Dynamic Programming

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Algorithm Design Techniques

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4. Dynamic programming
   - More powerful than greedy and divide-and-conquer strategies.
   - *Implicitly* explore space of all possible solutions.
   - Solve multiple sub-problems and build up correct solutions to larger and larger sub-problems.
   - Careful analysis needed to ensure number of sub-problems solved is polynomial in the size of the input.

History of Dynamic Programming

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- The Secretary of Defense at that time was hostile to mathematical research.
- Bellman sought an impressive name to avoid confrontation.
  - "it's impossible to use dynamic in a pejorative sense"
  - "something not even a Congressman could object to" (Bellman, R. E., *Eye of the Hurricane, An Autobiography*).

Applications of Dynamic Programming

- Computational biology: Smith-Waterman algorithm for sequence alignment.
- Operations research: Bellman-Ford algorithm for shortest path routing in networks.
- Control theory: Viterbi algorithm for hidden Markov models.
- Computer science (theory, graphics, AI, ...): Unix `diff` command for comparing two files.

Review: Interval Scheduling

INTERVAL SCHEDULING

INSTANCE: Nonempty set $\{(s_i, f_i), 1 \leq i \leq n\}$ of start and finish times of n jobs.

SOLUTION: The largest subset of mutually compatible jobs.

- Two jobs are *compatible* if they do not overlap.

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SOLUTION: The largest subset of mutually compatible jobs.

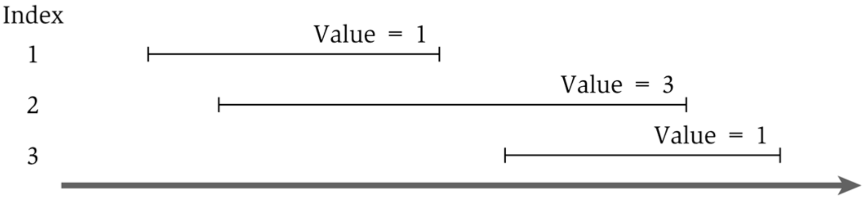
- Two jobs are *compatible* if they do not overlap.
- Greedy algorithm: sort jobs in increasing order of finish times. Add next job to current subset only if it is compatible with previously-selected jobs.

Weighted Interval Scheduling

WEIGHTED INTERVAL SCHEDULING

INSTANCE: Nonempty set $\{(s_i, f_i), 1 \leq i \leq n\}$ of start and finish times of n jobs and a weight $v_i \geq 0$ associated with each job.

SOLUTION: A set S of mutually compatible jobs such that $\sum_{i \in S} v_i$ is maximised.



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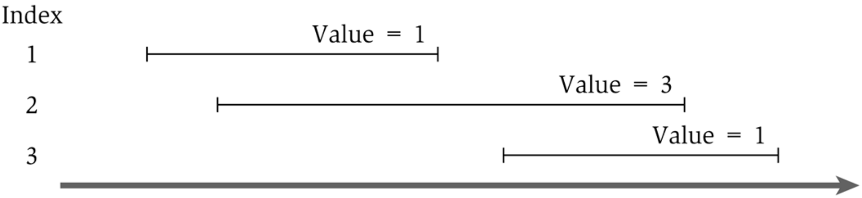
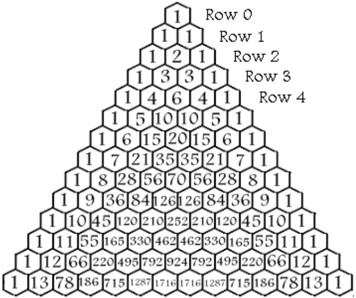


Figure 6.1 A simple instance of weighted interval scheduling.

- Greedy algorithm can produce arbitrarily bad results for this problem.

Detour: a Binomial Identity



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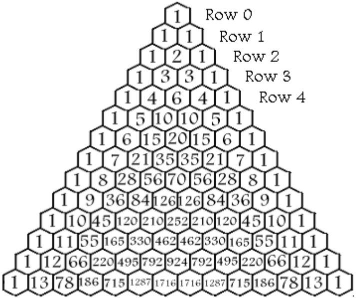

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  - Each element is a binomial co-efficient.
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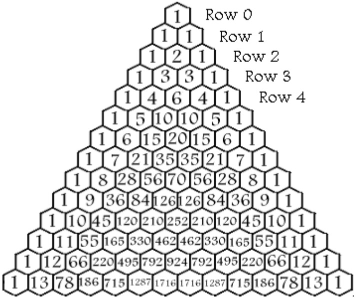

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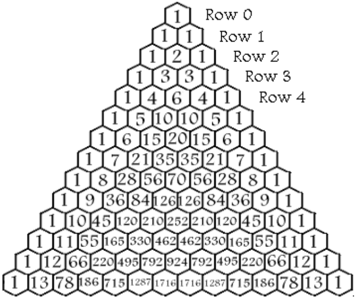

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$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

- Proof: either we select the nth element or not ...

Approach

- Sort jobs in increasing order of finish time and relabel: $f_1 \leq f_2 \leq \ldots \leq f_n$.
- Job i comes before job j if $i < j$.
- $p(j)$ is the largest index $i < j$ such that job i is compatible with job j. $p(j) = 0$ if there is no such job i.
- All jobs that come before job $p(j)$ are also compatible with job j.

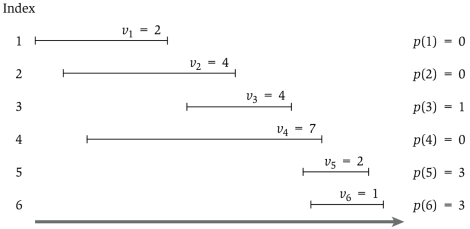

Figure 6.2 An instance of weighted interval scheduling with the functions $p(j)$ defined for each interval j.

- We will develop optimal algorithm from obvious statements about the problem.

Sub-problems

- Let $\mathcal{O}$ be the optimal solution: it contains a subset of the input jobs. Two cases to consider.
  - Case 1 job n is not in $\mathcal{O}$.
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- $\mathcal{O}$ must be the best of these two choices!
- Suggests finding optimal solution for sub-problems consisting of jobs $\{1, 2, \ldots, j-1, j\}$, for all values of j.

