Coping with NP-Completeness

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How Do We Tackle an $\mathcal{NP}$-Complete Problem?

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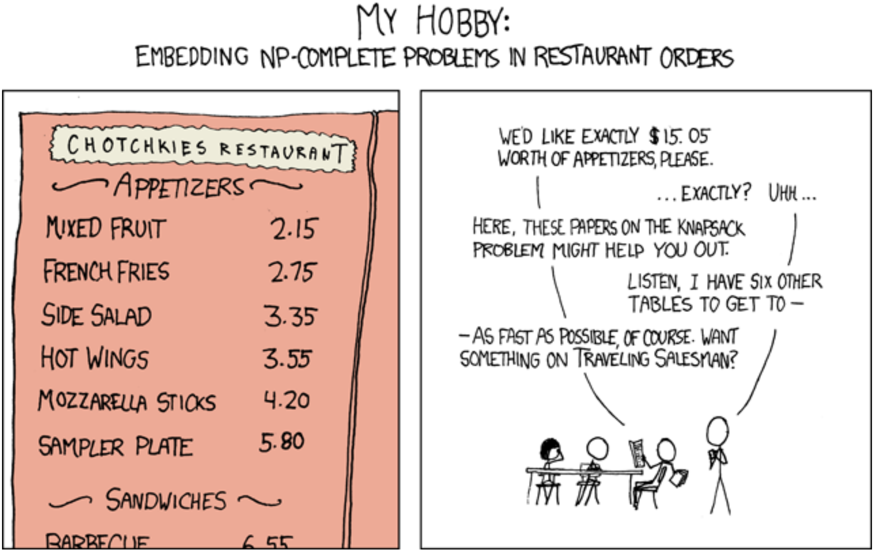

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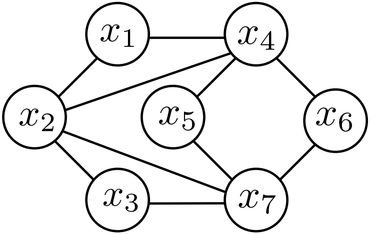
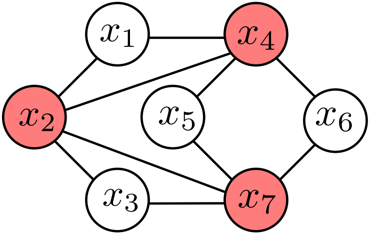
- $\mathcal{NP}$-Complete means that a problem is hard to solve in the *worst case*. Can we come up with better solutions at least in *some* cases?
  - Develop algorithms that are exponential in one parameter in the problem.
  - Consider special cases of the input, e.g., graphs that "look like" trees.
  - Develop algorithms that can provably compute a solution close to the optimal.

Vertex Cover Problem





VERTEX COVER

INSTANCE: Undirected graph *G* and an integer *k*

QUESTION: Does *G* contain a vertex cover of size at most *k*?

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- What is the running time of a brute-force algorithm?

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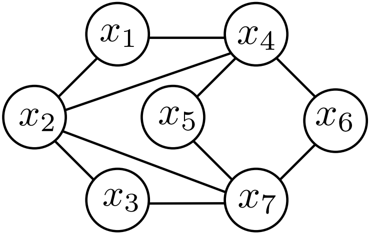

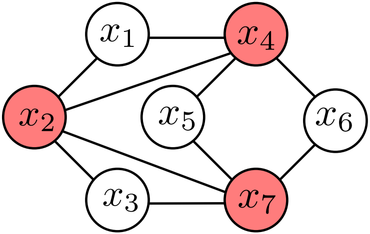

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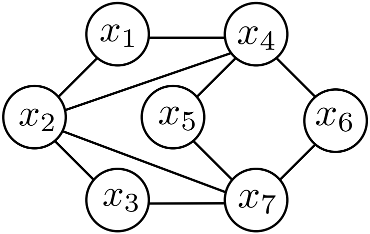

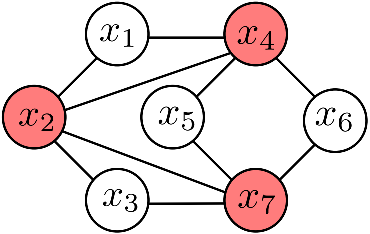

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- What is the running time of a brute-force algorithm? $O(kn\binom{n}{k}) = O(kn^{k+1})$.
- Can we devise an algorithm whose running time is exponential in k but polynomial in n, e.g., $O(2^k n)$?

Designing the Vertex Cover Algorithm

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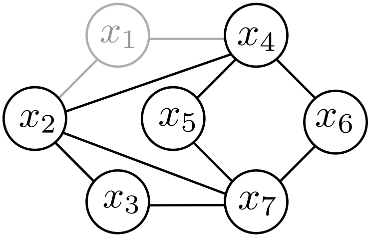

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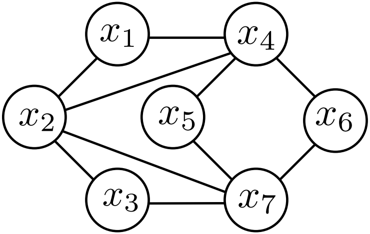

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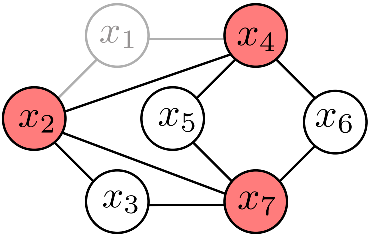
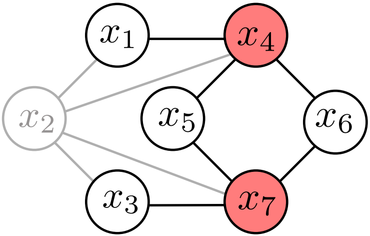
- $G - \{u\}$ is the graph G without node u and the edges incident on u.
- Consider an edge (u, v). Either u or v must be in the vertex cover.
- Claim: G has a vertex cover of size at most k iff for any edge (u, v) either $G - \{u\}$ or $G - \{v\}$ has a vertex cover of size at most $k - 1$.





