Analysis of Algorithms

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January 23, 2017

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What is Algorithm Analysis?

- Measure resource requirements: how does the amount of time and space an algorithm uses scale with increasing input size?
- How do we put this notion on a concrete footing?
- What does it mean for one function to grow faster or slower than another?

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Goal

Develop algorithms that provably run quickly and use low amounts of space.

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- Why worst-case? Why not average-case or on random inputs?
- *Input size* = number of elements in the input. *Values* in the input do not matter, except for specific algorithms.
- Assume all elementary operations take unit time: assignment, arithmetic on a fixed-size number, comparisons, array lookup, following a pointer, etc.

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- An algorithm has a *polynomial* running time if there exist constants $c > 0$ and $d > 0$ such that on every input of size n, the running time of the algorithm is bounded by cn^d steps.

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Definition

An algorithm is *efficient* if it has a polynomial running time.

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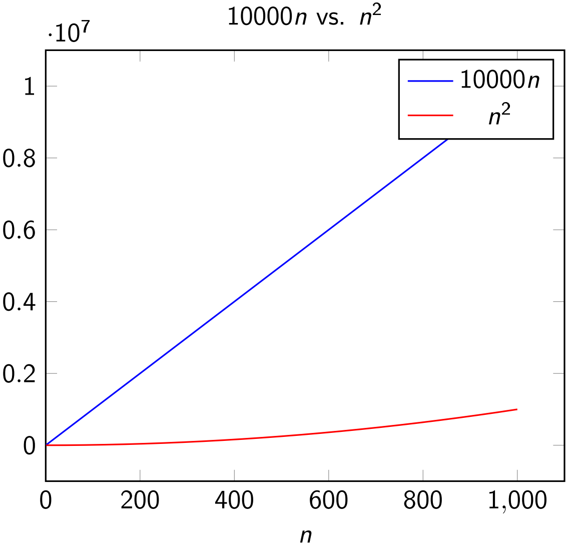
- How can make statements such as the following, in order to compare the running times of different algorithms?
  - $100n \log_2 n \leq n^2$
  - $10000n \leq n^2$
  - $5n^2 - 4n \geq 1000n \log n$

"$10000n \leq n^2$"



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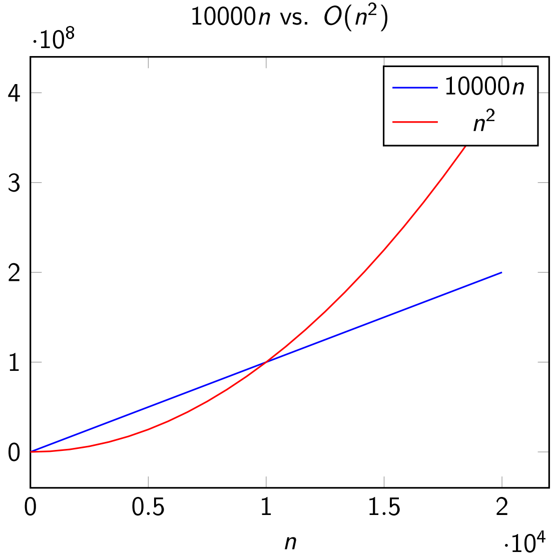

Upper Bound

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Asymptotic upper bound: A function $f(n)$ is $O(g(n))$ if

for all n , $f(n) \leq$ $g(n)$.