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- When does request j belong to $\mathcal{O}_j$? If and only if $v_j + \text{OPT}(p(j)) \geq \text{OPT}(j-1)$.

Recursive Algorithm

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Compute-Opt(j)
  If j = 0 then
    Return 0
  Else
    Return max(v_j+Compute-Opt(p(j)), Compute-Opt(j − 1))
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- Correctness of algorithm follows by induction (see textbook for proof).

Example of Recursive Algorithm

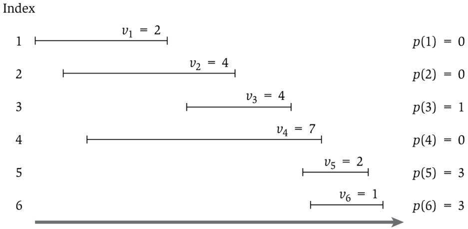

Figure 6.2 An instance of weighted interval scheduling with the functions $p(j)$ defined for each interval j.

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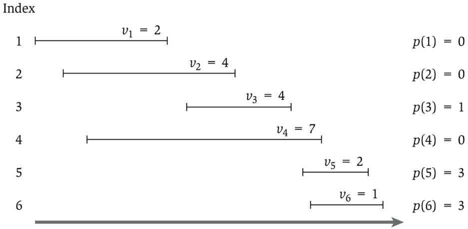

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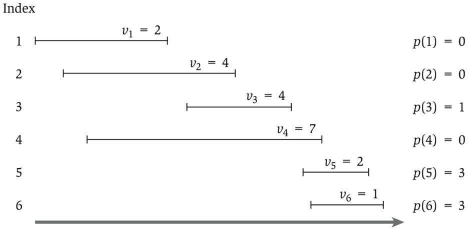

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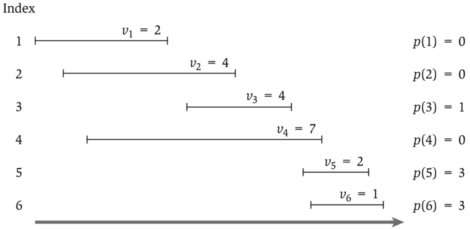

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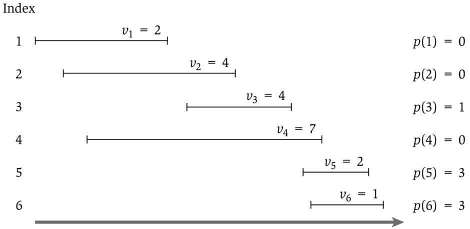

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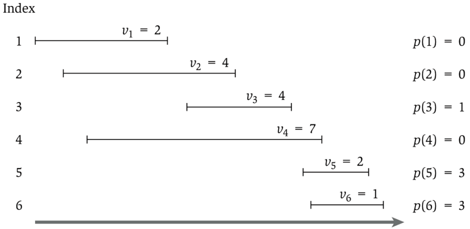

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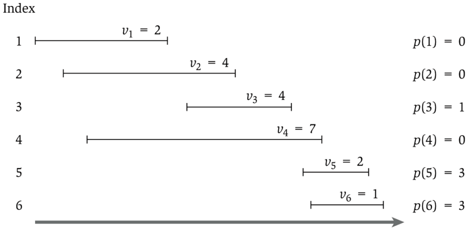

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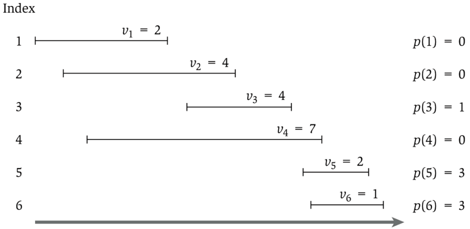

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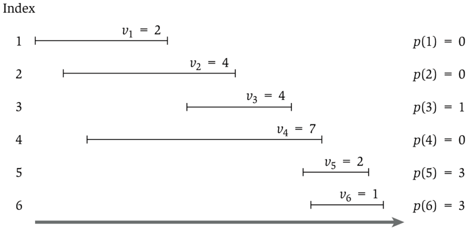

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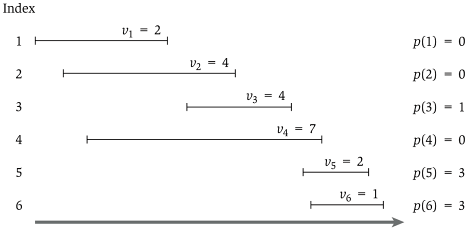

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\text{OPT}(2) &= \max(v_2 + \text{OPT}(p(2)), \text{OPT}(1)) = \max(4 + \text{OPT}(0), \text{OPT}(1)) = 4 \\
\text{OPT}(1) &= v_1 = 2 \\
\text{OPT}(0) &= 0
\end{aligned}$$

Example of Recursive Algorithm

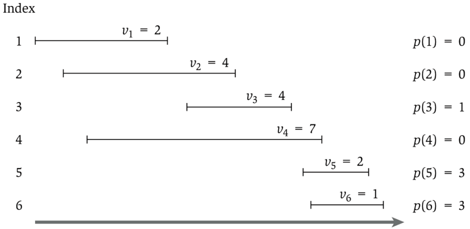

Figure 6.2 An instance of weighted interval scheduling with the functions $p(j)$ defined for each interval j.

$$
\begin{aligned}
\mathrm{OPT}(6) &= \max(v_6 + \mathrm{OPT}(p(6)), \mathrm{OPT}(5)) = \max(1 + \mathrm{OPT}(3), \mathrm{OPT}(5))\\
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$$

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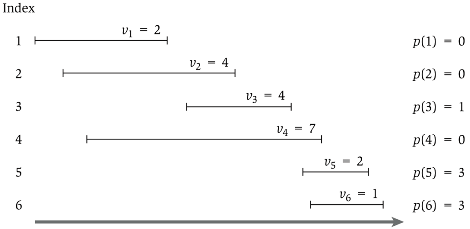

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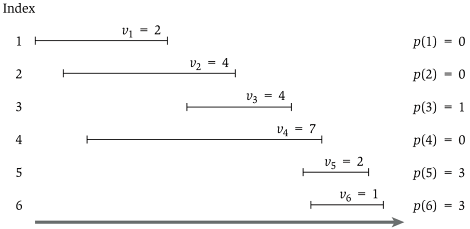

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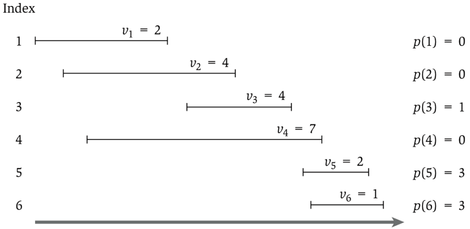
- Optimal solution is

Example of Recursive Algorithm



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\end{aligned}$$

- Optimal solution is job 5, job 3, and job 1.

