Greedy Graph Algorithms

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February 18, 23, and 25, 2016

Shortest Paths Problem

- $G(V, E)$ is a connected directed graph. Each edge e has a length $l_e \geq 0$.
- V has n nodes and E has m edges.
- *Length of a path* P is the sum of the lengths of the edges in P.
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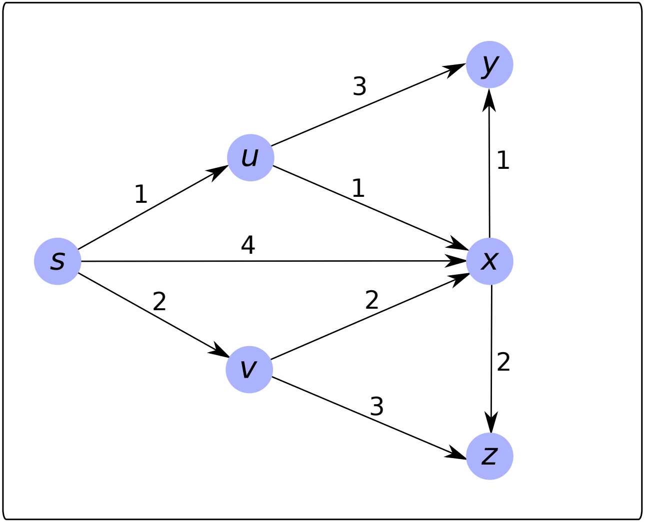
SHORTEST PATHS

INSTANCE: A directed graph $G(V, E)$, a function $l : E \rightarrow \mathbb{R}^+$, and a node $s \in V$

SOLUTION: A set $\{P_u, u \in V\}$, where P_u is the shortest path in G from s to u.

Shortest Paths Problem Instance



Dijkstra's Algorithm

- Maintain a set S of explored nodes.
  - For each node $u \in S$, compute a value $d(u)$, which (we will prove) is the length of the shortest path from s to u.
  - For each node $x \notin S$, maintain a value $d'(x)$, which is the length of the shortest path from s to x using only the nodes in S (and x, of course).

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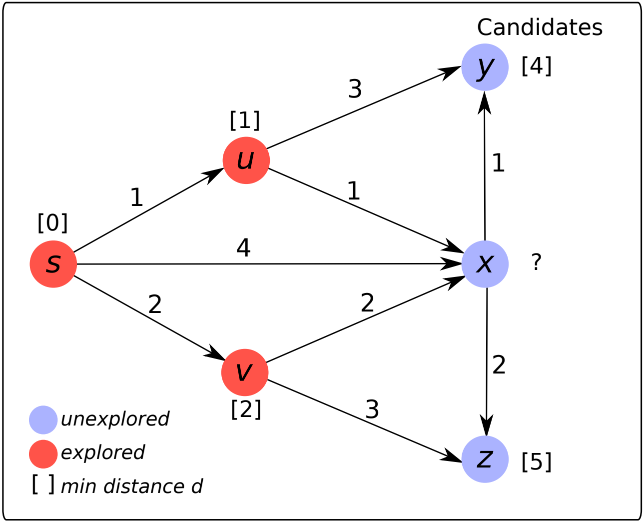

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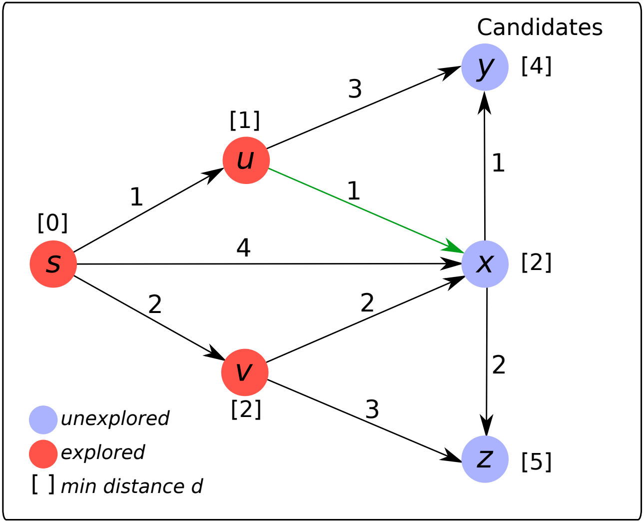

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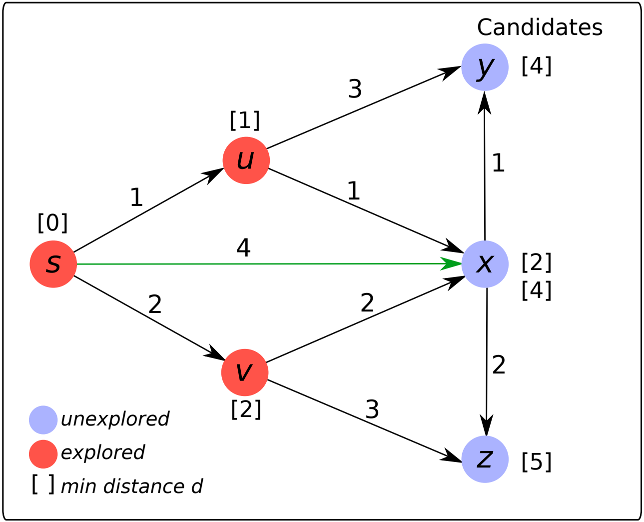

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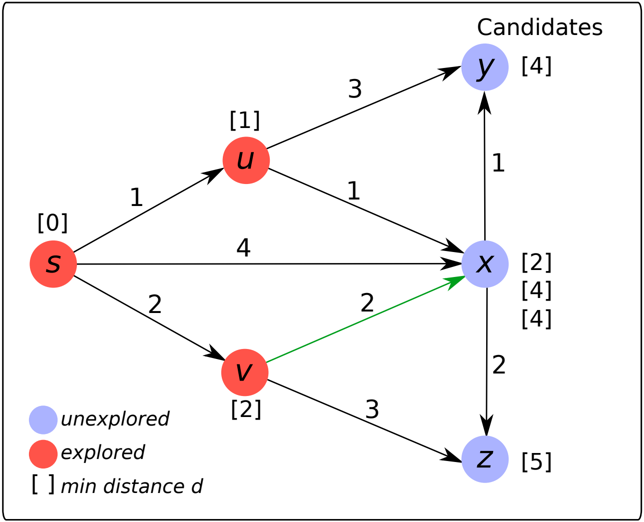

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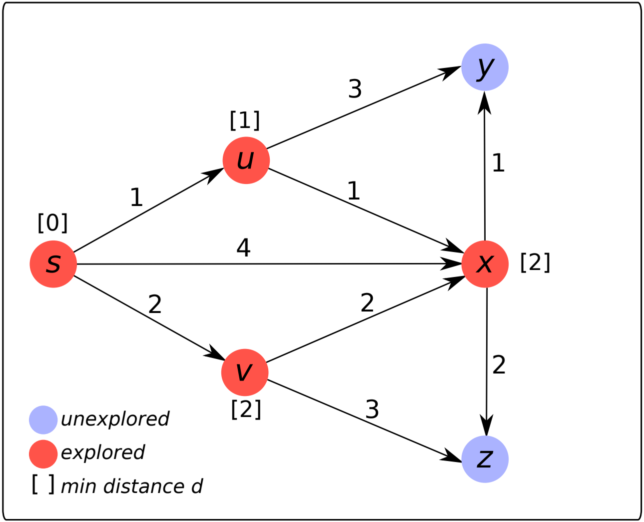