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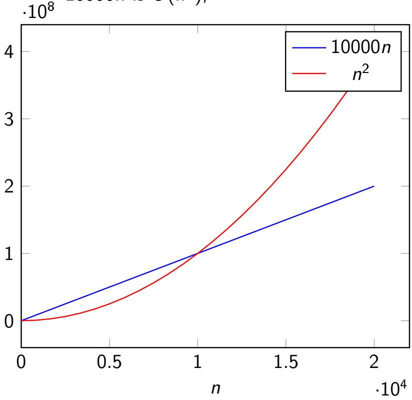

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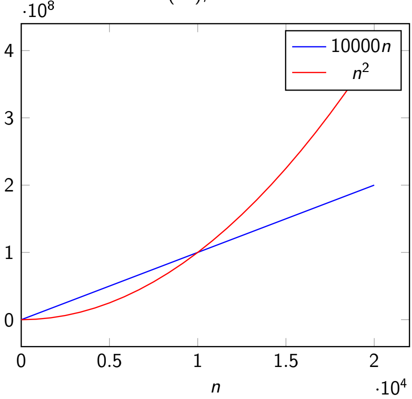

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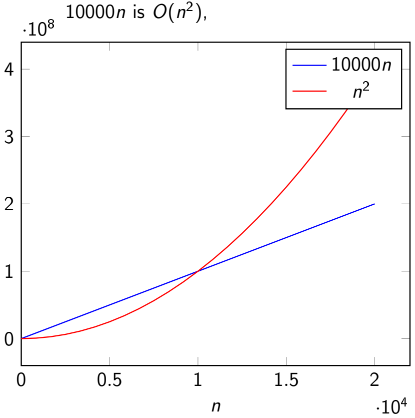

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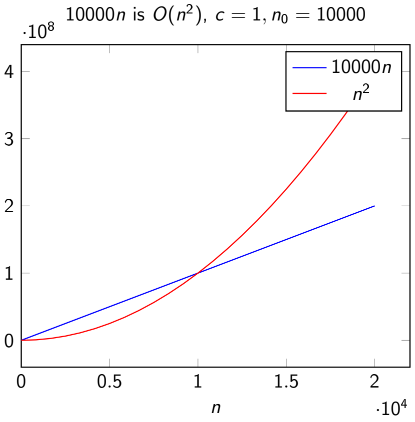

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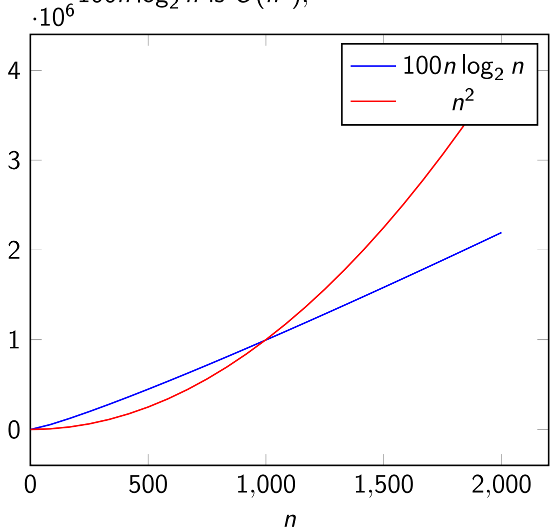

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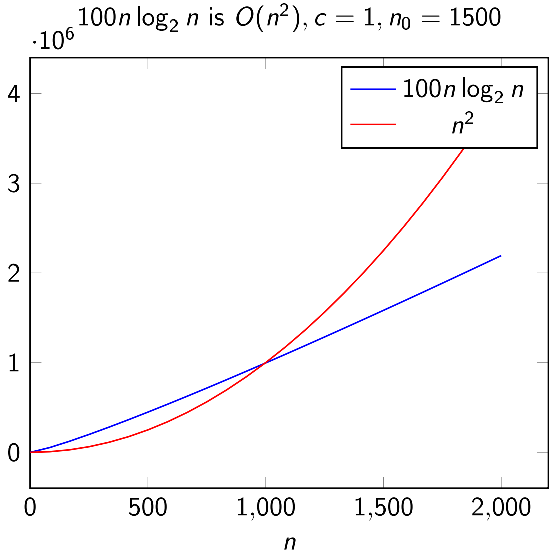