Running Time of Recursive Algorithm

```
Compute-Opt(j)
  If j = 0 then
    Return 0
  Else
    Return max(v_j+Compute-Opt(p(j)), Compute-Opt(j − 1))
  Endif
```

Running Time of Recursive Algorithm

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  If j = 0 then
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  Endif
```

- What is the running time of the algorithm? Can be exponential in n.
- When $p(j) = j - 2$, for all $j \geq 2$: recursive calls are for $j - 1$ and $j - 2$.

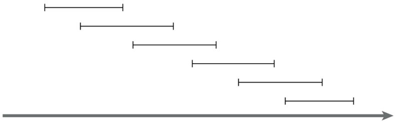

Figure 6.4 An instance of weighted interval scheduling on which the simple Compute-Opt recursion will take exponential time. The values of all intervals in this instance are 1.

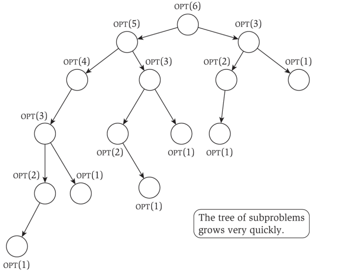

Figure 6.3 The tree of subproblems called by Compute-Opt on the problem instance of Figure 6.2.

Memoisation

- Store $\text{OPT}(j)$ values in a cache and reuse them rather than recompute them.

Memoisation

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```
M-Compute-Opt(j)
  If j = 0 then
    Return 0
  Else if M[j] is not empty then
    Return M[j]
  Else
   Define M[j] = max(v_j+M-Compute-Opt(p(j)), M-Compute-Opt(j - 1))
    Return M[j]
   Endif
```

Running Time of Memoisation

```
M-Compute-Opt(j)
  If j = 0 then
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  Else
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  Endif
```

- Claim: running time of this algorithm is $O(n)$ (after sorting).

Running Time of Memoisation

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    Return M[j]
  Endif
```

- Claim: running time of this algorithm is $O(n)$ (after sorting).
- Time spent in a single call to `M-Compute-Opt` is $O(1)$ apart from time spent in recursive calls.
- Total time spent is the order of the number of recursive calls to `M-Compute-Opt`.
- How many such recursive calls are there in total?

Running Time of Memoisation

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- Time spent in a single call to `M-Compute-Opt` is $O(1)$ apart from time spent in recursive calls.
- Total time spent is the order of the number of recursive calls to `M-Compute-Opt`.
- How many such recursive calls are there in total?
- Use number of filled entries in M as a measure of progress.
- Each time `M-Compute-Opt` issues two recursive calls, it fills in a new entry in M.
- Therefore, total number of recursive calls is $O(n)$.

Computing $\mathcal{O}$ in Addition to OPT(n)

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- Explicitly store $\mathcal{O}_j$ in addition to OPT(j).

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- Recall: request j belong to $\mathcal{O}_j$ if and only if $v_j + \text{OPT}(p(j)) \geq \text{OPT}(j-1)$.
- Can recover $\mathcal{O}_j$ from values of the optimal solutions in $O(j)$ time.

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- Can recover $\mathcal{O}_j$ from values of the optimal solutions in $O(j)$ time.

```
Find-Solution(j)
  If j = 0 then
    Output nothing
  Else
    If v_j + M[p(j)] ≥ M[j − 1] then
      Output j together with the result of Find-Solution(p(j))
    Else
      Output the result of Find-Solution(j − 1)
    Endif
  Endif
```

From Recursion to Iteration

- Unwind the recursion and convert it into iteration.
- Can compute values in M iteratively in $O(n)$ time.
- `Find-Solution` works as before.