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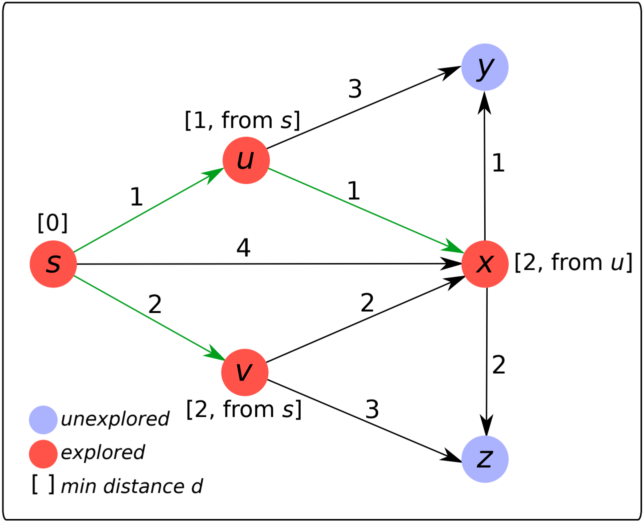

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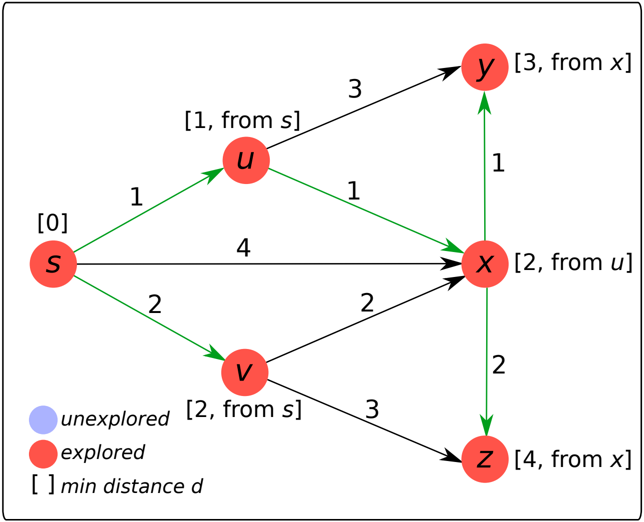

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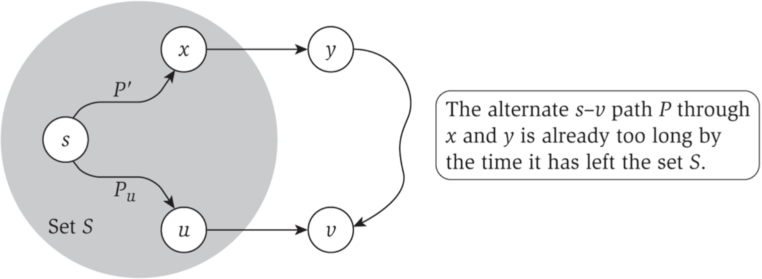

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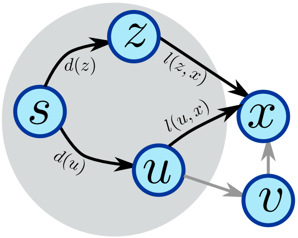

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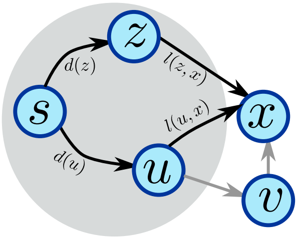

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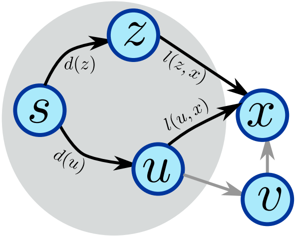

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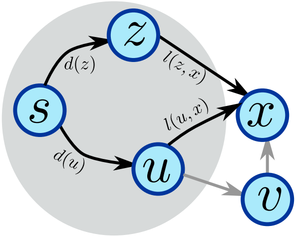

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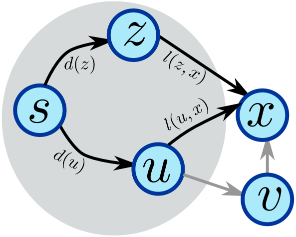

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- Running time per iteration is $O(m)$, since the algorithm processes each edge (u, x) in the graph exactly once (when computing $d'(x)$).
- The overall running time is $O(nm)$.

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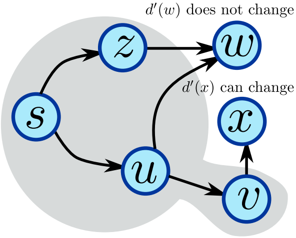

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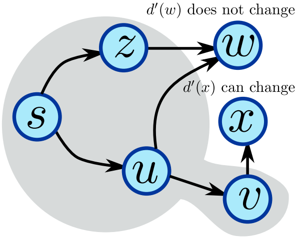

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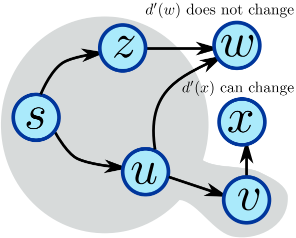

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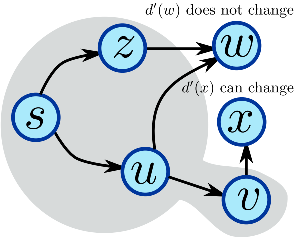

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- Idea: For each node $x \in V - S$, store the current value of $d'(x)$. Upon adding a node v to S, update $d'()$ only for neighbours of v.
- How do we efficiently compute $v = \arg\min_{x \in V-S} d'(x)$?

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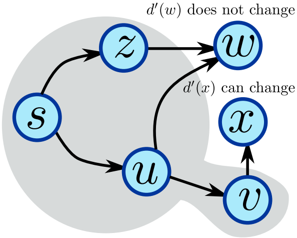

- Observation: If we add v to S, $d'(x)$ changes only if (v, x) is an edge in G.
- Idea: For each node $x \in V - S$, store the current value of $d'(x)$. Upon adding a node v to S, update $d'()$ only for neighbours of v.
- How do we efficiently compute $v = \arg\min_{x \in V-S} d'(x)$?
- Use a priority queue!

Faster Dijkstra's Algorithm

DIJKSTRA'S ALGORITHM(G, l, s)

1: INSERT$(Q, s, 0)$.
2: **while** $S \neq V$ **do**
3: $(v, d'(v)) =$ EXTRACTMIN(Q)
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8: CHANGEKEY$(Q, x, d'(x))$

- For each node $x \in V - S$, store the pair $(x, d'(x))$ in a priority queue Q with $d'(x)$ as the key.
- Determine the next node v to add to S using EXTRACTMIN (line 3).
- After adding v to S, for each node $x \in V - S$ such that there is an edge from v to x, check if $d'(x)$ should be updated, i.e., if there is a shortest path from s to x via v (lines 5–8).
- In line 8, if x is not in Q, simply insert it.

Running Time of Faster Dijkstra's Algorithm

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- How many times does the algorithm invoke CHANGEKEY? At most m times.
- What is total running time of the algorithm? $O(m \log n)$.
- State of the art: Fibonacci heaps achieve a running time of $O(m)$ for all CHANGEKEY operations, for a running time of $O(n \log n + m)$.