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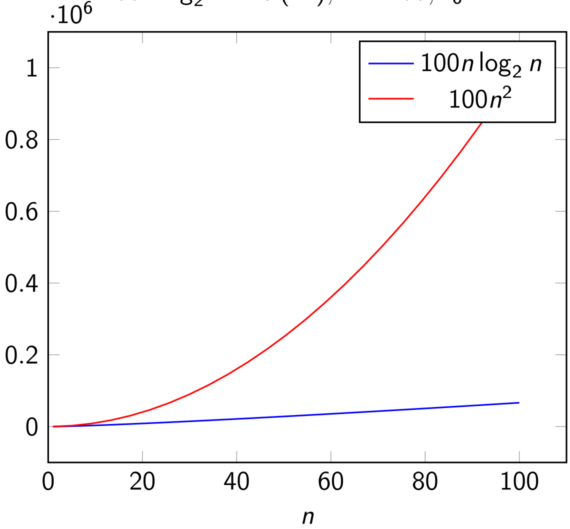
$100n \log_2 n$ is $O(n^2)$, $c = 100$, $n_0 = 1$



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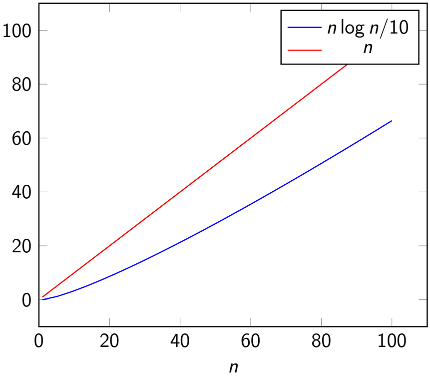
$n \log_2 n/10$ and $\Omega(n)$



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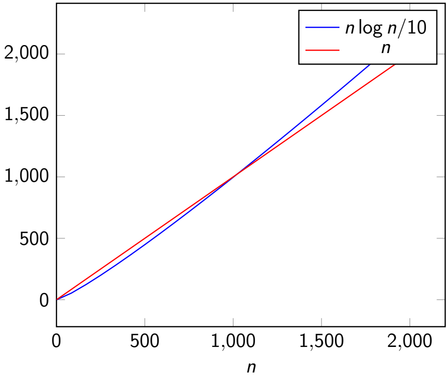

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- Problems: The problem of sorting n numbers has a lower bound of $\Omega(n \log n)$. For *any* comparison-based sorting algorithm, there is at least one input for which that algorithm will take $\Omega(n \log n)$ steps.

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- In all these definitions, c and n_0 are constants independent of n.
- Abuse of notation: say $g(n) = O(f(n))$, $g(n) = \Omega(f(n))$, $g(n) = \Theta(f(n))$.