```
Iterative-Compute-Opt
  M[0] = 0
  For j = 1, 2, ..., n
    M[j] = max(v_j + M[p(j)], M[j − 1])
  Endfor
```

Basic Outline of Dynamic Programming

- To solve a problem, we need a collection of sub-problems that satisfy a few properties:
  1. There are a polynomial number of sub-problems.
  2. The solution to the problem can be computed easily from the solutions to the sub-problems.
  3. There is a natural ordering of the sub-problems from "smallest" to "largest".
  4. There is an easy-to-compute recurrence that allows us to compute the solution to a sub-problem from the solutions to some smaller sub-problems.

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  4. There is an easy-to-compute recurrence that allows us to compute the solution to a sub-problem from the solutions to some smaller sub-problems.
- Difficulties in designing dynamic programming algorithms:
  1. Which sub-problems to define?
  2. How can we tie together sub-problems using a recurrence?
  3. How do we order the sub-problems (to allow iterative computation of optimal solutions to sub-problems)?

Least Squares Problem

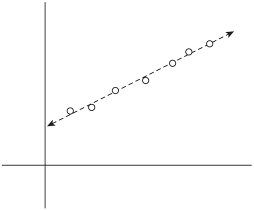
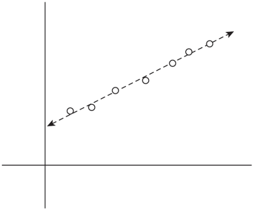


Figure 6.6 A "line of best fit."

- Given scientific or statistical data plotted on two axes.
- Find the "best" line that "passes" through these points.

Least Squares Problem



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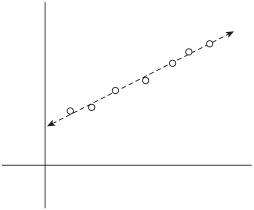
Least Squares Problem



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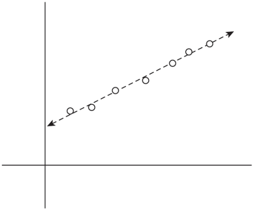
- How do we formalise the problem?

LEAST SQUARES
INSTANCE: Set $P = \{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$ of n points.
SOLUTION: Line $L : y = ax + b$ that minimises

$$\text{Error}(L, P) = \sum_{i=1}^{n} (y_i - ax_i - b)^2.$$

Least Squares Problem



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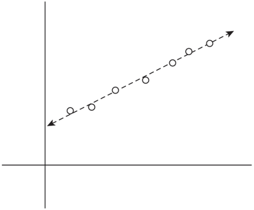

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$$\text{Error}(L, P) = \sum_{i=1}^{n} (y_i - ax_i - b)^2.$$

- Minimisation is over all possible choices of a and b.
- Solution is achieved by

$$a = \frac{n \sum_i x_i y_i - \left(\sum_i x_i\right)\left(\sum_i y_i\right)}{n \sum_i x_i^2 - \left(\sum_i x_i\right)^2} \text{ and } b = \frac{\sum_i y_i - a \sum_i x_i}{n}$$

Segmented Least Squares

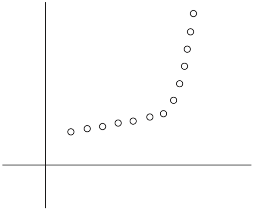
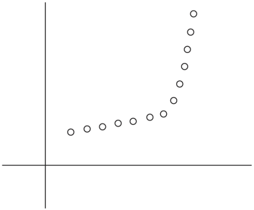


Figure 6.7 A set of points that lie approximately on two lines.

Segmented Least Squares



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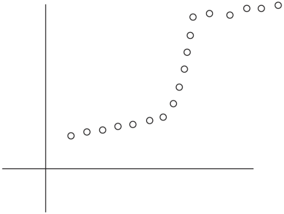
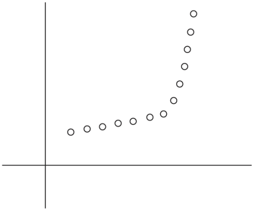


Figure 6.8 A set of points that lie approximately on three lines.

Segmented Least Squares



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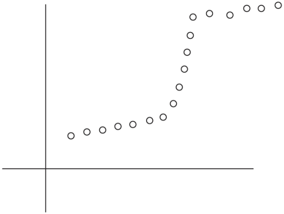
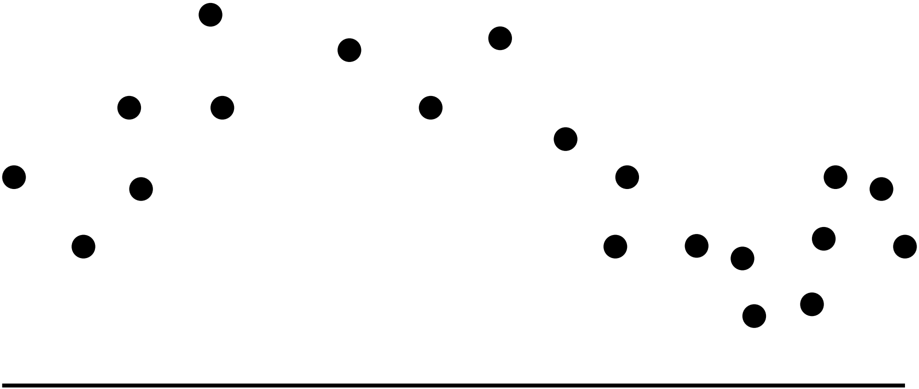


Figure 6.8 A set of points that lie approximately on three lines.

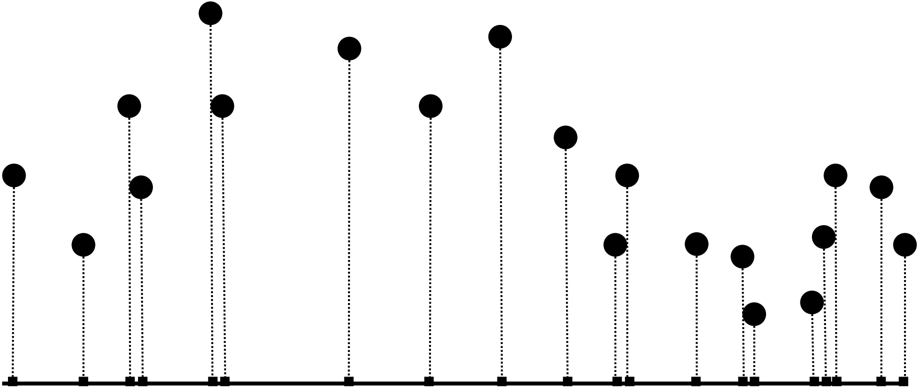
- Want to fit multiple lines through P.
- Each line must fit contiguous set of x-coordinates.
- Lines must minimise total error.

Example of Segmented Least Squares



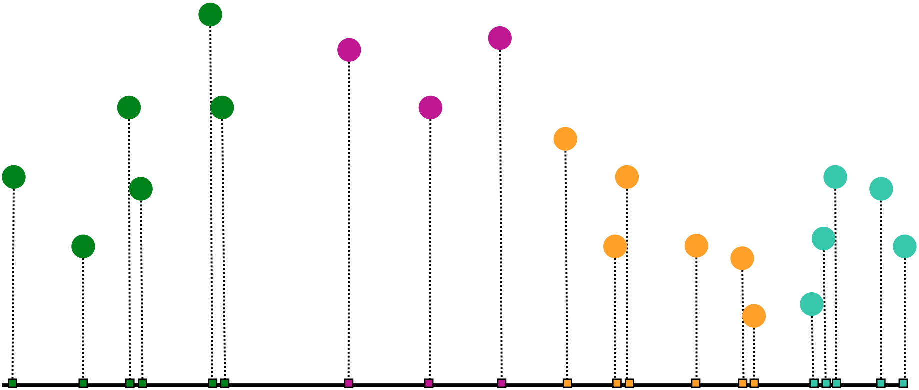
Input contains a set of two-dimensional points.

Example of Segmented Least Squares



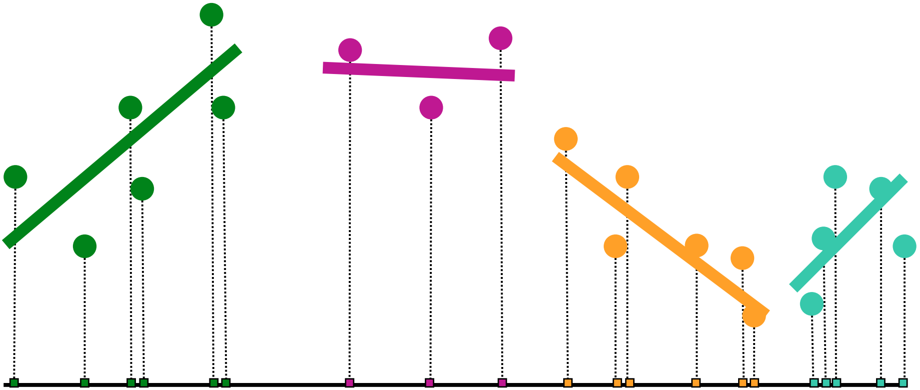
Consider the x-coordinates of the points in the input.

Example of Segmented Least Squares



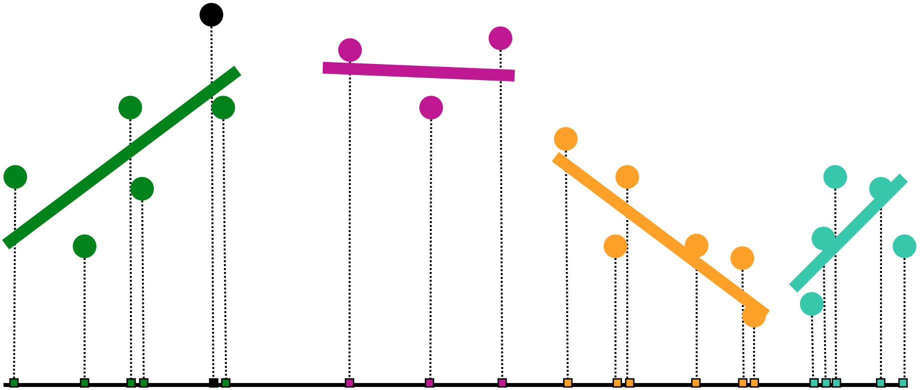