Network Design

- Connect a set of nodes using a set of edges with certain properties.
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Minimum Spanning Tree (MST)

- Given an undirected graph $G(V, E)$ with a cost $c_e > 0$ associated with each edge $e \in E$.
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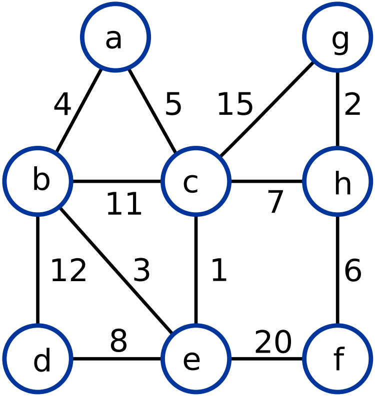

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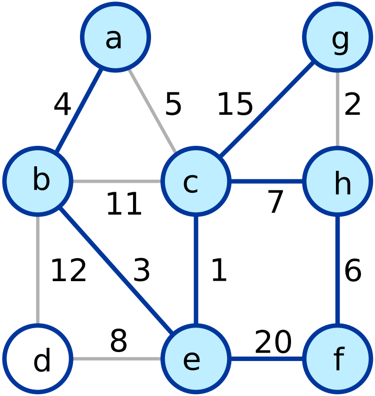

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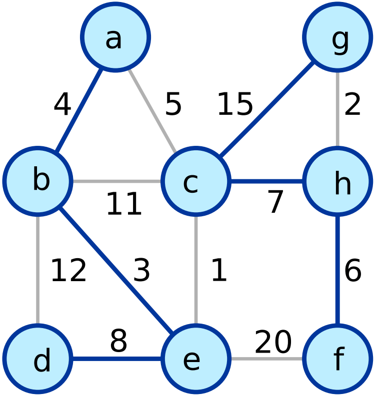

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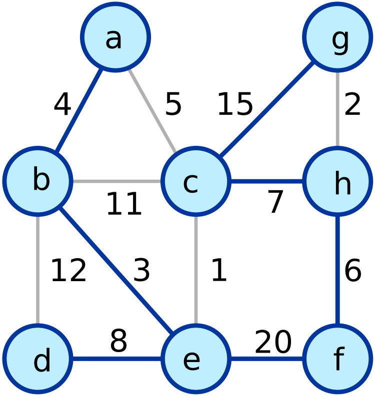

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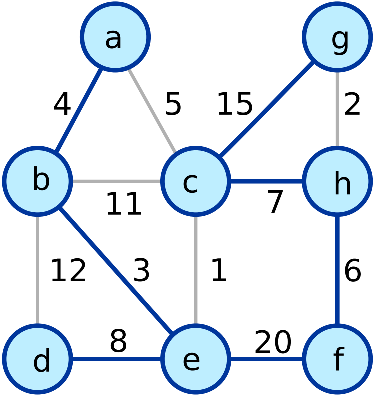

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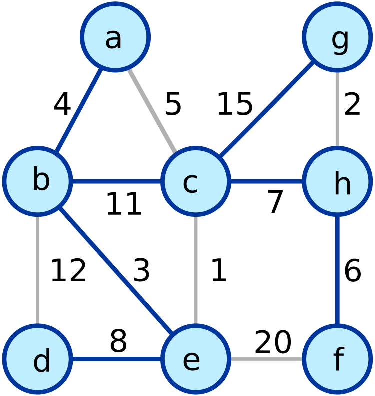

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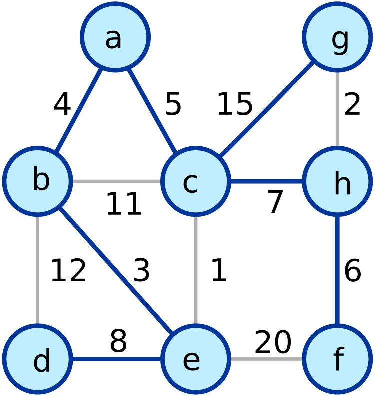

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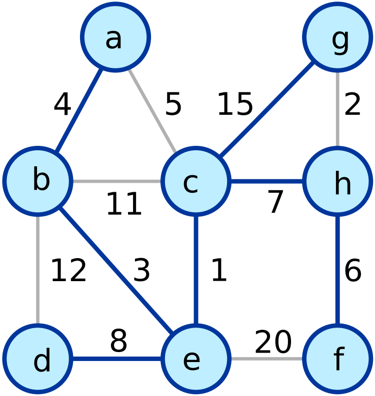

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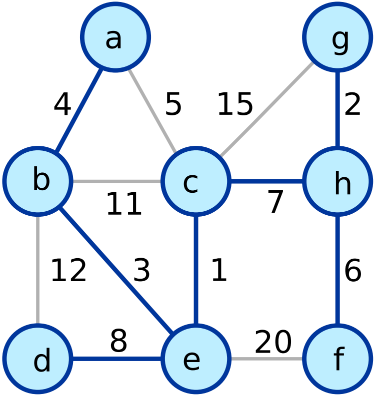

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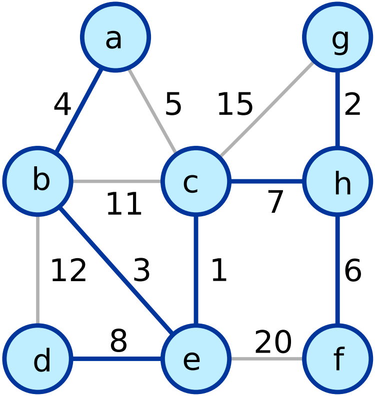

MINIMUM SPANNING TREE

INSTANCE: An undirected graph $G(V, E)$ and a function $c : E \to \mathbb{R}^+$

SOLUTION: A set $T \subseteq E$ of edges such that (V, T) is connected and the cost $\sum_{e \in T} c_e$ is as small as possible.

- Claim: If T is a minimum-cost solution to this problem then (V, T) is a tree.
- A subset T of E is a *spanning tree* of G if (V, T) is a tree.

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- Simplifying assumption: all edge costs are distinct.

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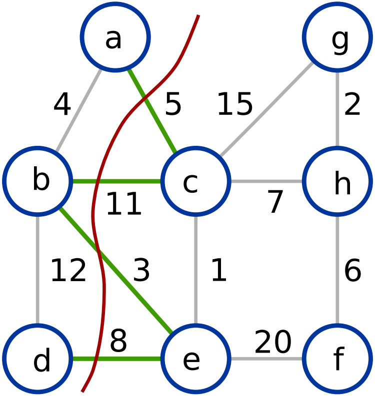
- Which edges cannot belong to an MST?
  - What happens when we add an edge to an MST?
  - We obtain a cycle.
  - Which edge in the cycle can we be sure does not belong to an MST?

Graph Cuts

- A *cut* in a graph $G(V, E)$ is a set of edges whose removal disconnects the graph (into two or more connected components).
- Every set $S \subset V$ (S cannot be empty or the entire set V) has a corresponding cut: $cut(S)$ is the set of edges (v, w) such that $v \in S$ and $w \in V - S$.



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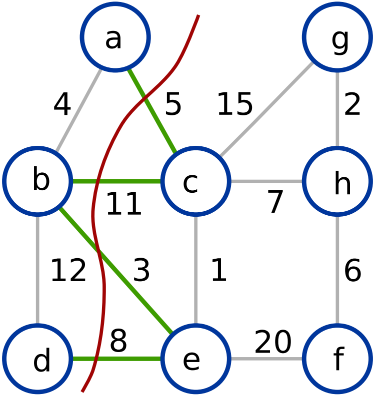

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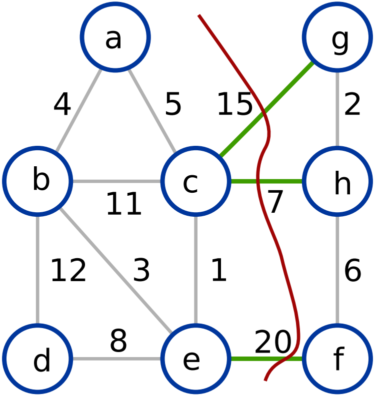

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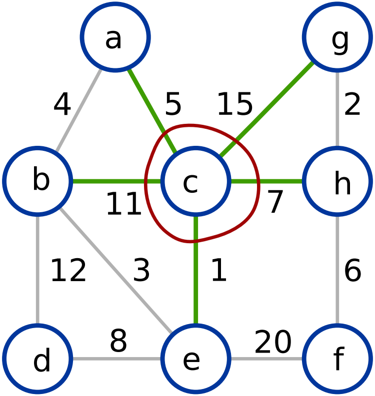

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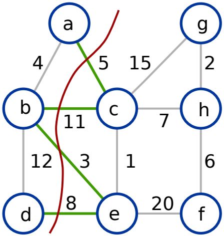

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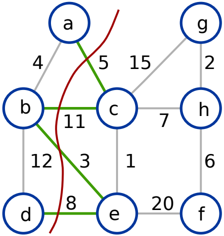

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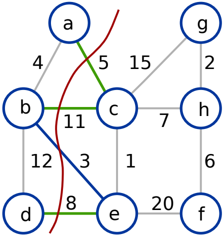
- Claim: every MST contains e.



