NP and Computational Intractability

T. M. Murali

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Algorithm Design

- Patterns
  - Greed. $O(n \log n)$ interval scheduling.
  - Divide-and-conquer. $O(n \log n)$ closest pair of points.
  - Dynamic programming. $O(n^3)$ RNA folding.
  - Duality. $O(n^3)$ maximum flow and minimum cuts.

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  - Local search.
  - Randomization.

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  - Reductions. $\textsc{Image segmentation} \leq_P \textsc{Minimum}$ s-t $\textsc{cut}$
  - Local search.
  - Randomization.
- "Anti-patterns"
  - NP-completeness. $O(n^k)$ algorithm unlikely.
  - PSPACE-completeness. $O(n^k)$ certification algorithm unlikely.
  - Undecidability. No algorithm possible.

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Polynomial time	**Probably not**
Shortest path	Longest path
Matching	3-D matching
Minimum cut	Maximum cut
2-SAT	3-SAT
Planar four-colour	Planar three-colour
Bipartite vertex cover	Vertex cover
Primality testing	Factoring

Problem Classification

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- Some extremely hard problems cannot be solved efficiently (e.g., chess on an n-by-n board).
- However, classification is unclear for a very large number of discrete computational problems.
- We can prove that these problems are fundamentally equivalent and are manifestations of the same problem!

Polynomial-Time Reduction

- Goal is to express statements of the type "Problem X is at least as hard as problem Y."
- Use the notion of *reductions*.
- *Y is polynomial-time reducible to X ($Y \leq_P X$)*

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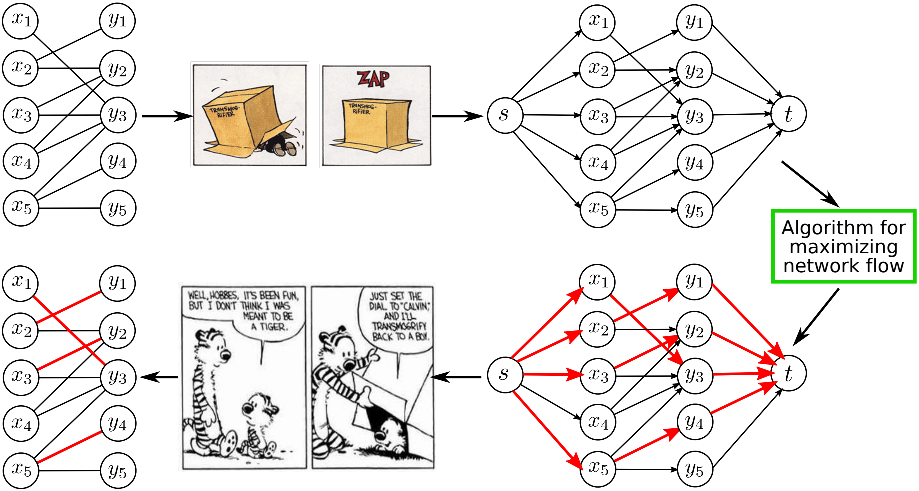
Polynomial-Time Reduction



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  - Image Segmentation $\leq_P$ Minimum s-t Cut

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  - MAXIMUM BIPARTITE MATCHING $\leq_P$ MAXIMUM s-t FLOW
  - IMAGE SEGMENTATION $\leq_P$ MINIMUM s-t CUT
- $Y \leq_P X$ implies that "X is at least as hard as Y."
- Such reductions are *Karp reductions*. *Cook reductions* allow a polynomial number of calls to the black box that solves X.

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- Claim: If $Y \leq_P X$ and X can be solved in polynomial time, then Y can be solved in polynomial time.

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- Claim: If $Y \leq_P X$ and X can be solved in polynomial time, then Y can be solved in polynomial time.
- Contrapositive: If $Y \leq_P X$ and Y cannot be solved in polynomial time, then X cannot be solved in polynomial time.
- Informally: If Y is hard, and we can show that Y reduces to X, then the hardness "spreads" to X.