Divide the points into segments; each segment contains consecutive points in the sorted order by x-coordinate.

Example of Segmented Least Squares



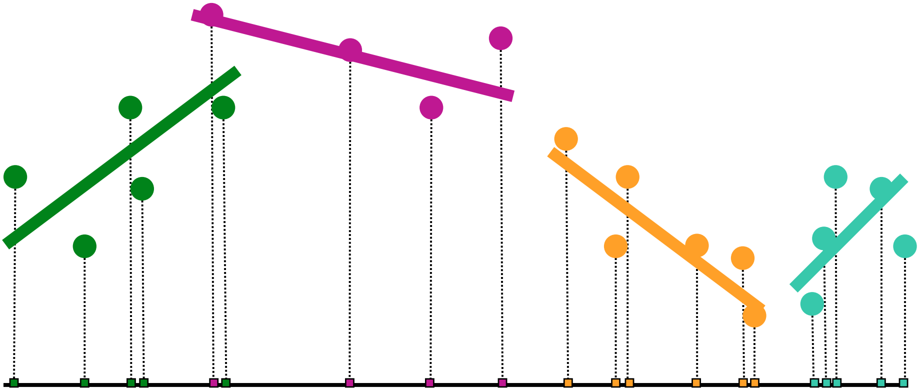
Fit the best line for each segment.

Example of Segmented Least Squares



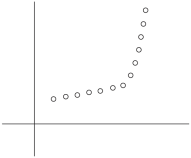
Illegal solution: black point is not in any segment.

Example of Segmented Least Squares



Illegal solution: leftmost purple point has x-coordinate between last two points in green segment.

Segmented Least Squares



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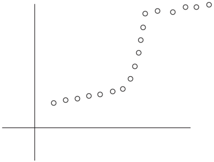

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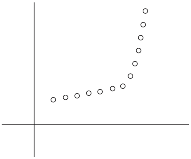

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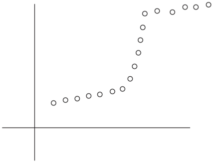

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SEGMENTED LEAST SQUARES

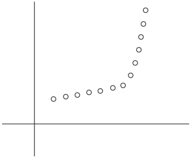
INSTANCE: Set $P = \{p_i = (x_i, y_i), 1 \leq i \leq n\}$ of n points, $x_1 < x_2 < \cdots < x_n$.

SOLUTION: A integer k, a partition of P into k segments $\{P_1, P_2, \ldots, P_k\}$, k lines $L_j : y = a_j x + b_j, 1 \leq j \leq k$ that minimise

$$\sum_{j=1}^{k} \text{Error}(L_j, P_j)$$

- A subset P' of P is a *segment* if $1 \leq i < j \leq n$ exist such that $P' = \{(x_i, y_i), (x_{i+1}, y_{i+1}), \ldots, (x_{j-1}, y_{j-1}), (x_j, y_j)\}$.

Segmented Least Squares



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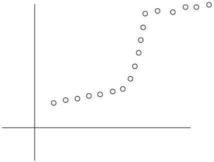

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INSTANCE: Set $P = \{p_i = (x_i, y_i), 1 \leq i \leq n\}$ of n points, $x_1 < x_2 < \cdots < x_n$ and a parameter $C > 0$.

SOLUTION: A integer k, a partition of P into k segments $\{P_1, P_2, \ldots, P_k\}$, k lines $L_j : y = a_j x + b_j, 1 \leq j \leq k$ that minimise

$$\sum_{j=1}^{k} \text{Error}(L_j, P_j) + Ck$$

- A subset P' of P is a *segment* if $1 \leq i < j \leq n$ exist such that $P' = \{(x_i, y_i), (x_{i+1}, y_{i+1}), \ldots, (x_{j-1}, y_{j-1}), (x_j, y_j)\}$.

Formulating the Recursion I

- Observation: p_n is part of some segment in the optimal solution. This segment starts at some point p_i.
- Let $OPT(i)$ be the optimal value for the points $\{p_1, p_2, \ldots, p_i\}$.
- Let $e_{i,j}$ denote the minimum error of a (single) line that fits $\{p_i, p_2, \ldots, p_j\}$.
- We want to compute $\mathrm{OPT}(n)$.

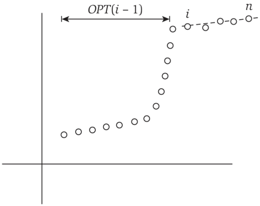

Figure 6.9 A possible solution: a single line segment fits points $p_i, p_{i+1}, \ldots, p_n$, and then an optimal solution is found for the remaining points $p_1, p_2, \ldots, p_{i-1}$.

- If the last segment in the optimal partition is $\{p_i, p_{i+1}, \ldots, p_n\}$, then

$$\mathrm{OPT}(n) = e_{i,n} + C + \mathrm{OPT}(i-1)$$

Formulating the Recursion II

- Consider the sub-problem on the points $\{p_1, p_2, \ldots p_j\}$
- To obtain $\text{OPT}(j)$, if the last segment in the optimal partition is $\{p_i, p_{i+1}, \ldots, p_j\}$, then

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- Since i can take only j distinct values,

$$\text{OPT}(j) = \min_{1 \leq i \leq j} \left(e_{i,j} + C + \text{OPT}(i-1) \right)$$

- Segment $\{p_i, p_{i+1}, \ldots p_j\}$ is part of the optimal solution for this sub-problem if and only if the minimum value of $\text{OPT}(j)$ is obtained using index i.

Dynamic Programming Algorithm

$$\mathrm{OPT}(j) = \min_{1 \le i \le j} \left(e_{i,j} + C + \mathrm{OPT}(i-1) \right)$$