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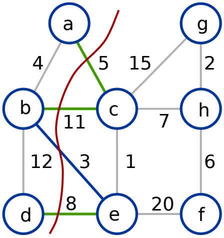

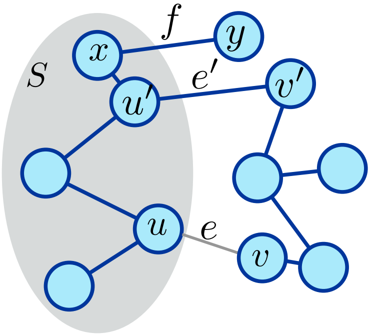

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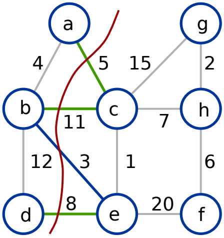

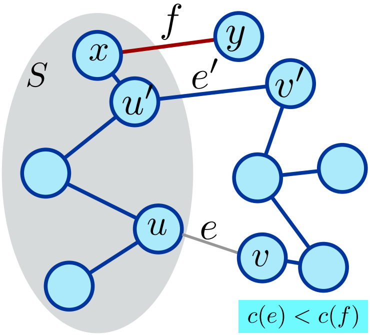

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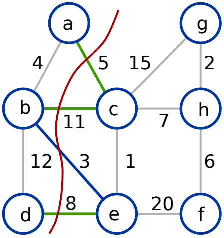

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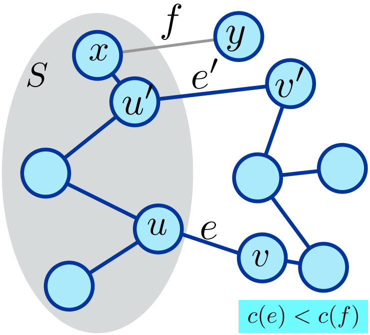

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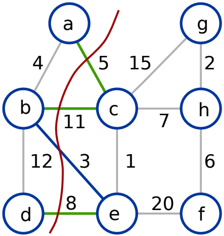

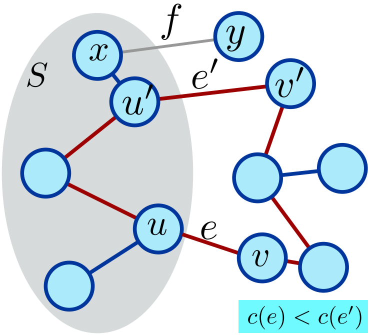

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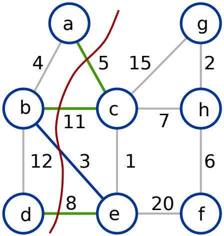

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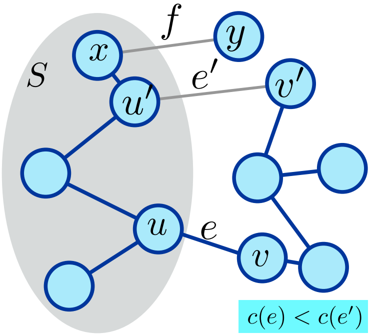
  - This *cycle* must contain an edge e' in cut(S).
  - $T - \{e'\} \cup \{e\}$ has smaller cost than T and is a spanning tree.



