this inversion counted by the algorithm?
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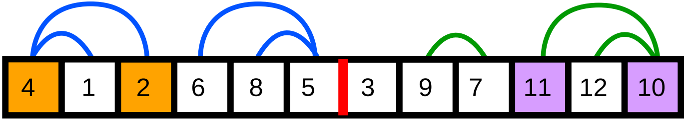

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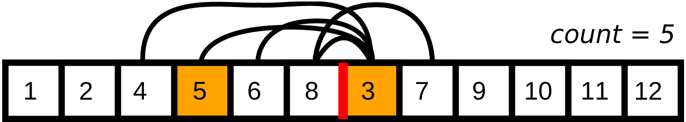

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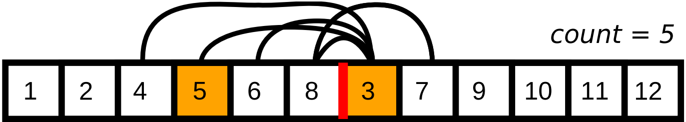

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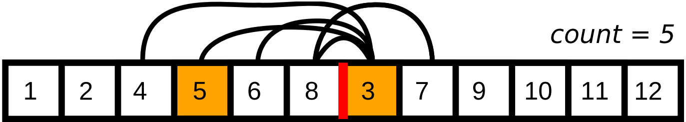

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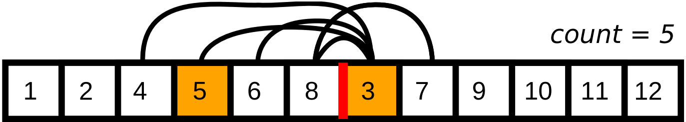

Integer Multiplication

MULTIPLY INTEGERS
INSTANCE: Two n-digit binary integers x and y
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(a) **(b)**

Figure 5.8 The elementary-school algorithm for multiplying two integers, in (a) decimal and (b) binary representation.

Integer Multiplication

MULTIPLY INTEGERS
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SOLUTION: The product xy

- Multiply two n-digit integers.
- Result has at most $2n$ digits.
- Algorithm we learnt in school takes $O(n^2)$ operations. Size of the input is not 2 but $2n$,

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$$
\begin{aligned}
xy &= (x_1 2^{n/2} + x_0)(y_1 2^{n/2} + y_0) \\
&= x_1 y_1\, 2^n + (\, x_1 y_0 \, + \, x_0 y_1 \,) 2^{n/2} + x_0 y_0
\end{aligned}
$$

n bits

$n/2$ bits

Divide-and-Conquer Algorithm

$$xy = x_1y_1\,2^n + (\,x_1y_0 + x_0y_1\,)2^{n/2} + x_0y_0$$

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- Algorithm:
  1. Compute x_1y_1, x_1y_0, x_0y_1, and x_0y_0 recursively.
  2. Merge the answers, i.e,.,
     1. Multiple x_1y_1 by 2^n
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$$xy \quad = \quad x_1 y_1\, 2^n + (\, x_1 y_0 \; + \; x_0 y_1 \,) 2^{n/2} + \; x_0 y_0$$

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$$T(n) \quad \leq \quad 4T(n/2) + cn \leq O(n^2)$$

Improving the Algorithm

- Four sub-problems lead to an $O(n^2)$ algorithm.
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- Four sub-problems lead to an $O(n^2)$ algorithm.
- How can we reduce the number of sub-problems?
  - No need to compute $x_1 y_0$ and $x_0 y_1$ independently; we just need their sum.

$$(x_0 + x_1)(y_0 + y_1) = x_1 y_1 + (x_1 y_0 + x_0 y_1) + x_0 y_0$$

$$\underbrace{(x_1 y_0 + x_0 y_1)}_{\text{Need this sum}} = (x_0 + x_1)(y_0 + y_1) - x_1 y_1 - x_0 y_0$$

$n/2$ bits

- Compute $x_1 y_1$, $x_0 y_0$ and $(x_0 + x_1)(y_0 + y_1)$ recursively and then compute $(x_1 y_0 + x_0 y_1)$ by subtraction.
- Strategy: simple arithmetic manipulations.