Properties of Asymptotic Growth Rates

Transitivity

- If $f = O(g)$ and $g = O(h)$, then $f = O(h)$.
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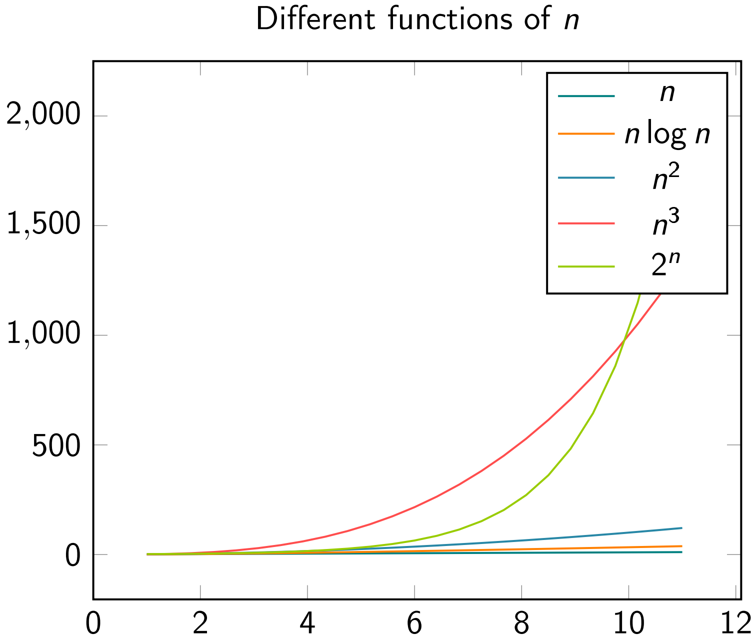
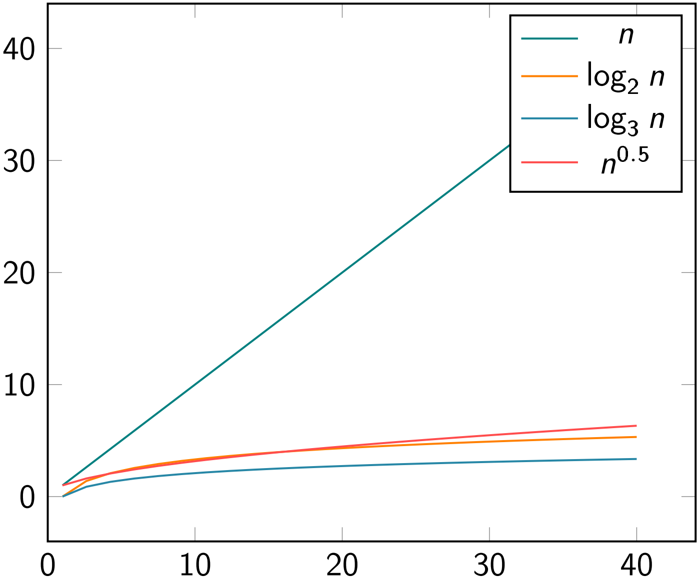
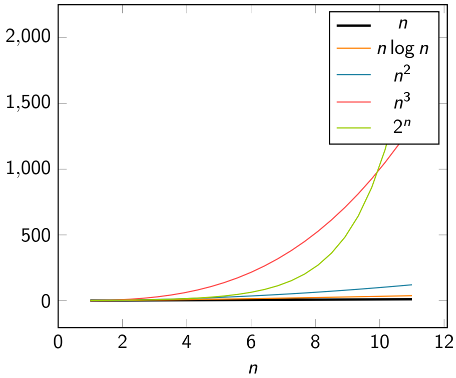
- For every constant $r > 1$ and every constant $d > 0$, $n^d = O(r^n)$.


Different functions of n
2,000
1,500
1,000
500
0
0
2
4
6
8
10
12
n
n log n
n2
n3
2n

More functions of n



Linear Time



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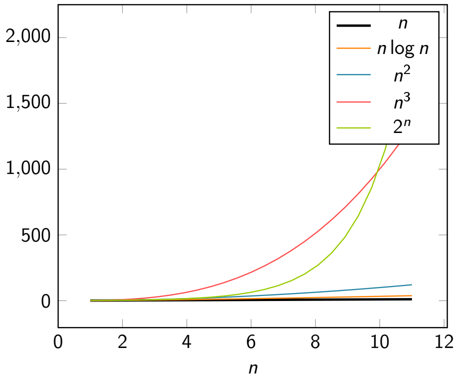

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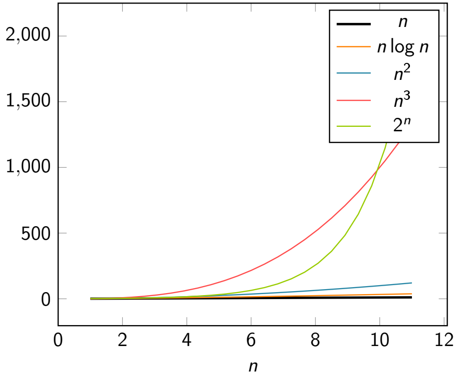