Prim's Algorithm

- Maintain a tree (S, T), i.e. a set of nodes and a set of edges, which we will show will always be a tree.
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PRIM'S ALGORITHM(G, c, s)

```
S = {s} and T = ∅
while S ≠ V do
    Compute (u, v) = min_{e=(u,v): u∈S, v∈V−S} c_e
    Add v to S and add e to T.
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Prim's Algorithm(G, c, s)

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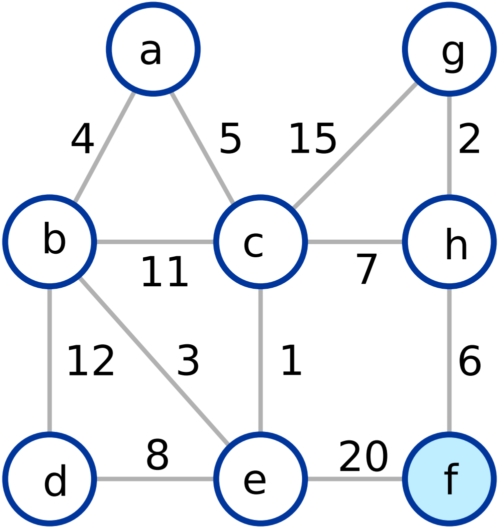
2: **while** $S \neq V$ **do**
3: Compute $(u, v) = \min_{e=(u,v):u \in S, v \in V-S} c_e$
4: Add v to S and add e to T.

- Note that

$$\min_{e=(u,v), u \in S, v \in V-S} c_e \equiv \min_{e \in \text{cut}(S)} c_e.$$

- In other words, in each step Prim's algorithm computes and adds the cheapest edge in the current value of cut(S).

Example of Prim's Algorithm



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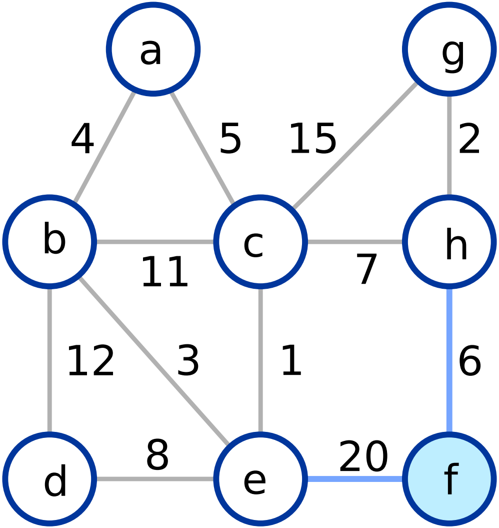

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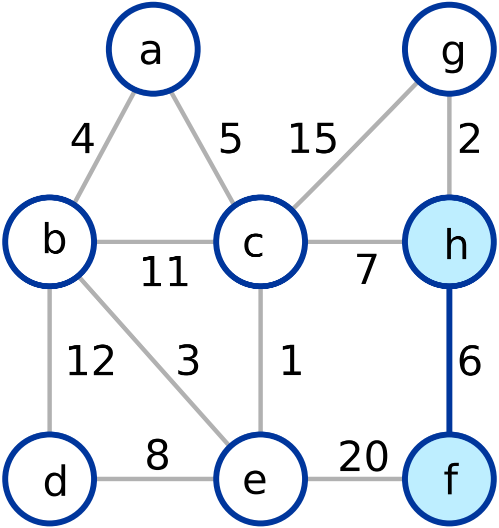

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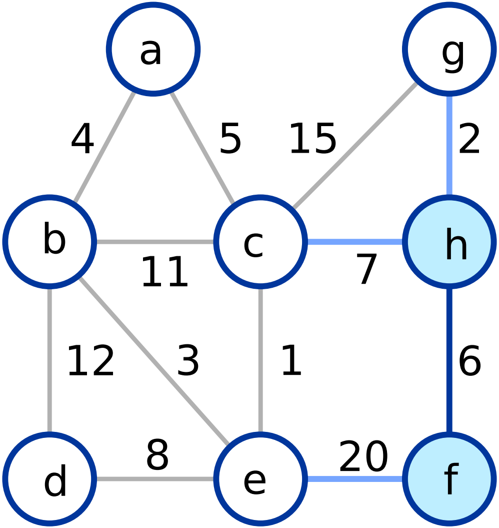

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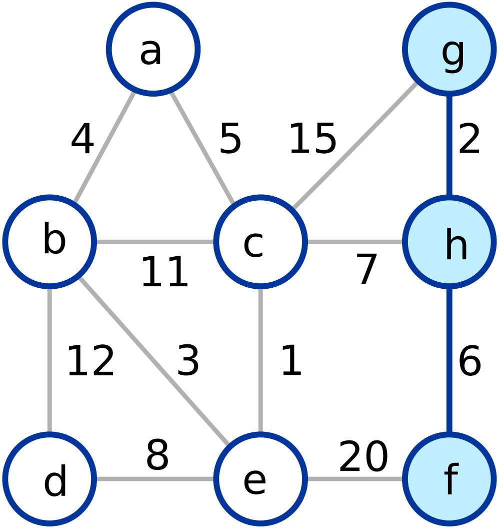

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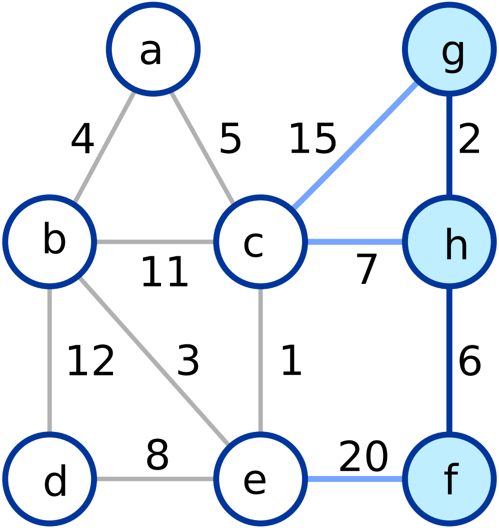

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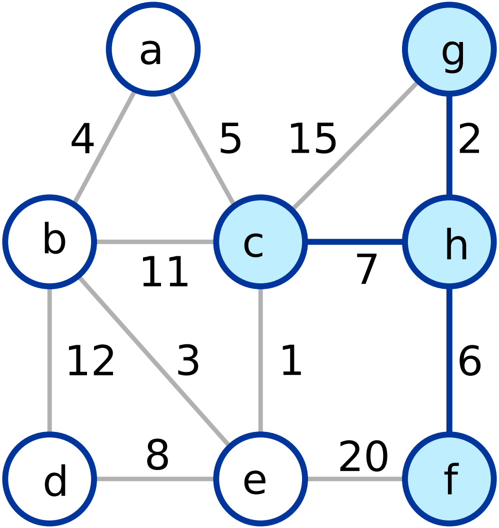

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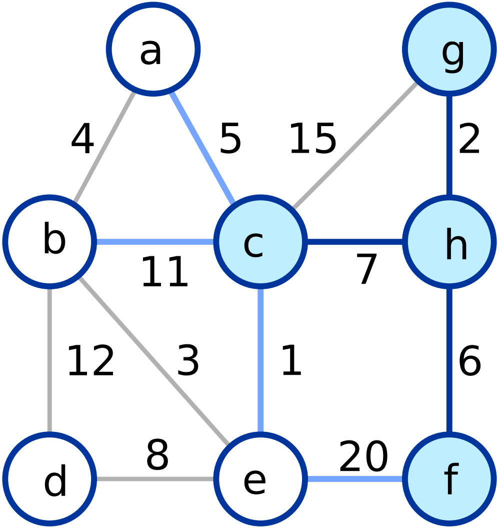

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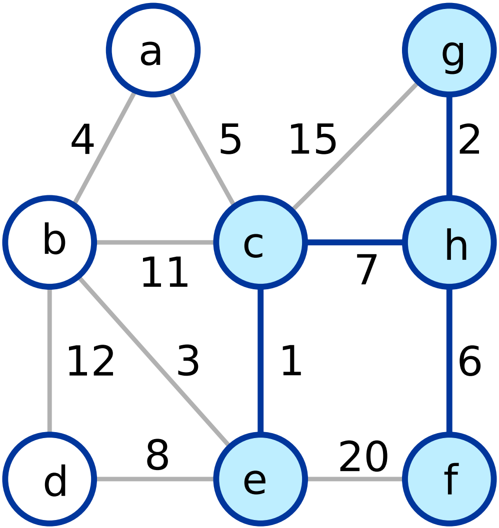

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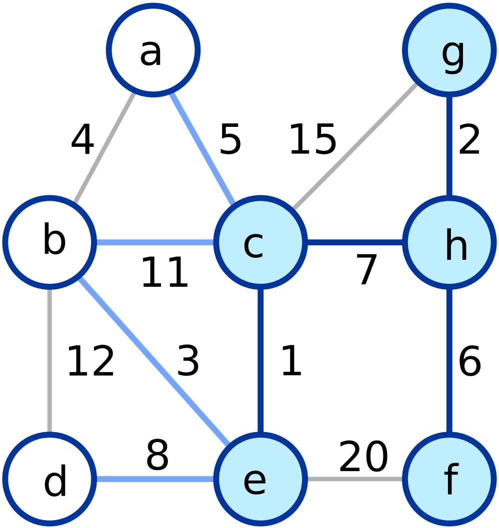

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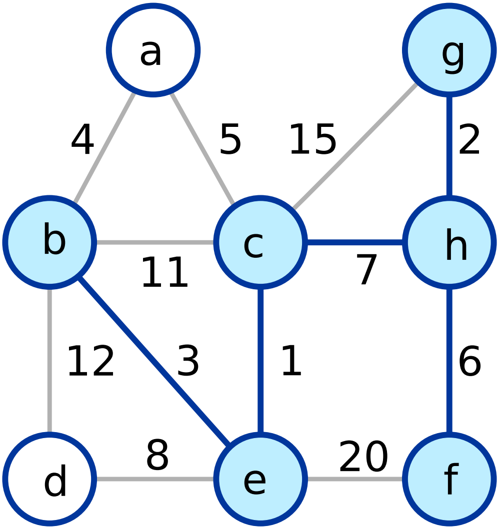

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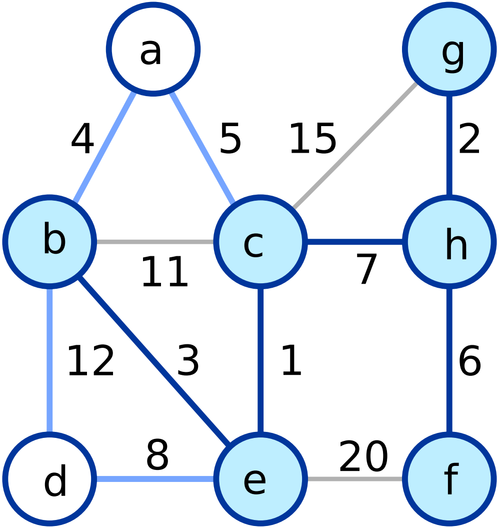

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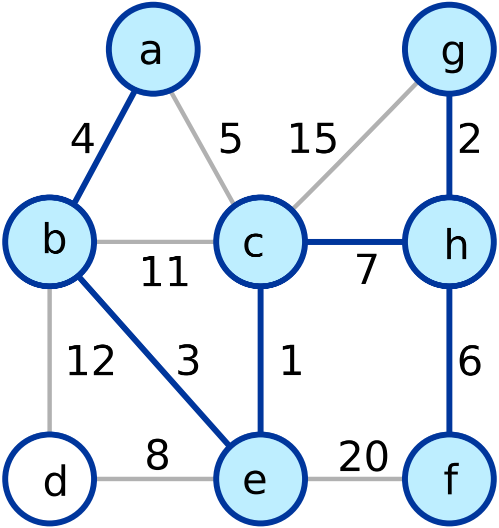

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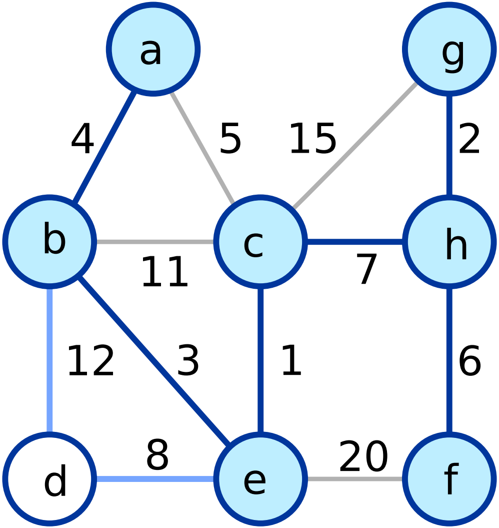

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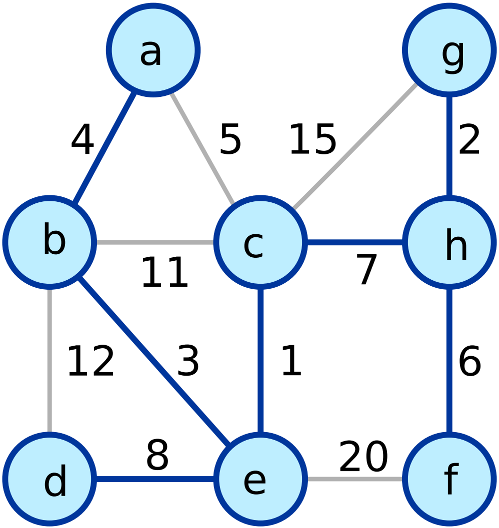

Optimality of Prim's Algorithm

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1: $S = \{s\}$ and and $T = \emptyset$
2: **while** $S \neq V$ **do**
3: Compute $(u, v) = \min_{e=(u,v) \in \text{cut}(S)} c_e$
4: Add v to S and add e to T.

- Claim: Prim's algorithm outputs an MST.

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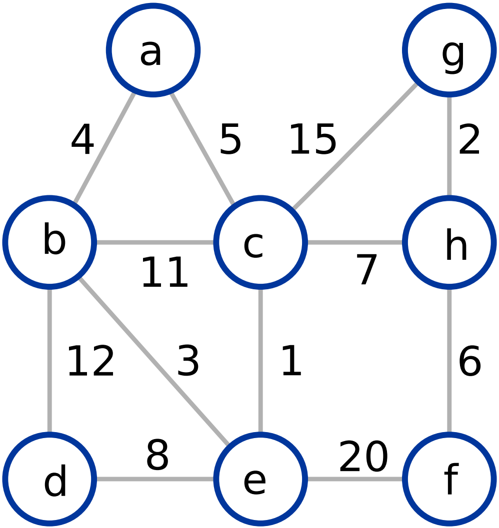

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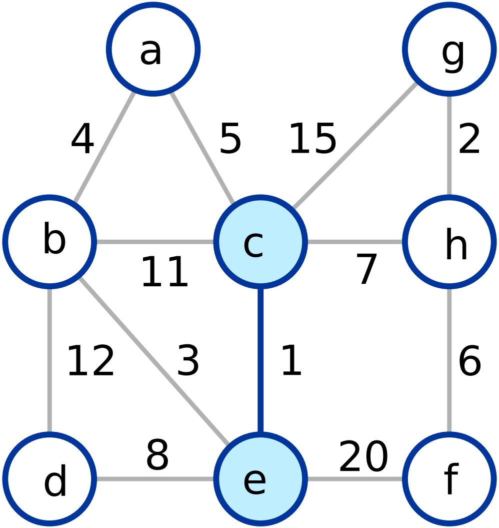

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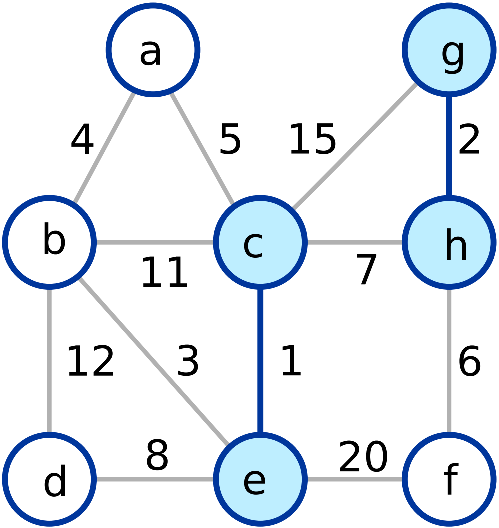

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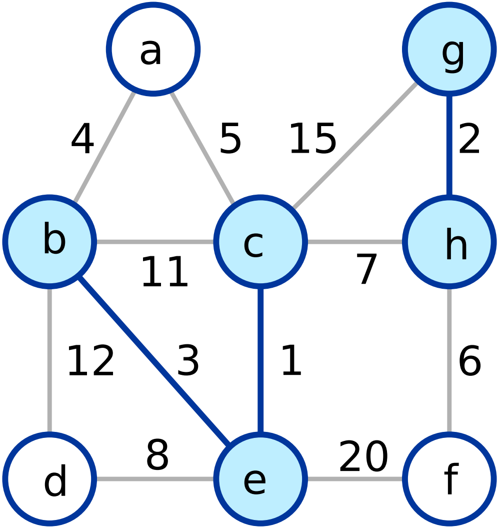

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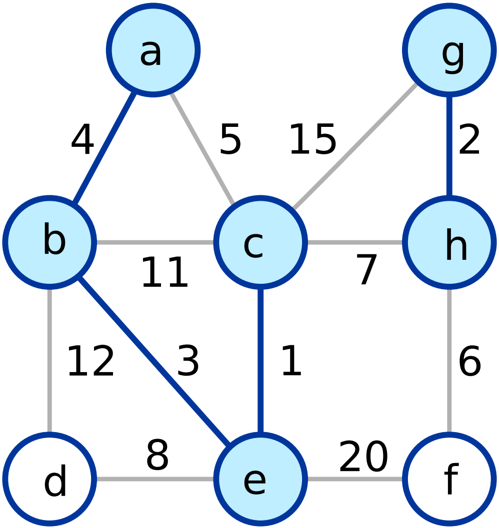

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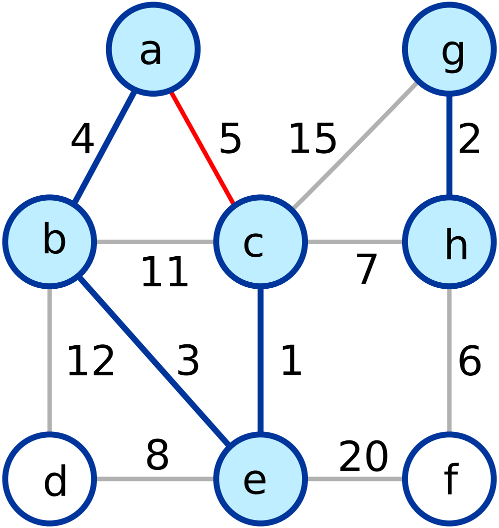