Final Algorithm

```
Recursive-Multiply(x,y):
  Write x = x_1 · 2^{n/2} + x_0
        y = y_1 · 2^{n/2} + y_0
  Compute x_1 + x_0 and y_1 + y_0
  p = Recursive-Multiply(x_1 + x_0, y_1 + y_0)
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$$T(n) \leq 3T(n/2) + cn$$

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$$\begin{aligned} T(n) &\leq 3T(n/2) + cn \\ &\leq O(n^{\log_2 3}) = O(n^{1.59}) \end{aligned}$$

Computational Geometry

- Algorithms for geometric objects: points, lines, segments, triangles, spheres, polyhedra, ldots.
- Started in 1975 by Shamos and Hoey.
- Problems studied have applications in a vast number of fields: ecology, molecular biology, statistics, computational finance, computer graphics, computer vision, . . .

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INSTANCE: A set P of n points in the plane

SOLUTION: The pair of points in P that are the closest to each other.

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- At first glance, it seems any algorithm must take $\Omega(n^2)$ time.
- Shamos and Hoey figured out an ingenious $O(n \log n)$ divide and conquer algorithm.

Closest Pair: Set-up

- Let $P = \{p_1, p_2, \ldots, p_n\}$ with $p_i = (x_i, y_i)$.
- Use $d(p_i, p_j)$ to denote the Euclidean distance between p_i and p_j. For a specific pair of points, can compute $d(p_i, p_j)$ in $O(1)$ time.
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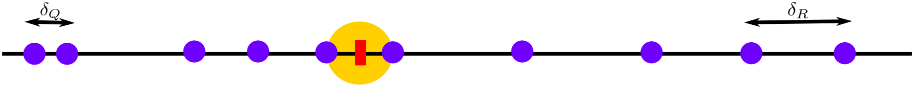

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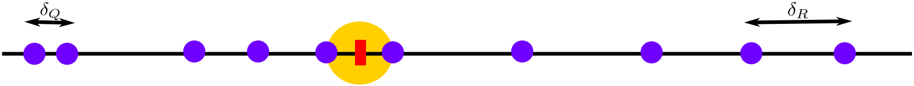
- Generalize the second idea to 2D.



Closest Pair: Algorithm Skeleton

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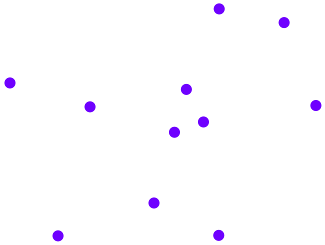

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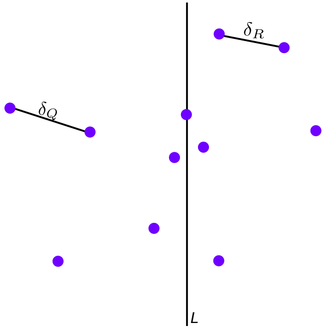
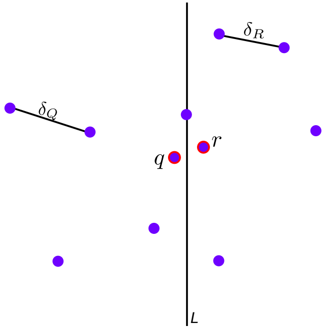
3. Let δ_Q be the distance computed for Q, δ_R be the distance computed for R, and $\delta = \min(\delta_Q, \delta_R)$.



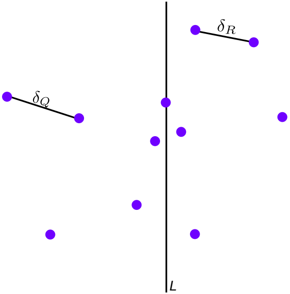
Closest Pair: Algorithm Skeleton

1. Divide P into two sets Q and R of $n/2$ points such that each point in Q has x-coordinate less than any point in R.