Vertex Cover Algorithm

To search for a k-node vertex cover in G:

If G contains no edges, then the empty set is a vertex cover

If G contains $> k\ |V|$ edges, then it has no k-node vertex cover

Else let $e = (u, v)$ be an edge of G

Recursively check if either of $G - \{u\}$ or $G - \{v\}$

has a vertex cover of size $k - 1$

If neither of them does, then G has no k-node vertex cover

Else, one of them (say, $G - \{u\}$) has a $(k-1)$-node vertex cover T

In this case, $T \cup \{u\}$ is a k-node vertex cover of G

Endif

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- $T(n, k) \leq 2T(n, k-1) + ckn$.
  - We need $O(kn)$ time to count the number of edges.
- Claim: $T(n, k) = O(2^k kn)$.

Solving $\mathcal{NP}$-Hard Problems on Trees

- "$\mathcal{NP}$-Hard": at least as hard as $\mathcal{NP}$-Complete. We will use $\mathcal{NP}$-Hard to refer to optimisation versions of decision problems.

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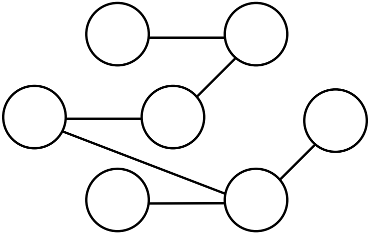
- "$\mathcal{NP}$-Hard": at least as hard as $\mathcal{NP}$-Complete. We will use $\mathcal{NP}$-Hard to refer to optimisation versions of decision problems.
- Many $\mathcal{NP}$-Hard problems can be solved efficiently on trees.
- Intuition: subtree rooted at any node v of the tree "interacts" with the rest of tree only through v. Therefore, depending on whether we include v in the solution or not, we can decouple solving the problem in v's subtree from the rest of the tree.

Designing Greedy Algorithm for Independent Set



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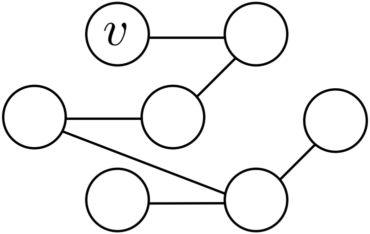

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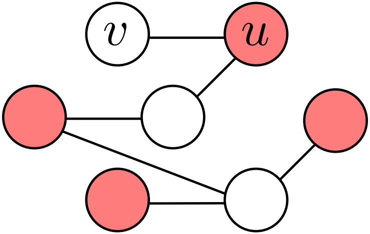

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  - Let S be a maximum-size independent set that does not contain v.
  - Let v be connected to u.
  - u must be in S; otherwise, we can add v to S, which means S is not maximum size.
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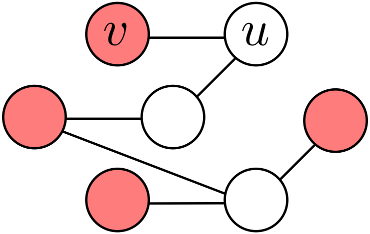

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- Claim: If a tree T has a a leaf v, then a maximum-size independent set in T is v and a maximum-size independent set in $T - \{v\}$.

Greedy Algorithm for Independent Set

- A *forest* is a graph where every connected component is a tree.

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To find a maximum-size independent set in a forest F:
  Let S be the independent set to be constructed (initially empty)
  While F has at least one edge
     Let e = (u, v) be an edge of F such that v is a leaf
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Greedy Algorithm for Independent Set

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- The algorithm works correctly on any graph for which we can repeatedly find a leaf.

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Maximum Weight Independent Set

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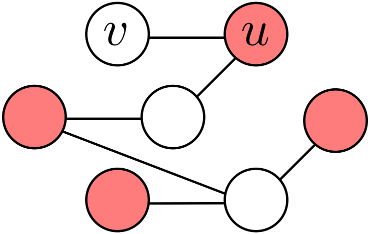

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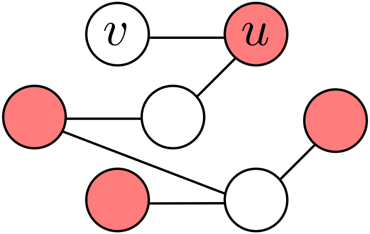

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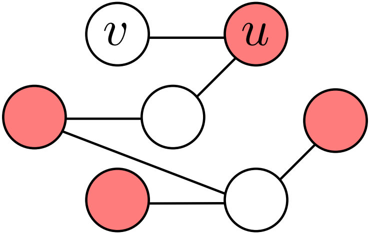

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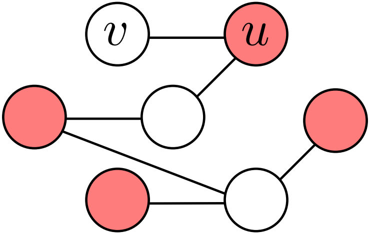

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- But there are still only two possibilities: either include u in the independent set or include *all* neighbours of u that are leaves.
- Suggests dynamic programming algorithm.

Designing Dynamic Programming Algorithm

- Dynamic programming algorithm needs a set of sub-problems, recursion to combine sub-problems, and order over sub-problems.
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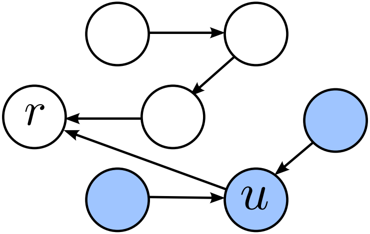
  - Sub-problems are T_u: subtree induced by u and all its descendants.