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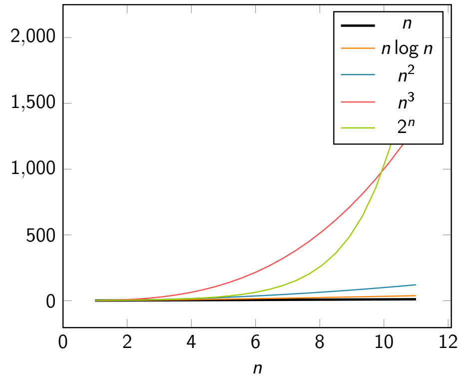

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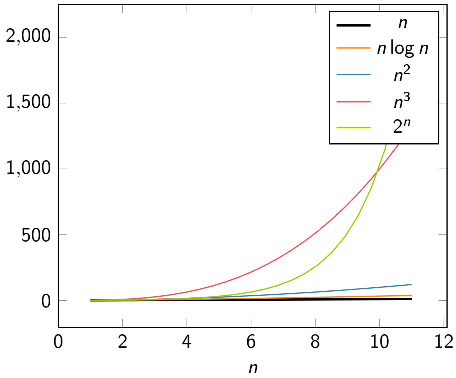

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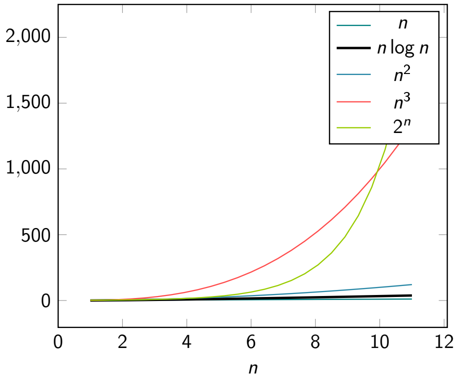
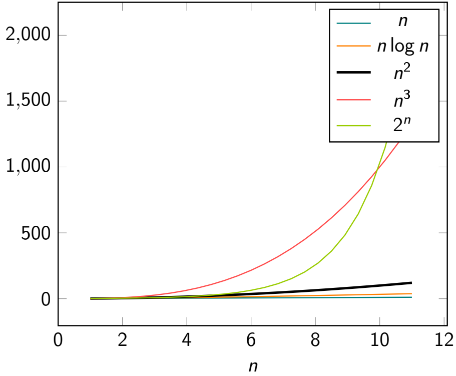
- Finding the minimum, merging two sorted lists.
- Computing the median (or kth smallest) element in an *unsorted* list. "Median-of-median" algorithm.
- Sub-linear time. Binary search in a sorted array of n numbers takes $O(\log n)$ time.

$O(n \log n)$ Time



- Any algorithm where the costliest step is sorting.

Quadratic Time



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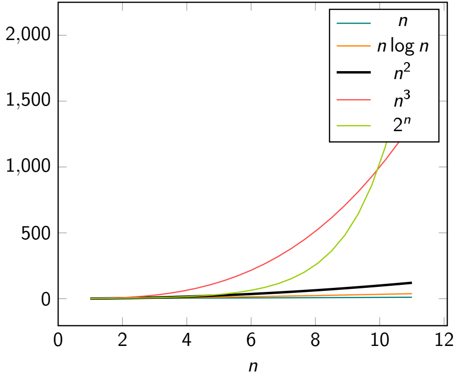
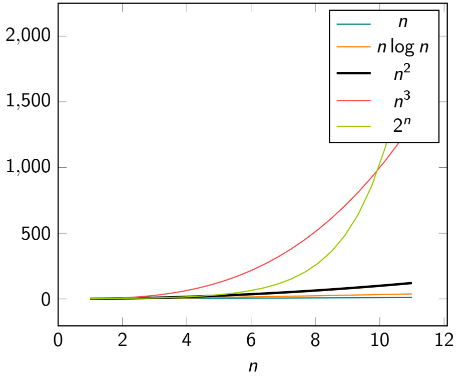


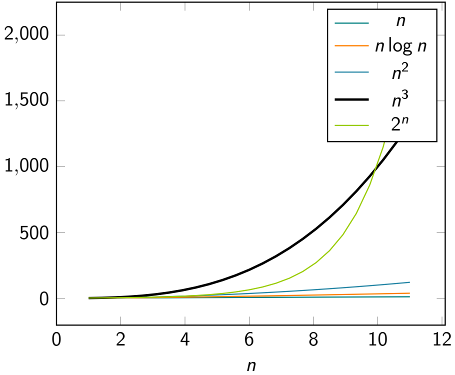
- Enumerate all pairs of elements.
- Given a set of n points in the plane, find the pair that are the closest.