Reduction Strategies

- Simple equivalence.
- Special case to general case.
- Encoding with gadgets.

Optimisation versus Decision Problems

- So far, we have developed algorithms that solve optimisation problems.
  - Compute the *largest* flow.
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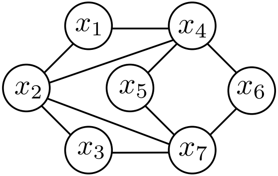
- So far, we have developed algorithms that solve optimisation problems.
  - Compute the *largest* flow.
  - Find the *closest* pair of points.
  - Find the schedule with the *least* completion time.
- Now, we will focus on *decision versions* of problems, e.g., is there a flow with value at least k, for a given value of k?
- Decision problem: answer to every input is `yes` or `no`.

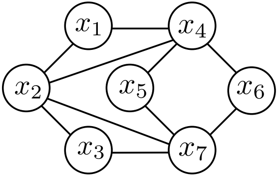
PRIMES

INSTANCE: A natural number n

QUESTION: Is n prime?

Independent Set and Vertex Cover





- Given an undirected graph $G(V, E)$, a subset $S \subseteq V$ is an *independent set* if no two vertices in S are connected by an edge.
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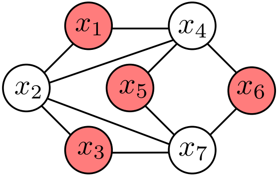

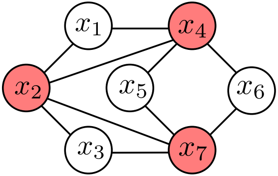

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INDEPENDENT SET

INSTANCE: Undirected graph G and an integer k

QUESTION: Does G contain an independent set of size $\geq k$?

VERTEX COVER

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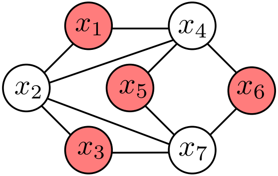

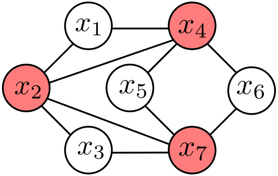

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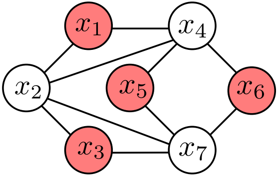

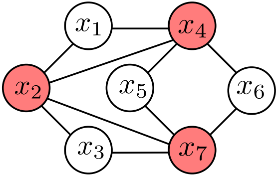

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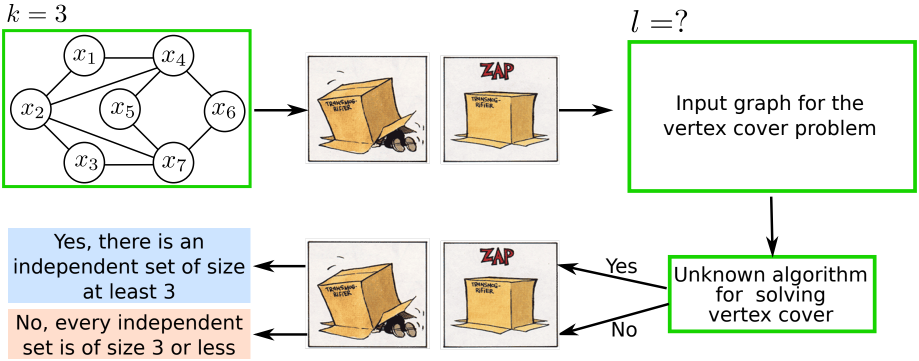
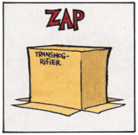
- Demonstrate simple equivalence between these two problems.
- Claim: INDEPENDENT SET $\leq_P$ VERTEX COVER and VERTEX COVER $\leq_P$ INDEPENDENT SET.