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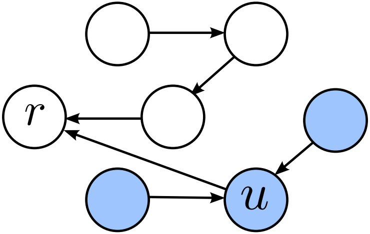
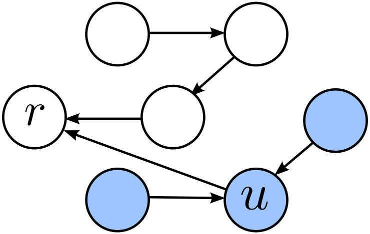
- Ordering the sub-problems: start at leaves and work our way up to the root.



Recursion for Dynamic Programming Algorithm



- Either we include u in an optimal solution or exclude u.
  - $OPT_{in}(u)$: maximum weight of an independent set in T_u that includes u.
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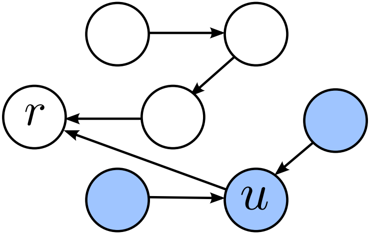

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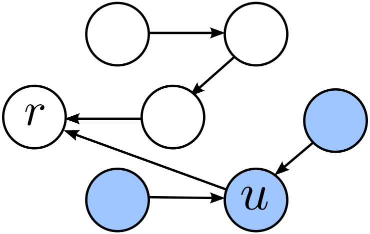

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- Base cases: For a leaf u, $\mathrm{OPT}_{\mathrm{in}}(u) = w_u$ and $\mathrm{OPT}_{\mathrm{out}}(u) = 0$.
- Recurrence: Include u or exclude u.

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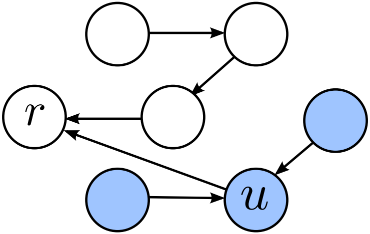

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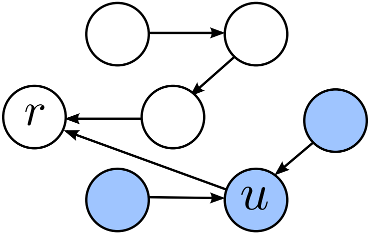

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     $\mathrm{OPT}_{\mathrm{in}}(u) = w_u + \sum_{v \in \mathrm{children}(u)} \mathrm{OPT}_{\mathrm{out}}(v)$
  2. If we exclude u, a child may or may not be excluded.
     $\mathrm{OPT}_{\mathrm{out}}(u) = \sum_{v \in \mathrm{children}(u)} \max\left(\mathrm{OPT}_{\mathrm{in}}(v), \mathrm{OPT}_{\mathrm{out}}(v)\right)$

Dynamic Programming Algorithm

```
To find a maximum-weight independent set of a tree T:
  Root the tree at a node r
  For all nodes u of T in post-order
    If u is a leaf then set the values:
```
$$M_{out}[u] = 0$$
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```
    Else set the values:
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$$M_{out}[u] = \sum_{v \in children(u)} \max(M_{out}[v], \quad M_{in}[v])$$
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    Endif
  Endfor
  Return max(M_out[r], M_in[r])
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Endif

Endfor

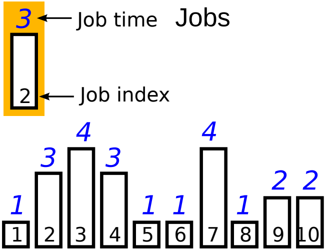
Return $\max(M_{out}[r], M_{in}[r])$

- Running time of the algorithm is $O(n)$.

Approximation Algorithms

- Methods for optimisation versions of $\mathcal{NP}$-Complete problems.
- Run in polynomial time.
- Solution returned is guaranteed to be within a small factor of the optimal solution

Load Balancing Problem



- Given set of m machines $M_1, M_2, \ldots M_m$.
- Given a set of n jobs: job j has processing time t_j.
- Assign each job to one machine so that the total time spent is minimised.

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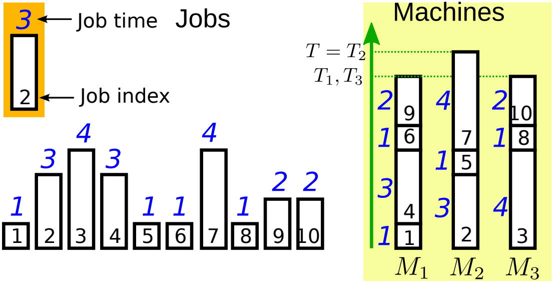

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- Assign each job to one machine so that the total time spent is minimised.
- Let $A(i)$ be the set of jobs assigned to machine M_i.
- Total time spent on machine i is $T_i = \sum_{k \in A(i)} t_k$.
- Minimise *makespan* $T = \max_i T_i$, the largest load on any machine.

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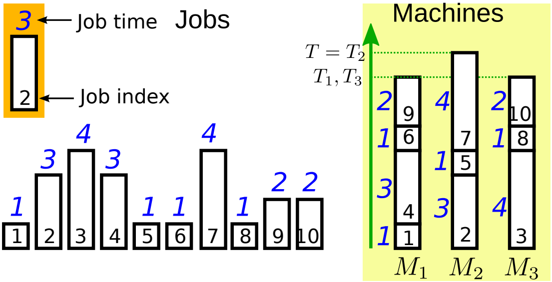

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- Minimise *makespan* $T = \max_i T_i$, the largest load on any machine.
- Minimising makespan is $\mathcal{NP}$-Complete.