```
Segmented-Least-Squares(n)
  Array M[0...n]
  Set M[0] = 0
  For all pairs i ≤ j
    Compute the least squares error e_{i,j} for the segment p_i, ..., p_j
  Endfor
  For j = 1, 2, ..., n
    Use the recurrence (6.7) to compute M[j]
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```

- Running time is $O(n^3)$, can be improved to $O(n^2)$.
- We can find the segments in the optimal solution by backtracking.

RNA Molecules

- RNA is a basic biological molecule. It is single stranded.
- RNA molecules fold into complex "secondary structures."
- Secondary structure often governs the behaviour of an RNA molecule.
- Various rules govern secondary structure formation:

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1. Pairs of bases match up; each base matches with ≤ 1 other base.
2. Adenine always matches with Uracil.
3. Cytosine always matches with Guanine.
4. There are no kinks in the folded molecule.
5. Structures are "knot-free".

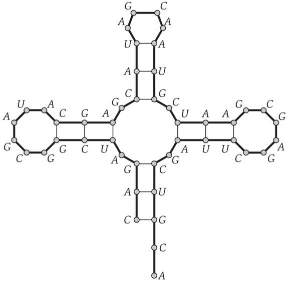

Figure 6.13 An RNA secondary structure. Thick lines connect adjacent elements of the sequence; thin lines indicate pairs of elements that are matched.

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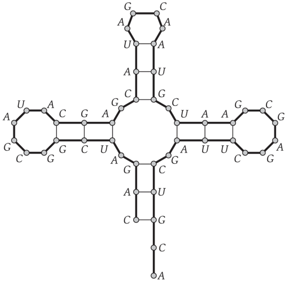

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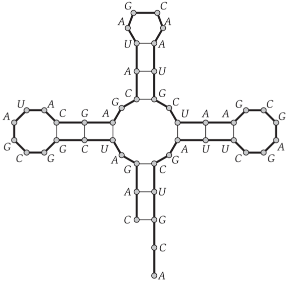

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- Problem: given an RNA molecule, predict its secondary structure.
- Hypothesis: In the cell, RNA molecules form the secondary structure with the lowest total free energy.

Formulating the Problem

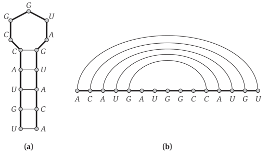

Figure 6.14 Two views of an RNA secondary structure. In the second view, (b), the string has been "stretched" lengthwise, and edges connecting matched pairs appear as noncrossing "bubbles" over the string.

- An *RNA molecule* is a string $B = b_1 b_2 \dots b_n$; each $b_i \in \{A, C, G, U\}$.
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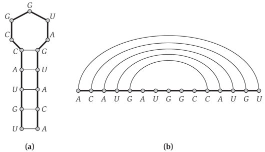
Formulating the Problem



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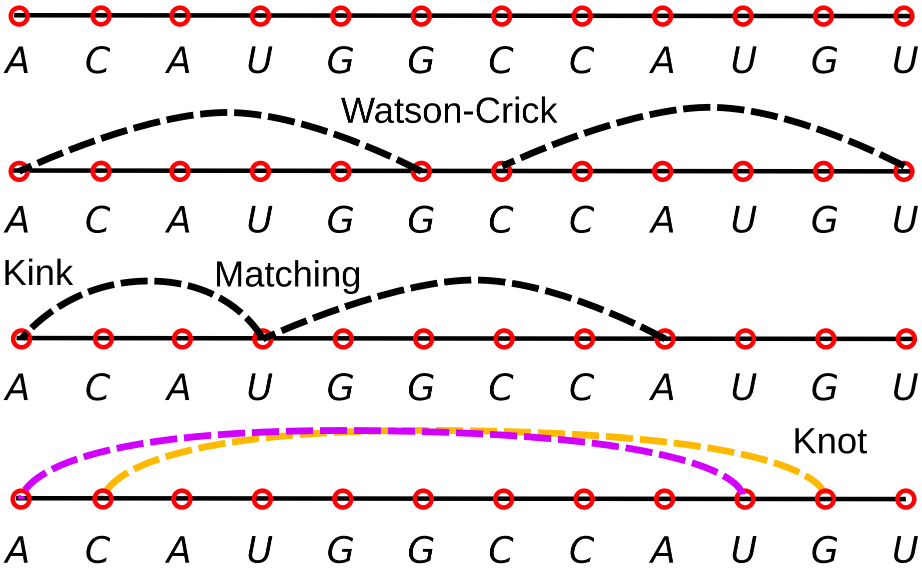
- A *secondary structure on B* is a set of pairs $S = \{(i, j)\}$, where $1 \leq i, j \leq n$ and
  1. *(No kinks.)* If $(i, j) \in S$, then $i < j - 4$.
  2. *(Watson-Crick)* The elements in each pair in S consist of either $\{A, U\}$ or $\{C, G\}$ (in either order).
  3. S is a *matching*: no index appears in more than one pair.
  4. *(No knots)* If (i, j) and (k, l) are two pairs in S, then we cannot have $i < k < j < l$.
- The *energy* of a secondary structure $\propto$ the number of base pairs in it.
- Problem: Compute the largest secondary structure, i.e., with the largest number of base pairs.