Strategy for Proving Indep. Set $\leq_P$ Vertex Cover



Strategy for Proving Indep. Set $\leq_P$ Vertex Cover

1. Start with an arbitrary instance of INDEPENDENT SET: an undirected graph $G(V, E)$ and an integer k.
2. From $G(V, E)$ and k, create an instance of VERTEX COVER: an undirected graph $G'(V', E')$ and an integer l.
   - G' related to G in some way.
   - l can depend upon k and size of G.



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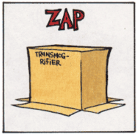

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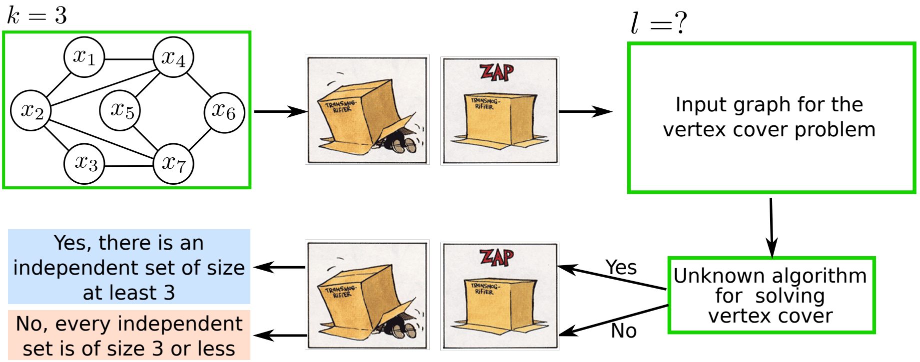
- Transformation and proof must be correct for all possible graphs $G(V, E)$ and all possible values of k.
- Why is the proof an iff statement?

Reason for Two-Way Proof



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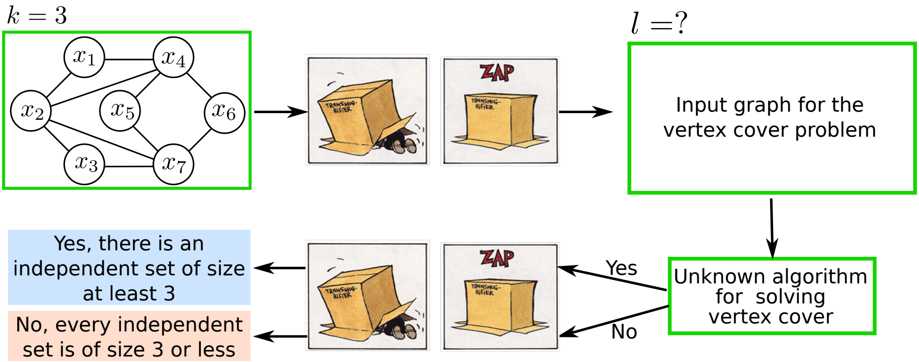
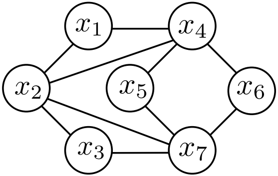
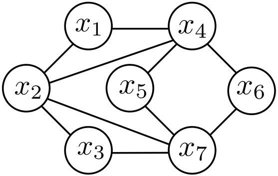


- Why is the proof an iff statement? In the reduction, we are using black box for Vertex Cover to solve Independent Set.
  (i) If there is an independent set size $\geq k$, we must be sure that there is a vertex cover of size $\leq l$, so that we know that the black box will find this vertex cover.
  (ii) If the black box finds a vertex cover of size $\leq l$, we must be sure we can construct an independent set of size $\geq k$ from this vertex cover.