Greedy-Balance Algorithm

- Adopt a greedy approach.
- Process jobs in *any* order.
- Assign next job to the processor that has smallest total load so far.

Greedy-Balance:

Start with no jobs assigned

Set $T_i = 0$ and $A(i) = \emptyset$ for all machines M_i

For $j = 1, \ldots, n$

  Let M_i be a machine that achieves the minimum $\min_k T_k$

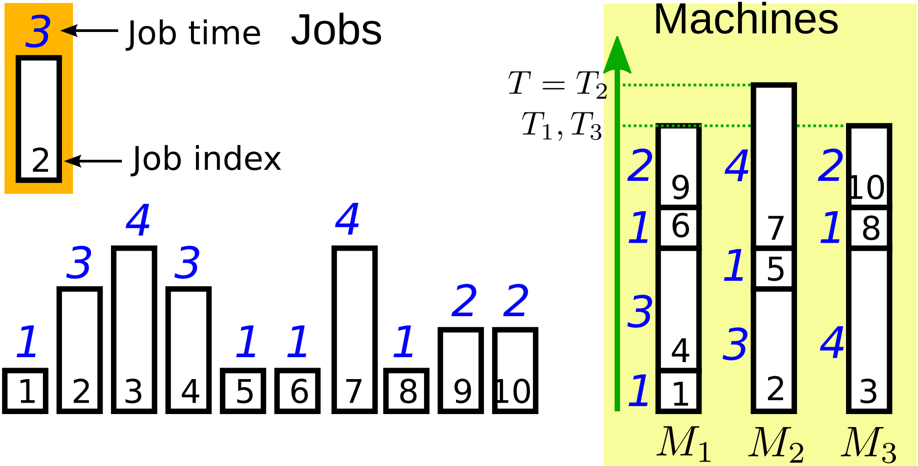
  Assign job j to machine M_i

  Set $A(i) \leftarrow A(i) \cup \{j\}$

  Set $T_i \leftarrow T_i + t_j$

EndFor

Example of Greedy-Balance Algorithm



Lower Bounds on the Optimal Makespan

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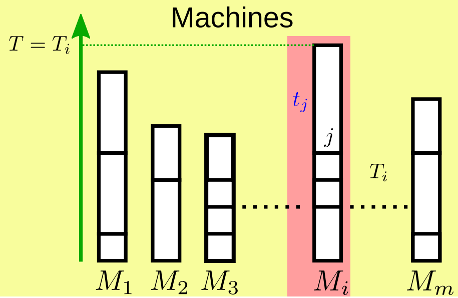
- The two bounds below will suffice:

$$T^* \geq \frac{1}{m}\sum_j t_j$$

$$T^* \geq \max_j t_j$$

Analysing Greedy-Balance



- Claim: Computed makespan $T \leq 2T^*$.

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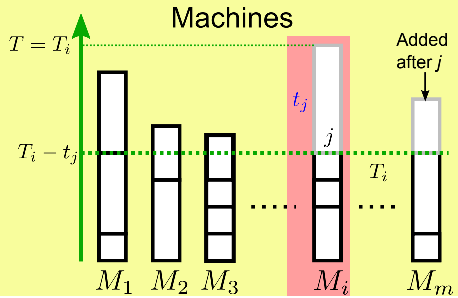

- Claim: Computed makespan $T \leq 2T^*$.
- Let M_i be the machine whose load is T and j be the last job placed on M_i.
- What was the situation just before placing this job?

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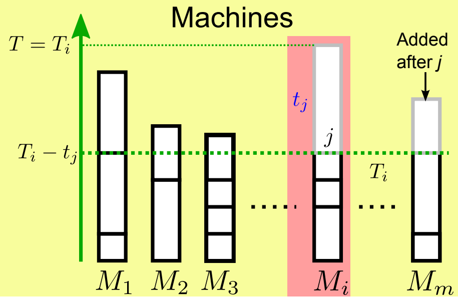
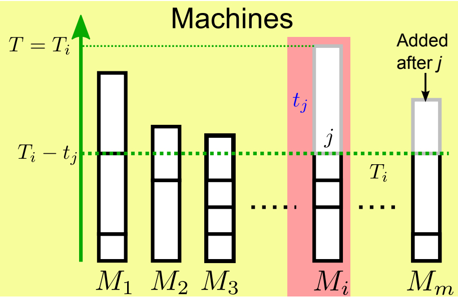


- Claim: Computed makespan $T \leq 2T^*$.
- Let M_i be the machine whose load is T and j be the last job placed on M_i.
- What was the situation just before placing this job?
- M_i had the smallest load and its load was $T - t_j$.
- For every machine M_k, load $T_k \geq T - t_j$.

Analysing Greedy-Balance



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$$\sum_k T_k \geq m(T - t_j), \text{ where } k \text{ ranges over all machines}$$

$$\sum_j t_j \geq m(T - t_j), \text{ where } j \text{ ranges over all jobs}$$

$$T - t_j \leq 1/m \sum_j t_j \leq T^*$$

$$T \leq 2T^*, \text{ since } t_j \leq T^*$$

Improving the Bound

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- How can we improve the algorithm?
- What if we process the jobs in decreasing order of processing time?