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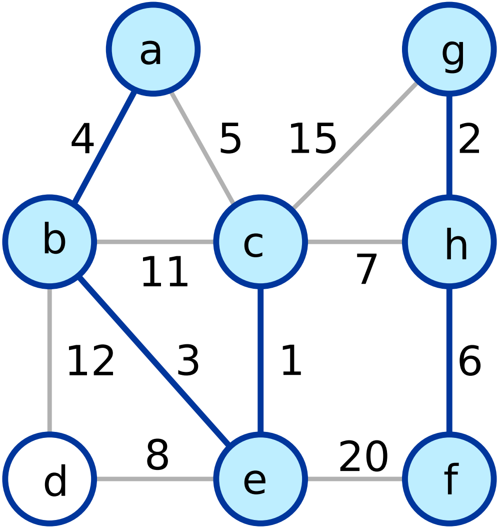

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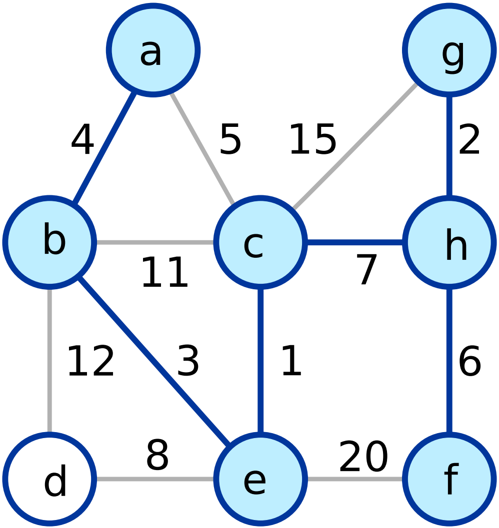

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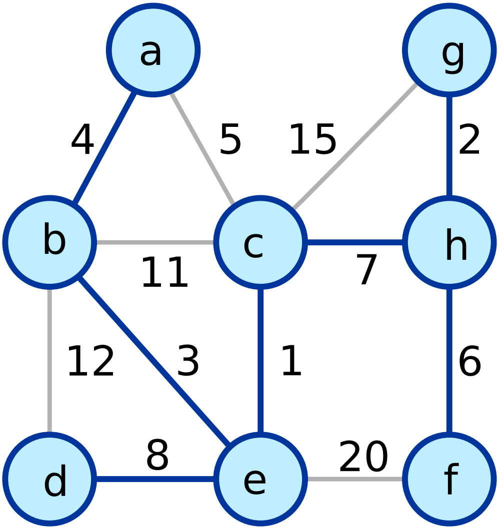

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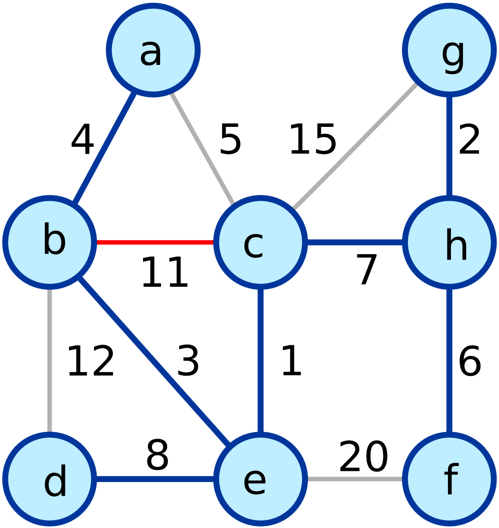

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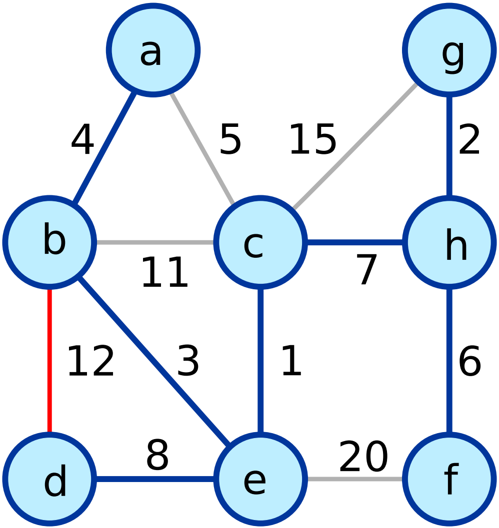

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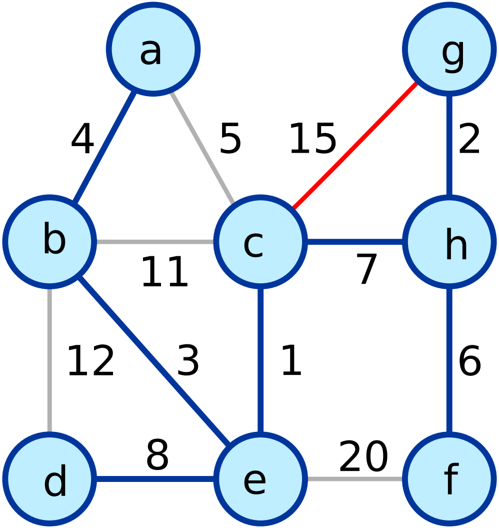

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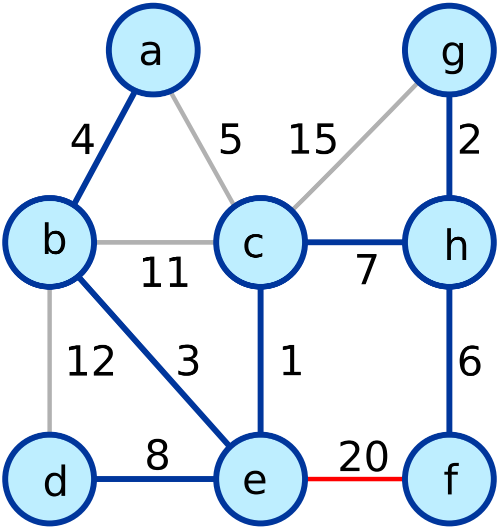

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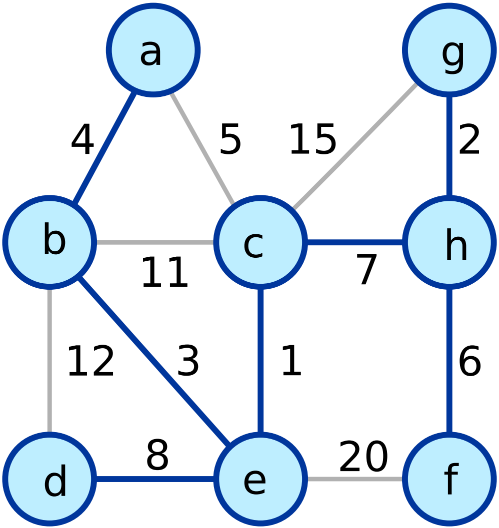

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- Kruskal's algorithm:
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     - (V, T) contains no cycles by construction.
     - If (V, T) is not connected, then exists a subset S of nodes not connected to $V - S$. What is the contradiction?

Cycle Property

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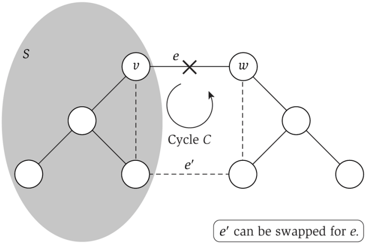

Figure 4.11 Swapping the edge e' for the edge e in the spanning tree T, as described in the proof of (4.20).

Optimality of the Reverse-Delete Algorithm

- Reverse-Delete algorithm: Maintain a set E' of edges.
  - Start with $E' = E$.
  - Process edges in decreasing order of cost.
  - Delete the next edge e from E' only if (V, E') is connected after deletion.
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     - (V, E') is connected at the end, by construction.
     - If (V, E') contains a cycle, consider the costliest edge in that cycle. The algorithm would have deleted that edge.

Comments on MST Algorithms

- To handle multiple edges with the same length, perturb each length by a random infinitesimal amount. Read the textbook.
- *Any* algorithm that constructs a spanning tree by including edges that satisfy the cut property and deleting edges that satisfy the cycle property will yield an MST!

Implementing Prim's Algorithm

PRIM'S ALGORITHM(G, c, s)

1: $S = \{s\}$ and and $U = \emptyset$
2: **while** $S \neq V$ **do**
3: Compute $(u, v) = \min_{e=(u,v):u \in S, v \in V-S} c_e$
4: Add v to S and add e to T.

- Implementation is very similar to Dijkstra's algorithm.
- Maintain S and store attachment costs $a(v) = \min_{e \in \text{cut}(S)} c_e$ for every node $v \in V - S$ in a priority queue.
- At each step, extract minimum v from priority queue and update the attachment costs of the neighbours of v.
- Total of $n - 1$ EXTRACTMIN and m CHANGEKEY operations, yielding a running time of $O(m \log n)$.

Implementing Kruskal's Algorithm

- Start with an empty set T of edges.
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- Sorting edges takes $O(m \log n)$ time.
- Key question: "Does adding $e = (u, v)$ to T create a cycle?"
  - Maintain set of connected components of T.
  - $\textsc{Find}(u)$: return the name of the connected component of T that u belongs to.