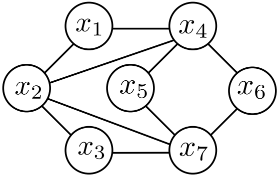
Quadratic Time



- Enumerate all pairs of elements.
- Given a set of n points in the plane, find the pair that are the closest. Surprising fact: will solve this problem in $O(n \log n)$ time later in the semester.

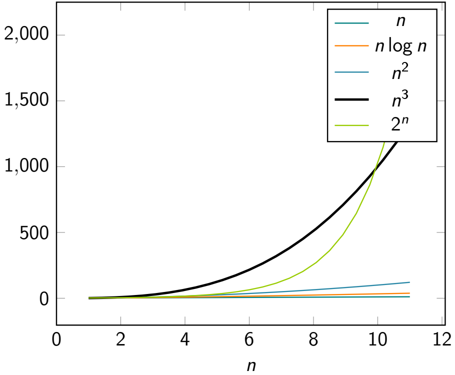
$O(n^k)$ Time

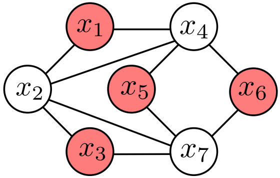




- Does a graph have an independent set of size k, where k is a constant, i.e. there are k nodes such that no two are joined by an edge?

$O(n^k)$ Time





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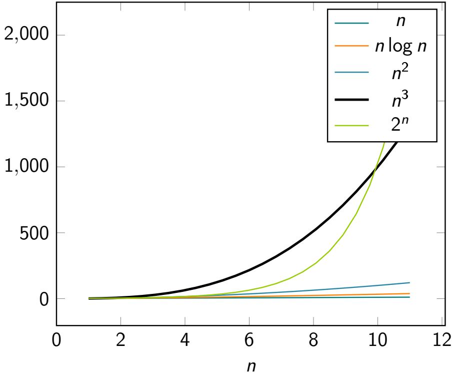
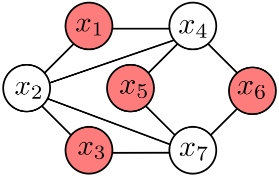
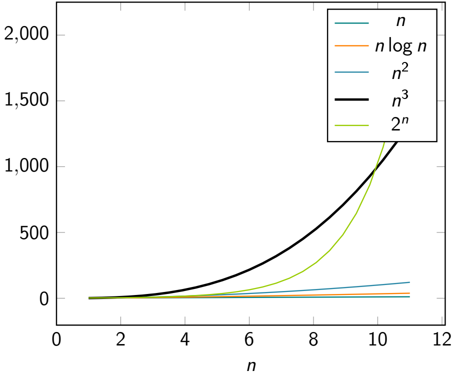
$O(n^k)$ Time

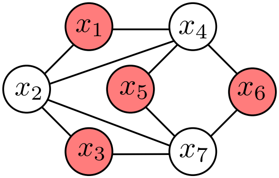




- Does a graph have an independent set of size k, where k is a constant, i.e. there are k nodes such that no two are joined by an edge?
- Algorithm: For each subset S of k nodes, check if S is an independent set. If the answer is yes, report it.