Proof that Independent Set $\leq_P$ Vertex Cover





1. Arbitrary instance of INDEPENDENT SET: an undirected graph $G(V, E)$ and an integer k.
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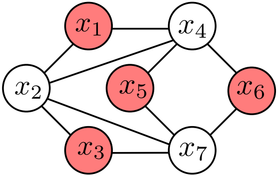

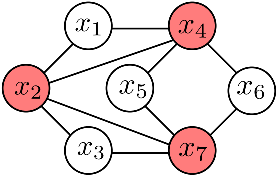

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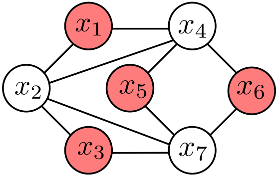

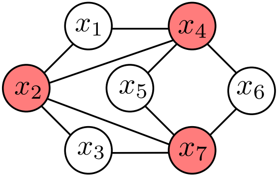

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- Same idea proves that VERTEX COVER $\leq_P$ INDEPENDENT SET

Vertex Cover and Set Cover

- INDEPENDENT SET is a "packing" problem: pack as many vertices as possible, subject to constraints (the edges).
- VERTEX COVER is a "covering" problem: cover all edges in the graph with as few vertices as possible.
- There are more general covering problems.

SET COVER

INSTANCE: A set U of n elements, a collection $S_1, S_2, \ldots, S_m$ of subsets of U, and an integer k.

QUESTION: Is there a collection of $\leq k$ sets in the collection whose union is U?

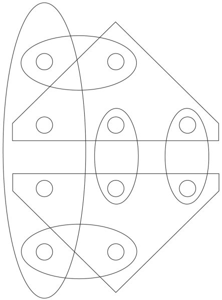
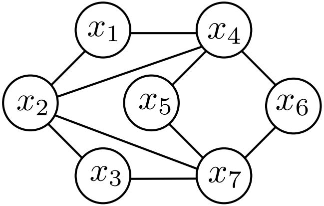


Figure 8.2 An instance of the Set Cover Problem.

Vertex Cover $\leq_P$ Set Cover



- Input to VERTEX COVER: an undirected graph $G(V, E)$ and an integer k.
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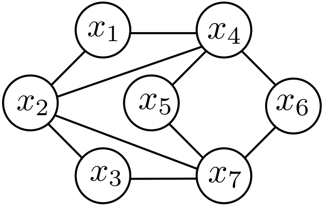

$$U = \{(x_1,x_2),(x_1,x_4),(x_2,x_3),(x_2,x_4),(x_2,x_7),(x_3,x_7),(x_4,x_5),(x_5,x_6),(x_5,x_7),(x_6,x_7)\}$$

$$S_1 = \{(x_1,x_2),(x_1,x_4)\}$$

$$S_2 = \{(x_1,x_2),(x_2,x_3),(x_2,x_4),(x_2,x_7)\}$$

S_3, S_4, S_5, S_6, and S_7 defined similarly.

- Input to VERTEX COVER: an undirected graph $G(V,E)$ and an integer k.
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  - $U = E$,
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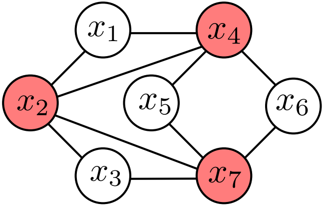
Vertex Cover $\leq_P$ Set Cover



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- Input to Vertex Cover: an undirected graph $G(V, E)$ and an integer k.
- Let $|V| = n$.
- Create an instance $\{U, \{S_1, S_2, \ldots S_n\}\}$ of Set Cover where
  - $U = E$,
  - for each vertex $i \in V$, create a set $S_i \subseteq U$ of the edges incident on i.
- Claim: U can be covered with fewer than k subsets iff G has a vertex cover with at most k nodes.
- Proof strategy:
  1. If $G(V, E)$ has a vertex cover of size at most k, then U can be covered with at most k subsets.
  2. If U can be covered with at most k subsets, then $G(V, E)$ has a vertex cover of size at most k.

Boolean Satisfiability

- Abstract problems formulated in Boolean notation.