  - $\textsc{Union}(A, B)$: merge connected components A and B.

Analysing Kruskal's Algorithm

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  - Each FIND takes $O(1)$ time, k invocations of UNION take $O(k \log k)$ time in total.
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- Total running time of Kruskal's algorithm is $O(m \log n)$.

Comments on Union-Find and MST

- The UNION-FIND data structure is useful to maintain the connected components of a graph as edges are added to the graph.
- The data structure does not support edge deletion efficiently.
- Current best algorithm for MST runs in $O(m\alpha(m, n))$ time (Chazelle 2000) and $O(m)$ randomised time (Karger, Klein, and Tarjan, 1995).
- Holy grail: $O(m)$ deterministic algorithm for MST.

Union-Find Data Structure

- Abstraction of the data structure needed by Kruskal's algorithm.
- Maintain disjoint subsets of elements from a universe U of n elements.
- Each subset has an name. We will set a set's name to be the identity of some element in it.
- Support three operations:
  1. MakeUnionFind(U): initialise the data structure with elements in U.
  2. Find(u): return the identity of the subset that contains u.
  3. Union(A, B): merge the sets named A and B into one set.

Union-Find Data Structure: Implementation 1

- Store all the elements of U in an array Component.
  - Assume identities of elements are integers from 1 to n.
  - Component[s] is the name of the set containing s.
- Implementing the operations:

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  1. MakeUnionFind(U): For each $s \in U$, set Component$[s] = s$ in $O(n)$ time.
  2. Find(s): return Component$[s]$ in $O(1)$ time.
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- Union is very slow because we cannot efficiently find the elements that belong to a set.

Union-Find Data Structure: Implementation 2

- Optimisation 1: Use an array Elements
  - Indices of Elements range from 1 to n.
  - Elements$[s]$ stores the elements in the subset named s in a list.
- Execute Union(A, B) by merging B into A in two steps:
  1. Updating Component for elements of B in $O(|B|)$ time.
  2. Append Elements$[B]$ to Elements$[A]$ in $O(1)$ time.
- Union takes $\Omega(n)$ in the worst-case.

Union-Find Data Structure: Implementation 2

- Optimisation 1: Use an array ELEMENTS
  - Indices of ELEMENTS range from 1 to n.
  - ELEMENTS$[s]$ stores the elements in the subset named s in a list.
- Execute UNION(A, B) by merging B into A in two steps:
  1. Updating COMPONENT for elements of B in $O(|B|)$ time.
  2. Append ELEMENTS$[B]$ to ELEMENTS$[A]$ in $O(1)$ time.
- UNION takes $\Omega(n)$ in the worst-case.
- Optimisation 2: Store size of each set in an array (say, SIZE). If SIZE$[B] \leq$ SIZE$[A]$, merge B into A. Otherwise merge A into B. Update SIZE.

Union-Find Data Structure: Analysis of Implementation

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- Find is fast in the worst case, Union is fast in an amortised sense. Can we make both operations worst-case efficient?