Illegal Secondary Structures



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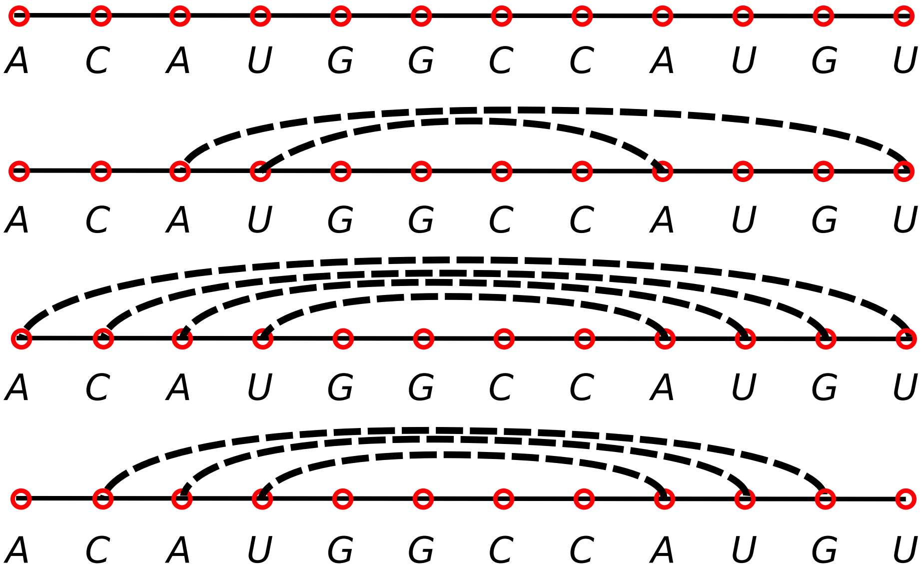

Dynamic Programming Approach

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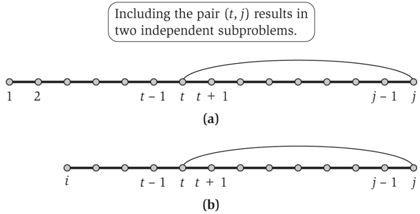

Figure 6.15 Schematic views of the dynamic programming recurrence using (a) one variable, and (b) two variables.

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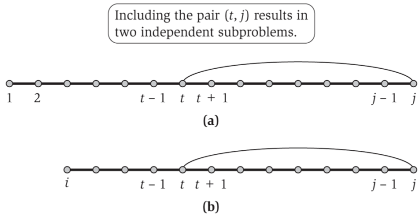

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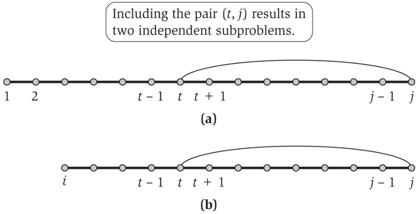

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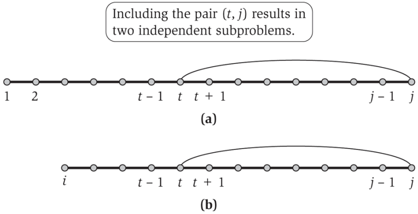
- Insight: need sub-problems indexed both by start and by end.



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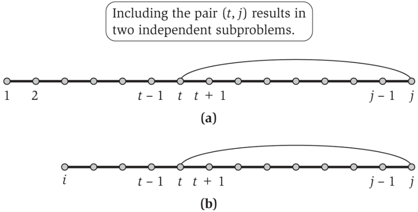

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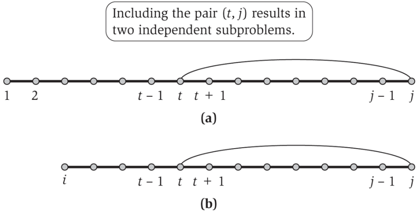

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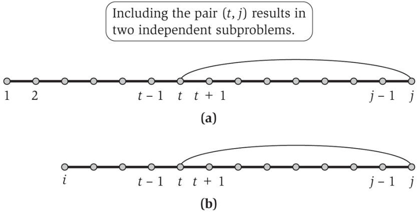

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$$\text{OPT}(i, j) = \max \left(\qquad \right)$$

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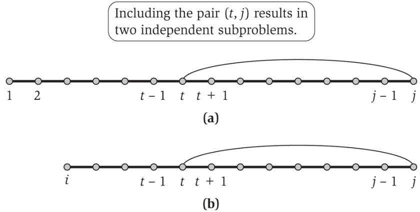

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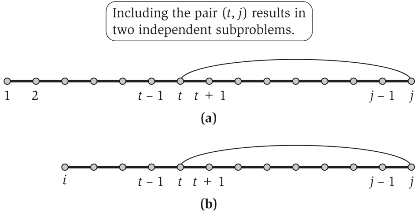

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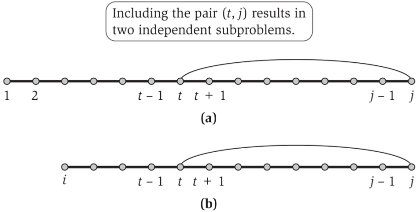