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- Given a set $X = \{x_1, x_2, \ldots, x_n\}$ of n Boolean variables.
- Each variable can take the value 0 or 1.
- *Term*: a variable x_i or its negation $\overline{x_i}$.
- *Clause* of *length* l: (`or`) of l distinct terms $t_1 \vee t_2 \vee \cdots t_l$.
- *Truth assignment* for X: is a function $\nu : X \to \{0, 1\}$.
- An assignment ν *satisfies* a clause C if it causes at least one term in C to evaluate to 1 (since C is an `or` of terms).
- An assignment *satisfies* a collection of clauses $C_1, C_2, \ldots C_k$ if it causes all clauses to evaluate to 1, i.e., $C_1 \wedge C_2 \wedge \cdots C_k = 1$.
  - ν is a *satisfying assignment* with respect to $C_1, C_2, \ldots C_k$.
  - set of clauses $C_1, C_2, \ldots C_k$ is *satisfiable*.

Example

- $X = \{x_1, x_2, x_3, x_4\}$
- Terms: $x_1, \overline{x_1}, x_2, \overline{x_2}, x_3, \overline{x_3}, x_4, \overline{x_4}$

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- Assignment: $x_1 = 1, x_2 = 0, x_3 = 1, x_4 = 0$

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  - Not a satisfying assignment
- Assignment: $x_1 = 1, x_2 = 0, x_3 = 0, x_4 = 0$

Example

- $X = \{x_1, x_2, x_3, x_4\}$
- Terms: $x_1, \overline{x_1}, x_2, \overline{x_2}, x_3, \overline{x_3}, x_4, \overline{x_4}$
- Clauses:

$$x_1 \vee \overline{x_2} \vee \overline{x_3}$$
$$x_2 \vee \overline{x_3} \vee x_4$$
$$x_3 \vee \overline{x_4}$$

- Assignment: $x_1 = 1, x_2 = 0, x_3 = 1, x_4 = 0$

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SAT and 3-SAT

SATISFIABILITY PROBLEM (SAT)

INSTANCE: A set of clauses $C_1, C_2, \ldots C_k$ over a set $X = \{x_1, x_2, \ldots x_n\}$ of n variables.

QUESTION: Is there a satisfying truth assignment for X with respect to C?

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3-SATISFIABILITY PROBLEM (SAT)

INSTANCE: A set of clauses $C_1, C_2, \ldots C_k$, each of length three, over a set $X = \{x_1, x_2, \ldots x_n\}$ of n variables.

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- SAT and 3-SAT are fundamental combinatorial search problems.
- We have to make n independent decisions (the assignments for each variable) while satisfying a set of constraints.
- Satisfying each constraint in isolation is easy, but we have to make our decisions so that all constraints are satisfied simultaneously.

Examples of 3-SAT

Example:

- $C_1 = x_1 \vee 0 \vee 0$
- $C_2 = x_2 \vee 0 \vee 0$
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4. Is $C_1 \wedge C_2 \wedge C_3$ satisfiable? No.

3-SAT and Independent Set

$$C_1 = x_1 \vee \overline{x_2} \vee \overline{x_3}$$
$$C_2 = \overline{x_1} \vee x_2 \vee x_4$$
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- We want to prove 3-SAT $\leq_P$ Independent Set.

3-SAT and Independent Set

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- Two ways to think about 3-SAT:
  1. Make an independent 0/1 decision on each variable and succeed if we achieve one of three ways in which to satisfy each clause.

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1. Select $x_1 = 1, x_2 = 1, x_3 = 1, x_4 = 1$.
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   - Choices of selected literals imply $x_1 = 0, x_2 = 0, x_4 = 1$.

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Proving 3-SAT $\leq_P$ Independent Set

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- Construct an instance of independent set: graph $G(V, E)$ with $3k$ nodes.