Sorted-Balance Algorithm

```
Sorted-Balance:
Start with no jobs assigned
Set T_i = 0 and A(i) = ∅ for all machines M_i
Sort jobs in decreasing order of processing times t_j
Assume that t_1 ≥ t_2 ≥ ... ≥ t_n
For j = 1, ..., n
  Let M_i be the machine that achieves the minimum min_k T_k
  Assign job j to machine M_i
  Set A(i) ← A(i) ∪ {j}
  Set T_i ← T_i + t_j
EndFor
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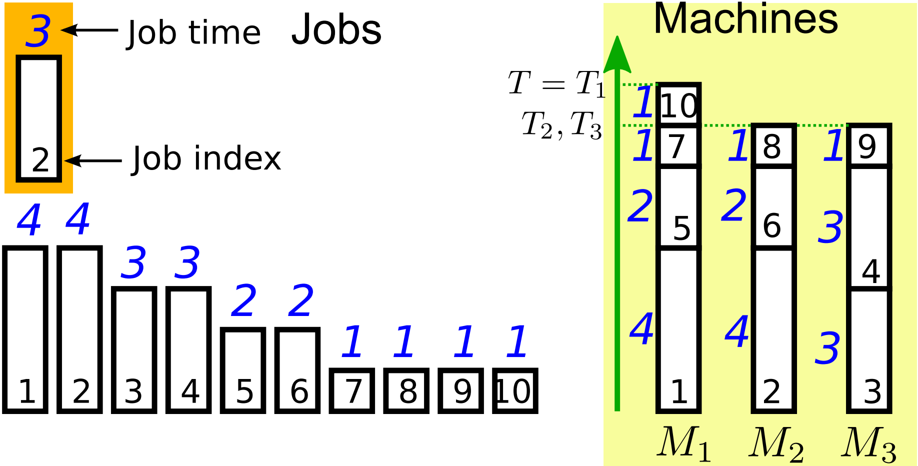
- This algorithm assigns the first m jobs to m distinct machines.

Example of Sorted-Balance Algorithm



Analyzing Sorted-Balance

- Claim: if there are fewer than m jobs, algorithm is optimal.
- Claim: if there are more than m jobs, then $T^* \geq 2t_{m+1}$.

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  - Consider *any* assignment of these $m+1$ jobs to machines.
  - Some machine must be assigned two jobs, each with processing time at least t_{m+1}.
  - This machine will have load at least $2t_{m+1}$.

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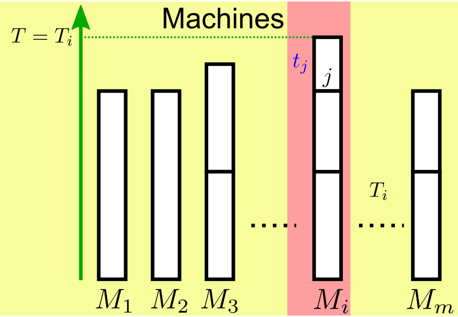

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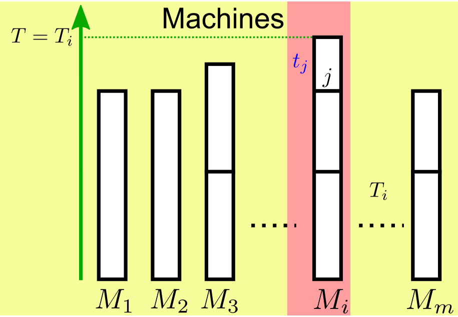
$$t_j \leq t_{m+1} \leq T^*/2, \text{ since } j \geq m+1$$
$$T - t_j \leq T^*, \text{ Greedy-Balance proof}$$
$$T \leq 3T^*/2$$

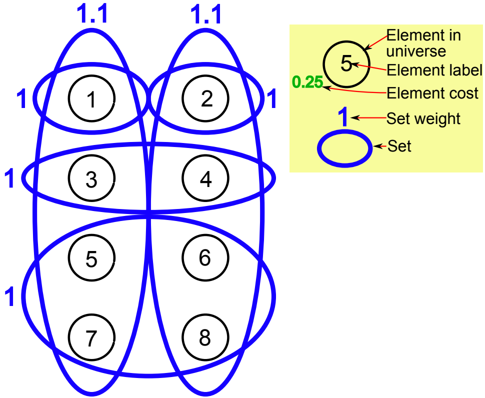


Set Cover

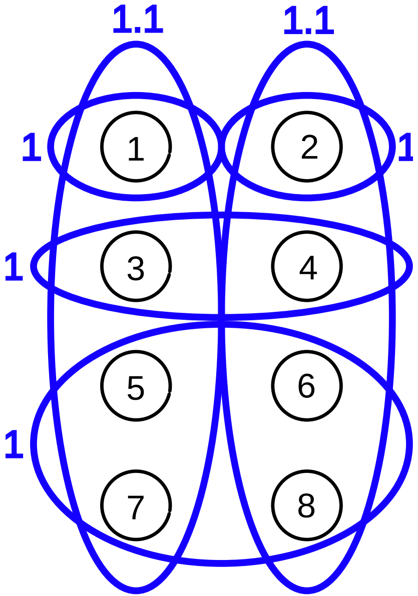
SET COVER

INSTANCE: A set U of n elements, a collection $S_1, S_2, \ldots, S_m$ of subsets of U, each with an associated weight w.

SOLUTION: A collection $\mathcal{C}$ of sets in the collection such that $\bigcup_{S_i \in C} S_i = U$ and $\sum_{S_i \in C} w_i$ is minimised.



Greedy Approach



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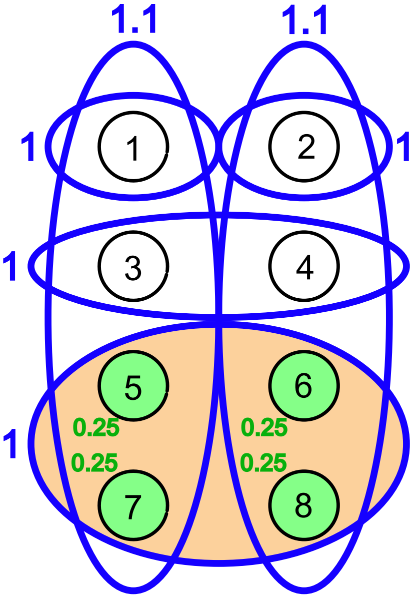

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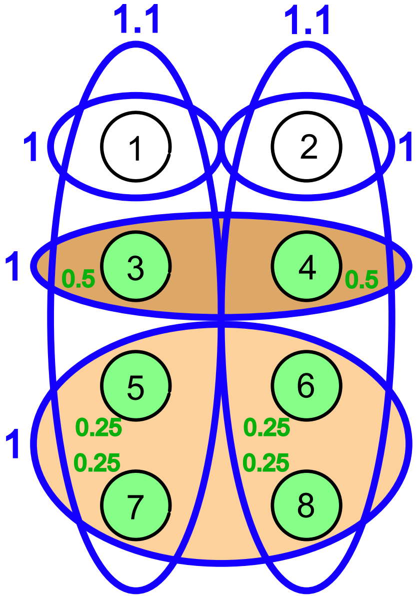

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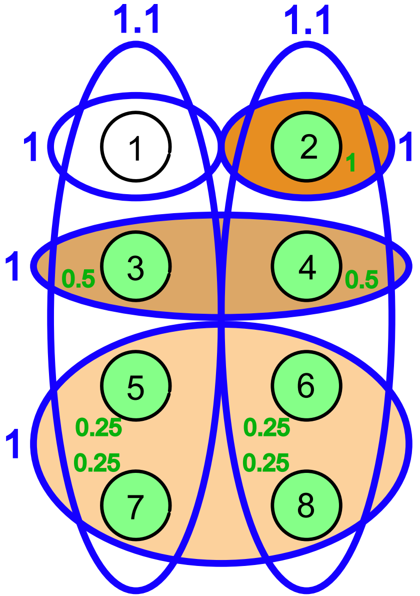

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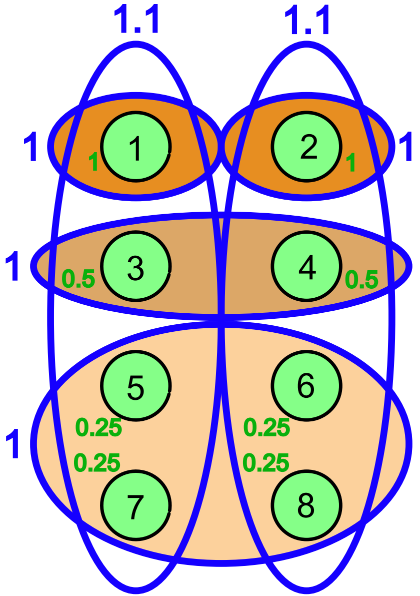

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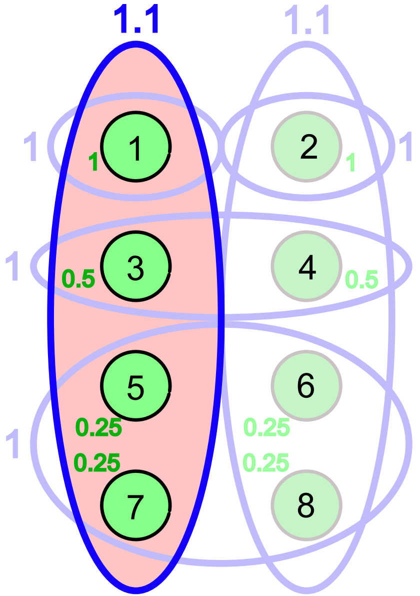

Greedy-Set-Cover

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Greedy-Set-Cover:

Start with $R = U$ and no sets selected

While $R \neq \emptyset$

  Select set S_i that minimizes $w_i/|S_i \cap R|$

  Delete set S_i from R

EndWhile

Return the selected sets

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Return the selected sets
```