2. Recursively compute closest pair in Q and in R, respectively.
3. Let δ_Q be the distance computed for Q, δ_R be the distance computed for R, and $\delta = \min(\delta_Q, \delta_R)$.
4. Compute pair (q, r) of points such that $q \in Q$, $r \in R$, $d(q, r) < \delta$ and $d(q, r)$ is the smallest possible.



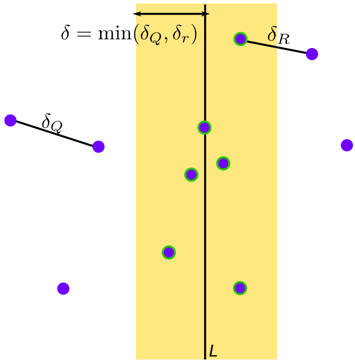
Closest Pair: Proof Sketch

- Prove by induction: Let (s, t) be the closest pair.
  (i) both are in Q: computed correctly by recursive call.
  (ii) both are in R: computed correctly by recursive call.
  (iii) one is in Q and the other is in R: computed correctly in $O(n)$ time by the procedure we will discuss.
- Strategy: Pairs of points for which we do not compute the distance between cannot be the closest pair.
- Overall running time is $O(n \log n)$.



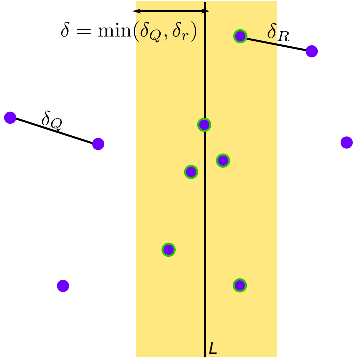
Closest Pair: Conquer Step

- Line L passes through right-most point in Q.
- Let S be the set of points within distance δ of L. (In image, $\delta = \delta_R$.)



Closest Pair: Conquer Step

- Line L passes through right-most point in Q.
- Let S be the set of points within distance δ of L. (In image, $\delta = \delta_R$.)
- Claim: There exist $q \in Q$, $r \in R$ such that $d(q, r) < \delta$ if and only if $q, r \in S$.



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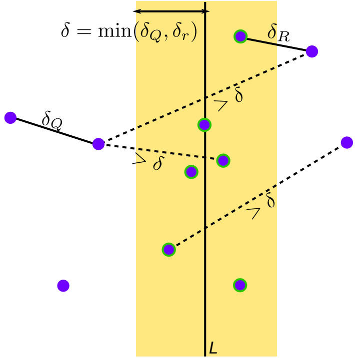
- Claim: There exist $q \in Q$, $r \in R$ such that $d(q, r) < \delta$ if and only if $q, r \in S$.
- Corollary: If $t \in Q - S$ or $u \in R - S$, then (t, u) cannot be the closest pair.



Closest Pair: Packing Argument

- Intuition: "too many" points in S that are closer than δ to each other $\Rightarrow$ there must be a pair in Q or in R that are less than δ apart.

Closest Pair: Packing Argument

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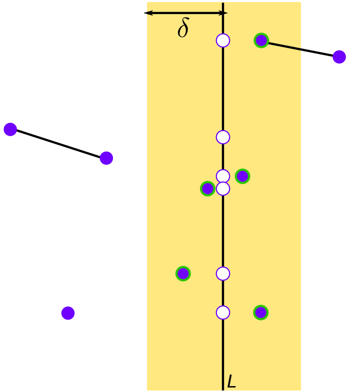
- Let S_y denote the set of points in S sorted by increasing y-coordinate and let s_y denote the y-coordinate of a point $s \in S$.



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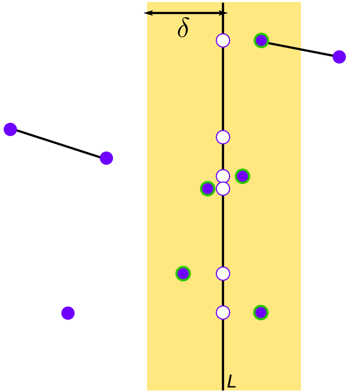