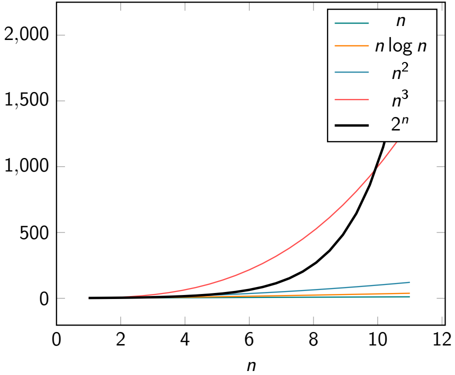
$O(n^k)$ Time

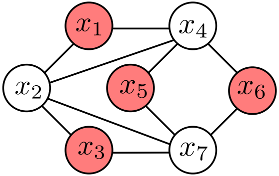




- Does a graph have an independent set of size k, where k is a constant, i.e. there are k nodes such that no two are joined by an edge?
- Algorithm: For each subset S of k nodes, check if S is an independent set. If the answer is yes, report it.
- Running time is $O(k^2\binom{n}{k}) = O(n^k)$.

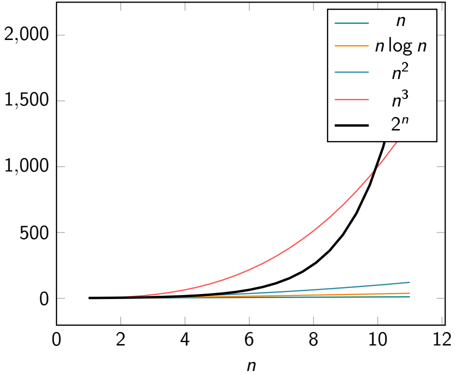
Beyond Polynomial Time

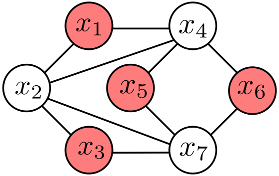




- What is the largest size of an independent set in a graph with n nodes?

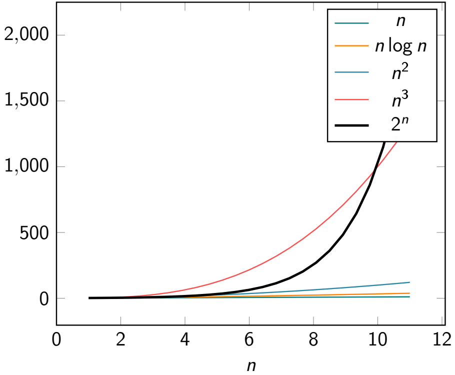
Beyond Polynomial Time

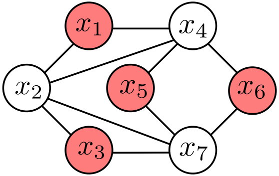




- What is the largest size of an independent set in a graph with n nodes?
- Algorithm: For each $1 \leq i \leq n$, check if the graph has an independent size of size i. Output largest independent set found.

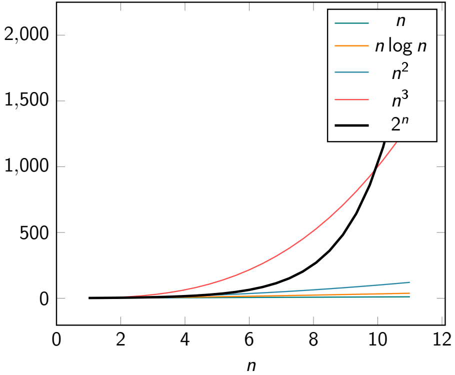
Beyond Polynomial Time

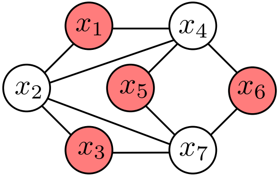




- What is the largest size of an independent set in a graph with n nodes?
- Algorithm: For each $1 \leq i \leq n$, check if the graph has an independent size of size i. Output largest independent set found.
- What is the running time?

Beyond Polynomial Time





- What is the largest size of an independent set in a graph with n nodes?
- Algorithm: For each $1 \le i \le n$, check if the graph has an independent size of size i. Output largest independent set found.
- What is the running time? $O(n^2 2^n)$.