- The algorithm computes a set cover whose weight is at most $O(\log n)$ times the optimal weight (Johnson 1974, Lovász 1975, Chvatal 1979).

Add Bookkeeping to Greedy-Set-Cover

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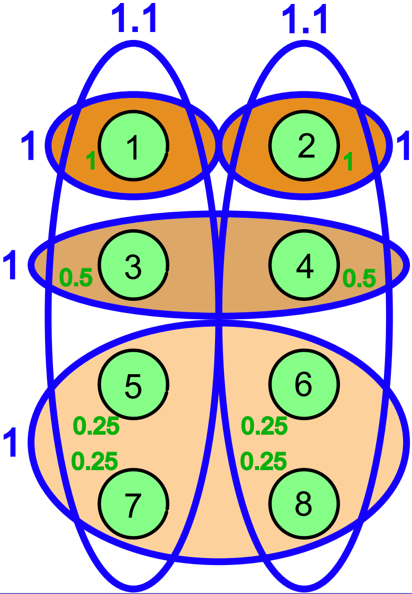
- In the algorithm, after selecting S_i, add the line

  `Define` $c_s = w_i/|S_i \cap R|$ `for all` $s \in S_i \cap R$.
- As each set S_i is selected, distribute its weight over the costs c_s of the *newly*-covered elements.
- Each element in the universe assigned cost exactly once.