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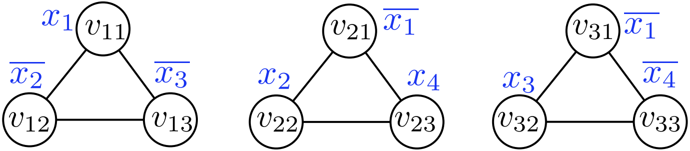

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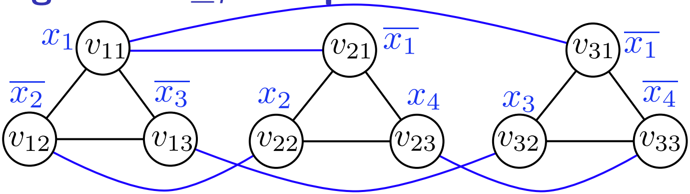

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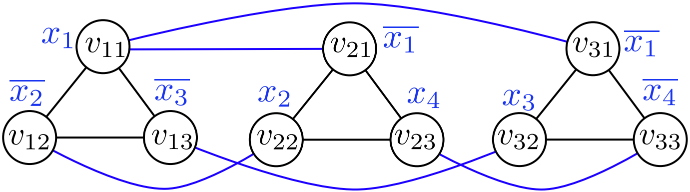

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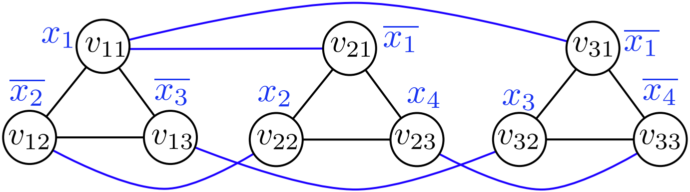

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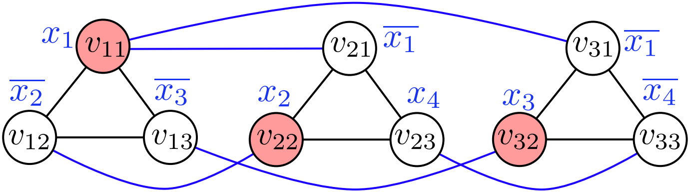

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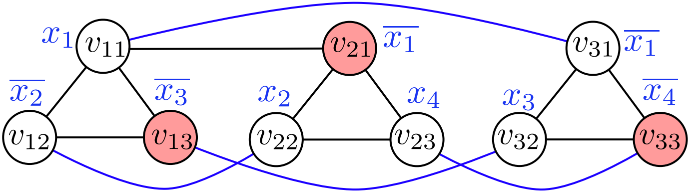

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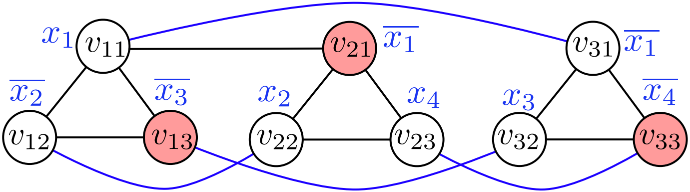

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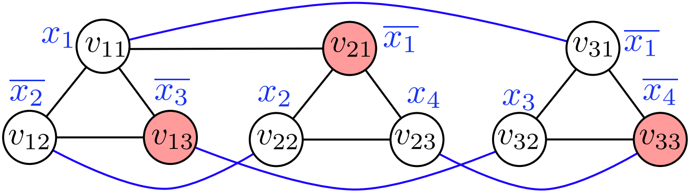

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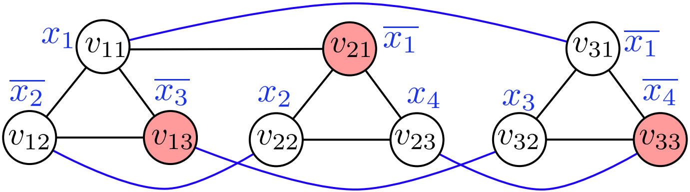

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  - If x_i is the label of a node in S, set $x_i = 1$; else set $x_i = 0$.
  - Why is each clause satisfied?

Transitivity of Reductions

- Claim: If $\mathrm{Z} \leq_P \mathrm{Y}$ and $\mathrm{Y} \leq_P \mathrm{X}$, then $\mathrm{Z} \leq_P \mathrm{X}$.

