Proof. Our general definition of instability has four parts: This means that we have to make sure that none of the four bad things happens.

First, suppose there is an instability of type (i), consisting of pairs (m, w) and (m', w') in S with the property that $(m, w') \notin F$, m prefers w' to w, and w' prefers m to m'. It follows that m must have proposed to w'; so w' rejected m, and thus she prefers her final partner to m—a contradiction.

Next, suppose there is an instability of type (ii), consisting of a pair $(m, w) \in S$, and a man m', so that m' is not part of any pair in the matching, $(m', w) \notin F$, and w prefers m' to m. Then m' must have proposed to w and been rejected; again, it follows that w prefers her final partner to m'—a contradiction.

Third, suppose there is an instability of type (iii), consisting of a pair $(m, w) \in S$, and a woman w', so that w' is not part of any pair in the matching, $(m, w') \notin F$, and m prefers w' to w. Then no man proposed to w' at all; in particular, m never proposed to w', and so he must prefer w to w'—a contradiction.

Finally, suppose there is an instability of type (iv), consisting of a man m and a woman w, neither of which is part of any pair in the matching, so that $(m, w) \notin F$. But for m to be single, he must have proposed to every nonforbidden woman; in particular, he must have proposed to w, which means she would no longer be single—a contradiction. ■

Exercises

1. Decide whether you think the following statement is true or false. If it is true, give a short explanation. If it is false, give a counterexample.

   True or false? In every instance of the Stable Matching Problem, there is a stable matching containing a pair (m, w) such that m is ranked first on the preference list of w and w is ranked first on the preference list of m.

2. Decide whether you think the following statement is true or false. If it is true, give a short explanation. If it is false, give a counterexample.

   True or false? Consider an instance of the Stable Matching Problem in which there exists a man m and a woman w such that m is ranked first on the preference list of w and w is ranked first on the preference list of m. Then in every stable matching S for this instance, the pair (m, w) belongs to S.

3. There are many other settings in which we can ask questions related to some type of "stability" principle. Here's one, involving competition between two enterprises.

Exercises

1. Suppose you have algorithms with the five running times listed below. (Assume these are the exact running times.) How much slower do each of these algorithms get when you (a) double the input size, or (b) increase the input size by one?
   (a) n^2
   (b) n^3
   (c) $100n^2$
   (d) $n \log n$
   (e) 2^n

2. Suppose you have algorithms with the six running times listed below. (Assume these are the exact number of operations performed as a function of the input size n.) Suppose you have a computer that can perform 10^{10} operations per second, and you need to compute a result in at most an hour of computation. For each of the algorithms, what is the largest input size n for which you would be able to get the result within an hour?
   (a) n^2
   (b) n^3
   (c) $100n^2$
   (d) $n \log n$
   (e) 2^n
   (f) 2^{2^n}

3. Take the following list of functions and arrange them in ascending order of growth rate. That is, if function $g(n)$ immediately follows function $f(n)$ in your list, then it should be the case that $f(n)$ is $O(g(n))$.

$$f_1(n) = n^{2.5}$$
$$f_2(n) = \sqrt{2n}$$
$$f_3(n) = n + 10$$
$$f_4(n) = 10^n$$
$$f_5(n) = 100^n$$
$$f_6(n) = n^2 \log n$$

4. Take the following list of functions and arrange them in ascending order of growth rate. That is, if function $g(n)$ immediately follows function $f(n)$ in your list, then it should be the case that $f(n)$ is $O(g(n))$.

$$g_1(n) = 2^{\sqrt{\log n}}$$
$$g_2(n) = 2^n$$
$$g_4(n) = n^{4/3}$$
$$g_3(n) = n(\log n)^3$$
$$g_5(n) = n^{\log n}$$
$$g_6(n) = 2^{2^n}$$
$$g_7(n) = 2^{n^2}$$

5. Assume you have functions f and g such that $f(n)$ is $O(g(n))$. For each of the following statements, decide whether you think it is true or false and give a proof or counterexample.
   (a) $\log_2 f(n)$ is $O(\log_2 g(n))$.
   (b) $2^{f(n)}$ is $O(2^{g(n)})$.
   (c) $f(n)^2$ is $O(g(n)^2)$.

6. Consider the following basic problem. You're given an array A consisting of n integers $A[1], A[2], \ldots, A[n]$. You'd like to output a two-dimensional n-by-n array B in which $B[i, j]$ (for $i < j$) contains the sum of array entries $A[i]$ through $A[j]$—that is, the sum $A[i] + A[i+1] + \cdots + A[j]$. (The value of array entry $B[i, j]$ is left unspecified whenever $i \geq j$, so it doesn't matter what is output for these values.)

   Here's a simple algorithm to solve this problem.

```
For i = 1, 2, ..., n
  For j = i + 1, i + 2, ..., n
    Add up array entries A[i] through A[j]
    Store the result in B[i, j]
  Endfor
Endfor
```

   (a) For some function f that you should choose, give a bound of the form $O(f(n))$ on the running time of this algorithm on an input of size n (i.e., a bound on the number of operations performed by the algorithm).
   (b) For this same function f, show that the running time of the algorithm on an input of size n is also $\Omega(f(n))$. (This shows an asymptotically tight bound of $\Theta(f(n))$ on the running time.)
   (c) Although the algorithm you analyzed in parts (a) and (b) is the most natural way to solve the problem—after all, it just iterates through