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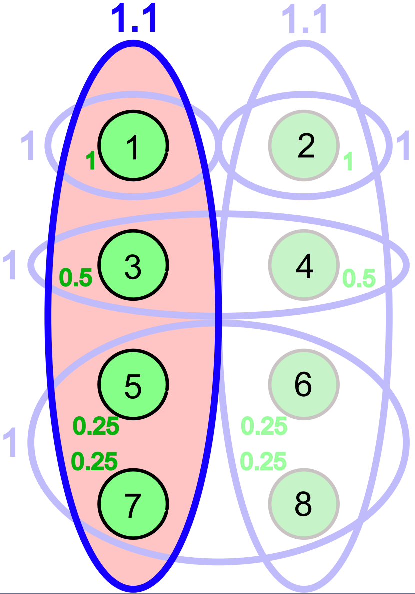

Starting the Analysis of Greedy-Set-Cover

- Let $\mathcal{C}$ be the set cover computed by GREEDY-SET-COVER.
- Claim: $\sum_{S_i \in \mathcal{C}} w_i = \sum_{s \in U} c_s$.

$$\sum_{S_i \in \mathcal{C}} w_i = \sum_{S_i \in \mathcal{C}} \left(\sum_{s \in S_i \cap R} c_s \right), \text{ by definition of } c_s$$
$$= \sum_{s \in U} c_s, \text{ since each element in the universe contributes exactly once}$$

- In other words, the total weight of the solution computed by GREEDY-SET-COVER is the total costs it assigns to the elements in the universe.
- Can "switch" between set-based weight of solution and element-based costs.
- Note: sets have weights whereas GREEDY-SET-COVER assigns costs to elements.

Intuition Behind the Proof

- Suppose $\mathcal{C}^*$ is the optimal set cover: $w^* = \sum_{S_j \in \mathcal{C}^*} w_j$.
- Goal is to relate total weight of sets in $\mathcal{C}$ to total weight of sets in $\mathcal{C}^*$.

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- Since $\mathcal{C}^*$ is a set cover, $\displaystyle\sum_{S_j \in \mathcal{C}^*} \left(\sum_{s \in S_j} c_s \right) \geq \sum_{s \in U} c_s = \sum_{S_i \in \mathcal{C}} w_i = w.$

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- In the sum on the left, S_j is a set in $\mathcal{C}^*$ (need not be a set in $\mathcal{C}$). How large can total cost of elements in such a set be?
- For *any* set S_k, suppose we can prove $\sum_{s \in S_k} c_s \leq \alpha w_k$, for some fixed $\alpha > 0$, i.e., total cost assigned by GREEDY-SET-COVER to the elements in S_k cannot be much larger than the weight of s_k.

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- Then $\displaystyle w \leq \sum_{S_j \in \mathcal{C}^*} \left(\sum_{s \in S_j} c_s \right) \leq \sum_{S_j \in \mathcal{C}^*} \alpha w_j = \alpha w^*.$

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- Then $\displaystyle w \leq \sum_{S_j \in \mathcal{C}^*} \left(\sum_{s \in S_j} c_s \right) \leq \sum_{S_j \in \mathcal{C}^*} \alpha w_j = \alpha w^*.$
- For every set S_k in the input, goal is to prove an upper bound on $\displaystyle\frac{\sum_{s \in S_k} c_s}{w_k}$.

Upper Bounding Cost-by-Weight Ratio

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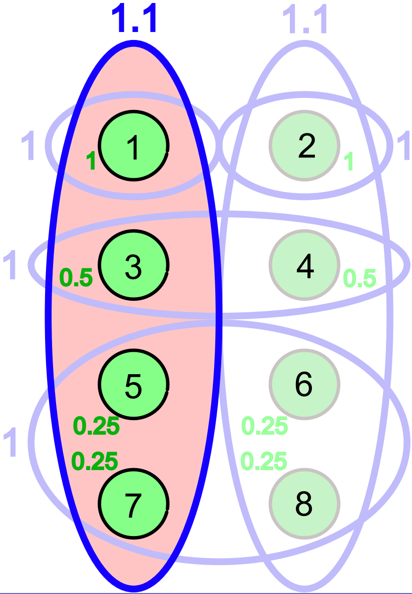
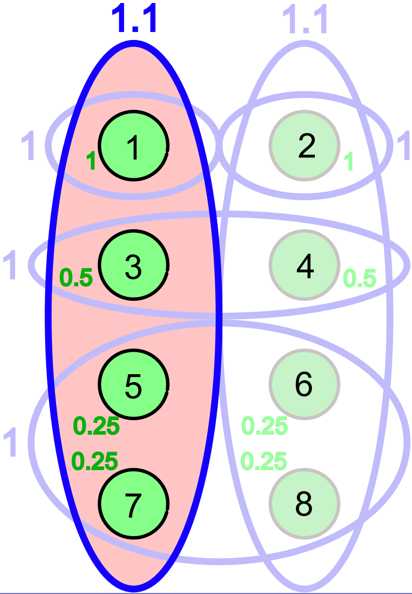
- Claim: For every set S_k, the sum $\sum_{s \in S_k} c_s \leq H(|S_K|) w_k$.



Renumbering Elements in S_k

- Renumber elements in U so that elements in S_k are the first $d = |S_k|$ elements of U, i.e., $S_k = \{s_1, s_2, \ldots, s_d\}$.
- Order elements of S in the order they get covered by the algorithm (i.e., when they get assigned a cost by Greedy-Set-Cover).