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- We have shown

$$\text{3-SAT} \leq_P \textsc{Independent Set} \leq_P \textsc{Vertex Cover} \leq_P \textsc{Set Cover}$$

Finding vs. Certifying

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- Is it easy to check if a given set of vertices in an undirected graph forms an independent set of size at least k?
- Is it easy to check if a particular truth assignment satisfies a set of clauses?
- We draw a contrast between *finding* a solution and *checking* a solution (in polynomial time).
- Since we have not been able to develop efficient algorithms to solve many decision problems, let us turn our attention to whether we can check if a proposed solution is correct.

Problems and Algorithms

PRIMES
INSTANCE: A natural number n
QUESTION: Is n prime?

- Decision problem X: for every input s, answer $X(s)$ is `yes` or `no`.

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  - There is an algorithm such that $p(|s|) = |s|^{12}$ for PRIMES (Agarwal, Kayal, Saxena, 2002, improved to $|s|^6$ by Pomerance and Lenstra, 2005).

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- $\mathcal{P}$: set of problems X for which there is a polynomial time algorithm.

A decision problem X is in $\mathcal{P}$ iff there is an algorithm A with polynomial running time that solves X.

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- Certifier does not care about how to find these certificates.

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  - Certifier B: .

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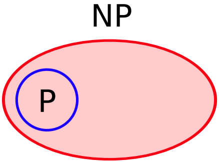

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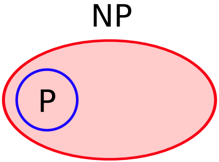

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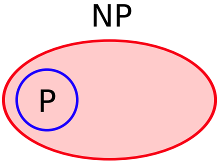

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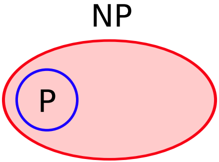

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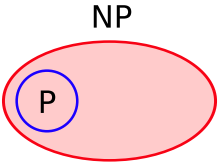
Summary



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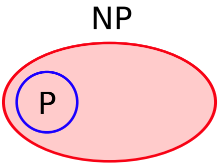

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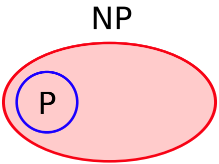

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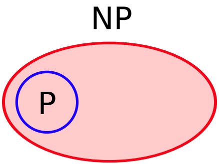

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$\mathcal{NP}$-Complete and $\mathcal{NP}$-Hard Problems

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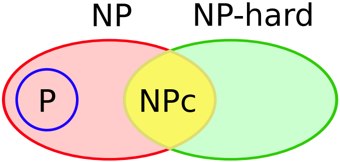

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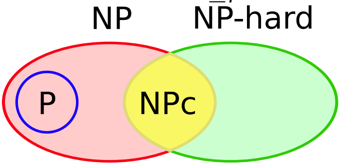

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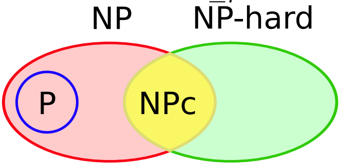

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- Does even one $\mathcal{NP}$-Complete problem exist?! If it does, how can we prove that *every* problem in $\mathcal{NP}$ reduces to this problem?

Circuit Satisfiability

- Cook-Levin Theorem: CIRCUIT SATISFIABILITY is $\mathcal{NP}$-Complete.

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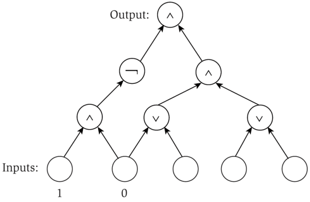

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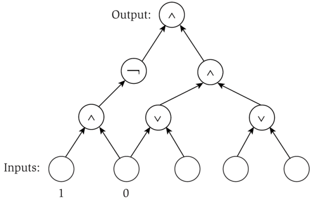

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CIRCUIT SATISFIABILITY

INSTANCE: A circuit K.

QUESTION: Is there a truth assignment to the inputs that causes the output to have value 1?