- Let S_y denote the set of points in S sorted by increasing y-coordinate and let s_y denote the y-coordinate of a point $s \in S$.
- Claim: If there exist $s, s' \in S$ such that $d(s, s') < \delta$ then s and s' are at most 15 indices apart in S_y.



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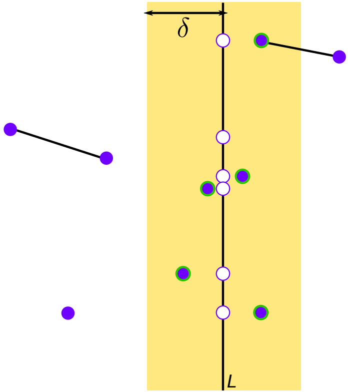
- Converse of the claim: If there exist $s, s' \in S$ such that s' appears 16 or more indices after s in S_y, then $s'_y - s_y \geq \delta$.



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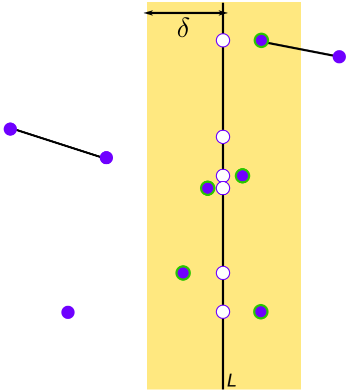
- Use the claim in the algorithm: For every point $s \in S_y$, compute distances only to the next 15 points in S_y.
- Other pairs of points cannot be candidates for the closest pair



Closest Pair: Proof of Packing Argument

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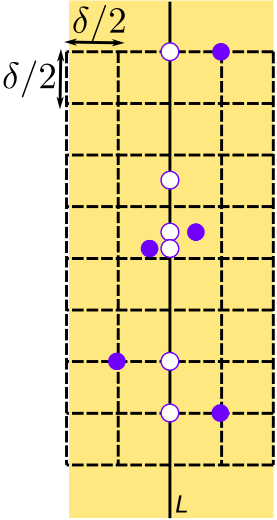
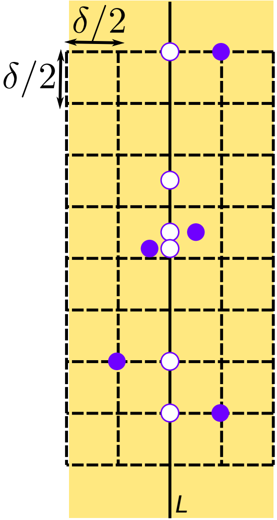


Closest Pair: Proof of Packing Argument

- Claim: If there exist $s, s' \in S$ such that s' appears 16 or more indices after s in S_y, then $s'_y - s_y \geq \delta$.
- Pack the plane with squares of side $\delta/2$.



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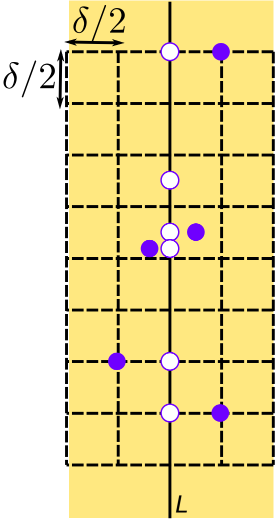
- Each square contains at most one point.



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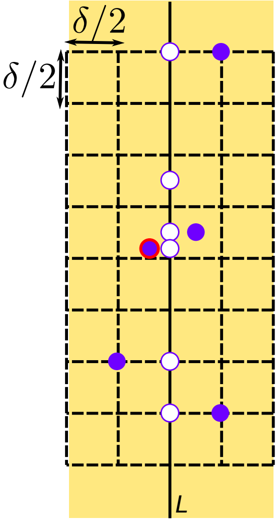
- Let s lie in one of the squares.



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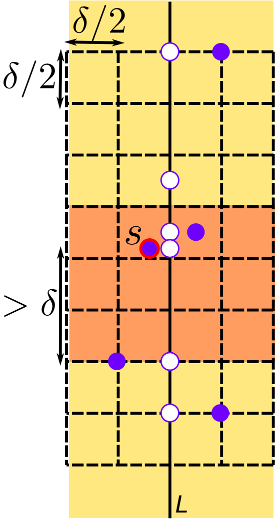
- Pack the plane with squares of side $\delta/2$.
- Each square contains at most one point.
- Let s lie in one of the squares.
- Any point in the third row of the packing below s has a y-coordinate at least δ more than s_y.



Closest Pair: Proof of Packing Argument

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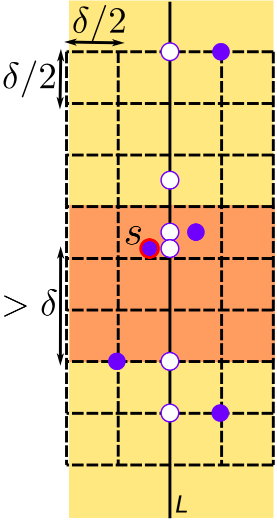
- Each square contains at most one point.
- Let s lie in one of the squares.
- Any point in the third row of the packing below s has a y-coordinate at least δ more than s_y.
- We get a count of 12 or more indices (textbook says 16).



Closest Pair: Final Algorithm