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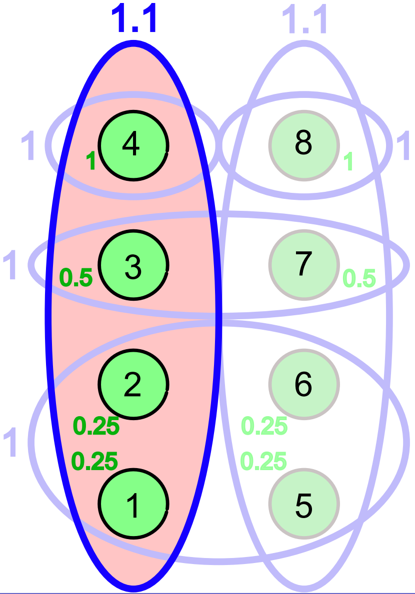

Proving $\sum_{s \in S_k} c_s \leq H(|S_K|)w_k$

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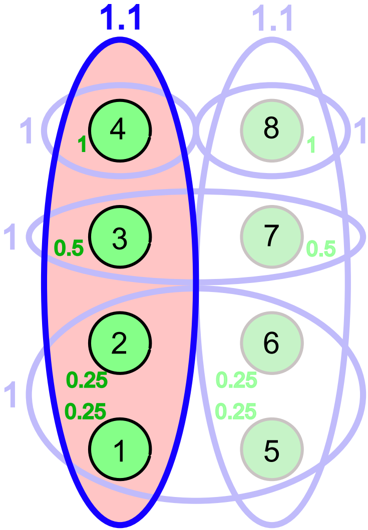

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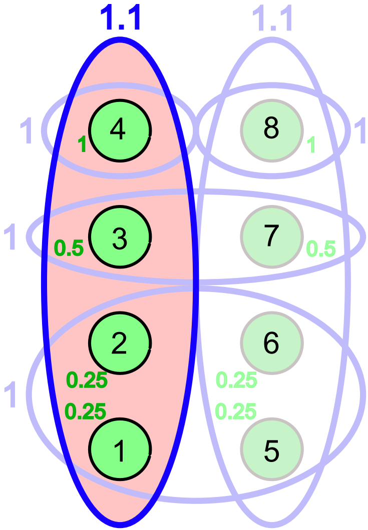
- Therefore, $\dfrac{w_k}{|S_k \cap R|} \leq \dfrac{w_k}{d - j + 1}$.



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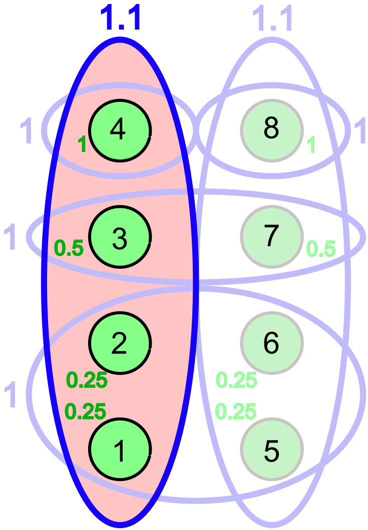

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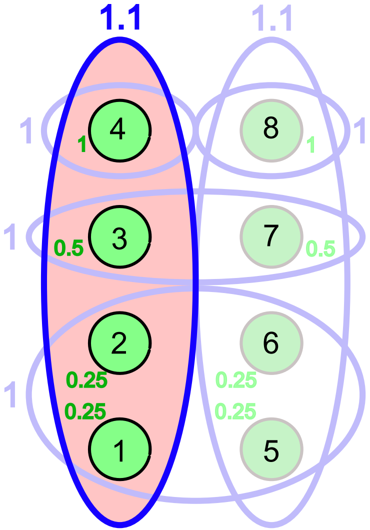
- We are done!

$$\sum_{s \in S_k} c_s = \sum_{j=1}^{d} c_{s_j} \le \sum_{j=1}^{d} \frac{w_k}{d - j + 1} = H(d)w_k.$$



Proving Upper Bound on Cost of Greedy-Set-Cover

- Let us assume $\sum_{s \in S_k} c_s \leq H(|S_K|)w_k$.
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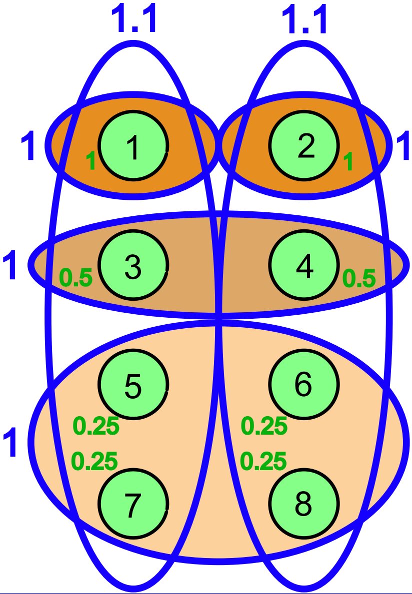
- We have proven that Greedy-Set-Cover computes a set cover whose weight is at most $H(d^*)$ times the optimal weight.

How Badly Can Greedy-Set-Cover Perform?



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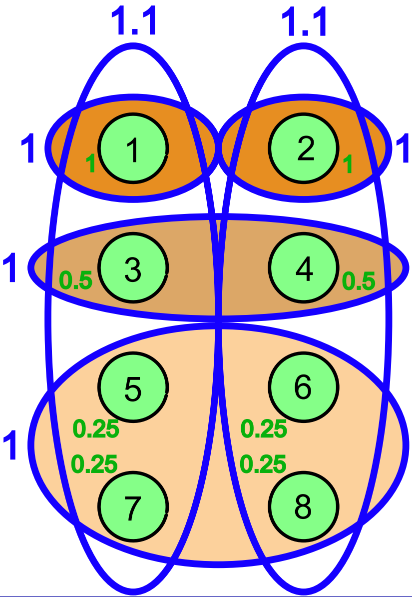

- Generalise this example to show that algorithm produces a set cover of weight $\Omega(\log n)$ even though optimal weight is $2 + \varepsilon$.
- More complex constructions show greedy algorithm incurs a weight close to $H(n)$ times the optimal weight.
- No polynomial time algorithm can achieve an approximation bound better than $H(n)$ times optimal unless $\mathcal{P} = \mathcal{NP}$ (Lund and Yannakakis, 1994).