▸ Skip proof; read textbook or Chapter 2.6 of Garey and Johnson.

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- $s \in X$ iff there is an assignment of the input bits of K that makes K satisfiable.

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- Does a graph G on n nodes have a two-node independent set?

Example of Transformation to Circuit Satisfiability

- Does a graph G on n nodes have a two-node independent set?
- s encodes the graph G with $\binom{n}{2}$ bits.
- t encodes the independent set with n bits.
- Certifier needs to check if
  1. at least two bits in t are set to 1 and
  2. no two bits in t are set to 1 if they form the ends of an edge (the corresponding bit in s is set to 1).

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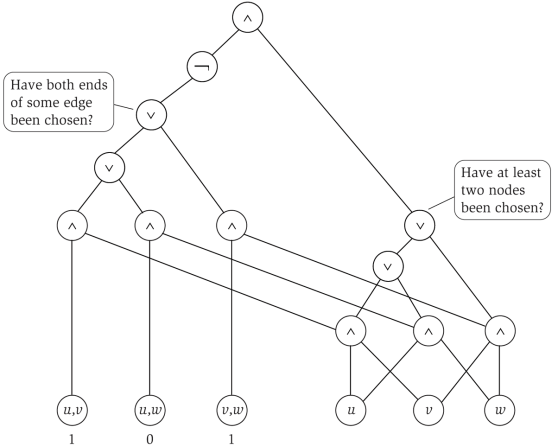

Figure 8.5 A circuit to verify whether a 3-node graph contains a 2-node independent set.

Asymmetry of Certification

- Definition of efficient certification and $\mathcal{NP}$ is fundamentally asymmetric:
  - An input s is a "yes" instance iff there exists a short certificate t such that $B(s, t) = \texttt{yes}$.
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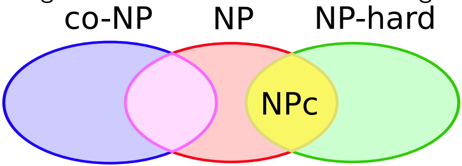

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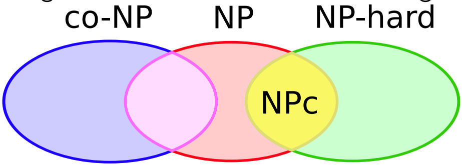

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Good Characterisations: the Class $\mathcal{NP} \cap$ co-$\mathcal{NP}$

- If a problem belongs to both $\mathcal{NP}$ and co-$\mathcal{NP}$, then
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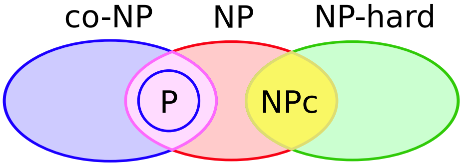
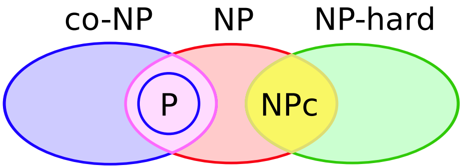


- Claim: $\mathcal{P} \subseteq \mathcal{NP} \cap$ co-$\mathcal{NP}$.

Good Characterisations: the Class $\mathcal{NP} \cap$ co-$\mathcal{NP}$

- If a problem belongs to both $\mathcal{NP}$ and co-$\mathcal{NP}$, then
  - When the answer is yes, there is a short proof.
  - When the answer is no, there is a short proof.
- Problems in $\mathcal{NP} \cap$ co-$\mathcal{NP}$ have a *good characterisation*.
- Example is the problem of determining if a flow network contains a flow of value at least ν, for some given value of ν.
  - Yes: construct a flow of value at least ν.
  - No: demonstrate a cut with capacity less than ν.



- Claim: $\mathcal{P} \subseteq \mathcal{NP} \cap$ co-$\mathcal{NP}$.
- Open problem: Is $\mathcal{P} = \mathcal{NP} \cap$ co-$\mathcal{NP}$?