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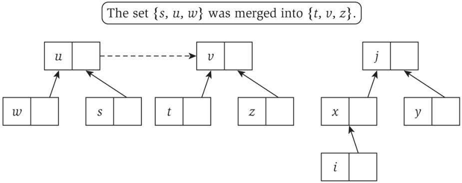

Figure 4.12 A Union-Find data structure using pointers. The data structure has only two sets at the moment, named after nodes v and j. The dashed arrow from u to v is the result of the last Union operation. To answer a Find query, we follow the arrows until we get to a node that has no outgoing arrow. For example, answering the query Find(i) would involve following the arrows i to x, and then x to j.

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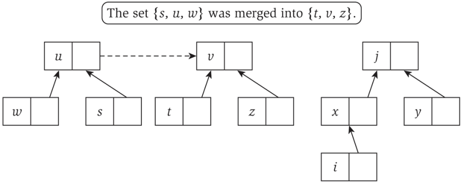

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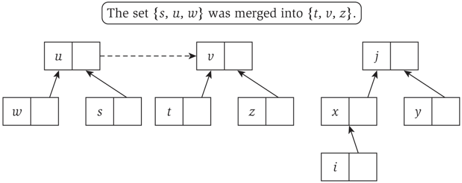
- Implementing UNION(A, B): make smaller tree's root a child of the larger tree's root. Takes $O(1)$ time.



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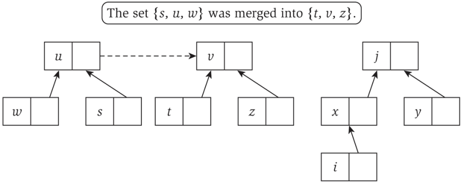

Figure 4.12 A `Union-Find` data structure using pointers. The data structure has only two sets at the moment, named after nodes v and j. The dashed arrow from u to v is the result of the last `Union` operation. To answer a `Find` query, we follow the arrows until we get to a node that has no outgoing arrow. For example, answering the query `Find`(i) would involve following the arrows i to x, and then x to j.

- Why does Find(u) take $O(\log n)$ time?

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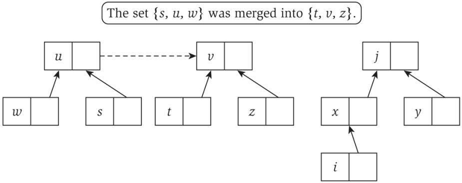

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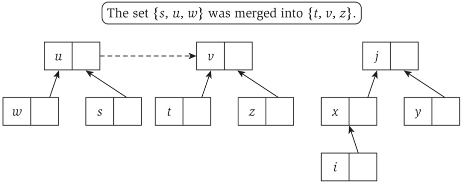
- Why does Find(u) take $O(\log n)$ time?
- Number of pointers followed equals the number of times the identity of the set containing u changed.
- Every time u's set's identity changes, the set at least doubles in size $\Rightarrow$ there are $O(\log n)$ pointers followed.

Union-Find Data Structure: Improving Implementation



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- Every time we invoke Find(u), we follow the same set of pointers.

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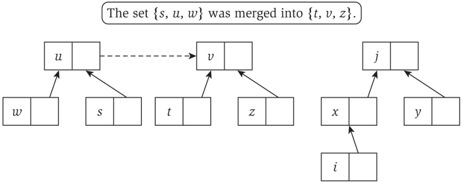

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- Every time we invoke $\text{Find}(u)$, we follow the same set of pointers.
- Path compression: make all nodes visited by $\text{Find}(u)$ children of the root.

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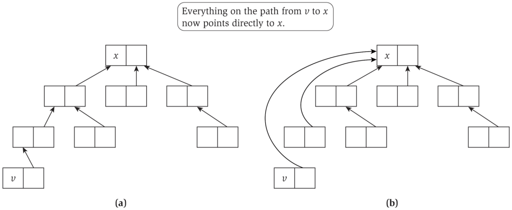

Figure 4.13 (a) An instance of a Union-Find data structure; and (b) the result of the operation Find(v) on this structure, using path compression.

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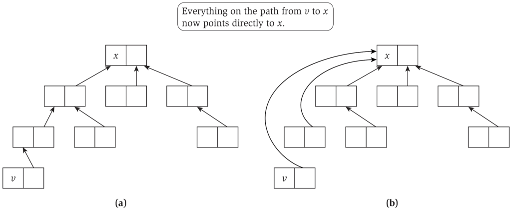

Figure 4.13 (a) An instance of a `Union-Find` data structure; and (b) the result of the operation `Find`(v) on this structure, using path compression.

- Every time we invoke FIND(u), we follow the same set of pointers.
- Path compression: make all nodes visited by FIND(u) children of the root.
- Can prove that total time taken by n FIND operations is $O(n\alpha(n))$, where $\alpha(n)$ is the inverse of the Ackermann function, and grows e-x-t-r-e-m-e-l-y s-l-o-w-l-y with n.