```
Closest-Pair(P)
  Construct Px and Py  (O(n log n) time)
  (p0*, p1*) = Closest-Pair-Rec(Px,Py)

Closest-Pair-Rec(Px, Py)
  If |P| ≤ 3 then
    find closest pair by measuring all pairwise distances
  Endif

  Construct Qx, Qy, Rx, Ry (O(n) time)
  (q0*,q1*) = Closest-Pair-Rec(Qx, Qy)
  (r0*,r1*) = Closest-Pair-Rec(Rx, Ry)

  δ = min(d(q0*,q1*), d(r0*,r1*))
  x* = maximum x-coordinate of a point in set Q
  L = {(x,y) : x = x*}
  S = points in P within distance δ of L.

  Construct Sy (O(n) time)
  For each point s ∈ Sy, compute distance from s
     to each of next 15 points in Sy
     Let s, s' be pair achieving minimum of these distances
     (O(n) time)

  If d(s,s') < δ then
     Return (s,s')
  Else if d(q0*,q1*) < d(r0*,r1*) then
     Return (q0*,q1*)
  Else
     Return (r0*,r1*)
  Endif
```

Closest Pair: Final Algorithm

Closest-Pair(P)

  Construct P_x and P_y ($O(n \log n)$ time)

  (p_0^*, p_1^*) = Closest-Pair-Rec(P_x,P_y)

Closest-Pair-Rec(P_x, P_y)

  If $|P| \leq 3$ then

    find closest pair by measuring all pairwise distances

  Endif

  Construct Q_x, Q_y, R_x, R_y ($O(n)$ time)

  (q_0^*, q_1^*) = Closest-Pair-Rec(Q_x, Q_y)

  (r_0^*, r_1^*) = Closest-Pair-Rec(R_x, R_y)

  $\delta = \min(d(q_0^*, q_1^*), d(r_0^*, r_1^*))$

  x^* = maximum x-coordinate of a point in set Q

Closest Pair: Final Algorithm

```
x* = maximum x-coordinate of a point in set Q
L = {(x,y) : x = x*}
S = points in P within distance δ of L.

Construct S_y (O(n) time)
For each point s ∈ S_y, compute distance from s
   to each of next 15 points in S_y
   Let s, s' be pair achieving minimum of these distances
   (O(n) time)

If d(s,s') < δ then
   Return (s,s')
Else if d(q0*,q1*) < d(r0*,r1*) then
   Return (q0*,q1*)
Else
   Return (r0*,r1*)
Endif
```