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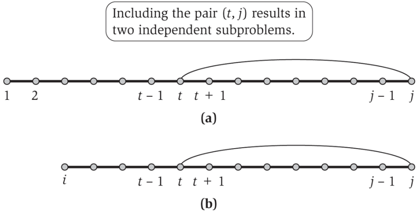

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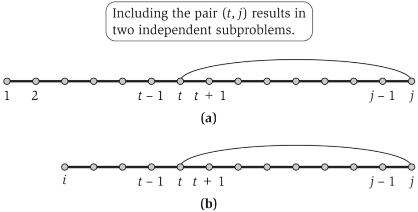

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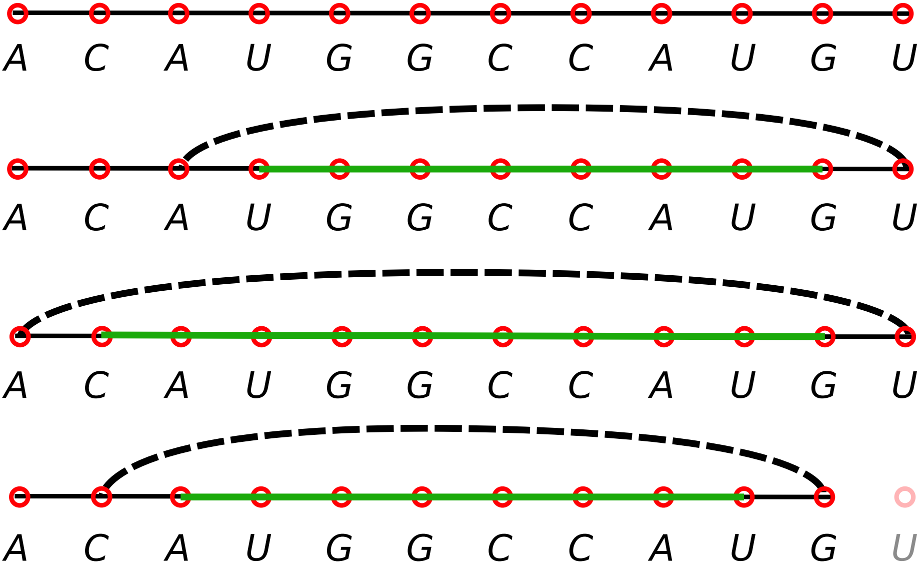
- In the "inner" maximisation, t runs over all indices between i and $j-5$ that are allowed to pair with j.

Example of Dynamic Programming Algorithm



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  For i = 1, 2, . . . n − k
    Set j = i + k
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  Endfor
Endfor
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- Running time of the algorithm is $O(n^3)$.

Example of Algorithm

RNA sequence *ACCGGUAGU*

4	0	0	0	
3	0	0		
2	0			
$i = 1$				
	$j = 6$	7	8	9

Initial values

4	0	0	0	0
3	0	0	1	
2	0	0		
$i = 1$	1			
	$j = 6$	7	8	9

Filling in the values for $k = 5$

4	0	0	0	0
3	0	0	1	1
2	0	0	1	
$i = 1$	1	1		
	$j = 6$	7	8	9

Filling in the values for $k = 6$

4	0	0	0	0
3	0	0	1	1
2	0	0	1	1
$i = 1$	1	1	1	
	$j = 6$	7	8	9

Filling in the values for $k = 7$

4	0	0	0	0
3	0	0	1	1
2	0	0	1	1
$i = 1$	1	1	1	2
	$j = 6$	7	8	9

Filling in the values for $k = 8$

Motivation

- Computational finance:
  - Each node is a financial agent.
  - The cost c_{uv} of an edge (u, v) is the cost of a transaction in which we buy from agent u and sell to agent v.
  - Negative cost corresponds to a profit.
- Internet routing protocols
  - Dijkstra's algorithm needs knowledge of the entire network.
  - Routers only know which other routers they are connected to.
  - Algorithm for shortest paths with negative edges is decentralised.
  - We will not study this algorithm in the class. See Chapter 6.9.

Problem Statement

- Input: a directed graph $G = (V, E)$ with a cost function $c : E \to \mathbb{R}$, i.e., c_{uv} is the cost of the edge $(u, v) \in E$.
- A *negative cycle* is a directed cycle whose edges have a total cost that is negative.
- Two related problems:
  1. If G has no negative cycles, find the *shortest s-t path*: a path of from source s to destination t with minimum total cost.
  2. Does G have a *negative cycle*?

Problem Statement

- Input: a directed graph $G = (V, E)$ with a cost function $c : E \rightarrow \mathbb{R}$, i.e., c_{uv} is the cost of the edge $(u, v) \in E$.
- A *negative cycle* is a directed cycle whose edges have a total cost that is negative.
- Two related problems:
    1. If G has no negative cycles, find the *shortest s-t path*: a path of from source s to destination t with minimum total cost.
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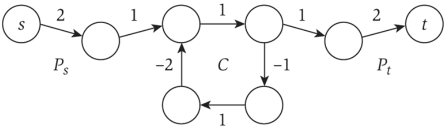

Figure 6.20 In this graph, one can find s-t paths of arbitrarily negative cost (by going around the cycle C many times).

Approaches for Shortest Path Algorithm

1. Dijsktra's algorithm.

2. Add some large constant to each edge.

Approaches for Shortest Path Algorithm

1. Dijsktra's algorithm. Computes incorrect answers because it is greedy.
2. Add some large constant to each edge. Computes incorrect answers because the minimum cost path changes.

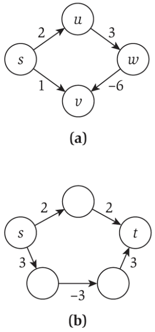

Figure 6.21 (a) With negative edge costs, Dijkstra's Algorithm can give the wrong answer for the Shortest-Path Problem. (b) Adding 3 to the cost of each edge will make all edges nonnegative, but it will change the identity of the shortest *s*-*t* path.

Dynamic Programming Approach

- Assume G has no negative cycles.
- Claim: There is a shortest path from s to t that is *simple* (does not repeat a node)

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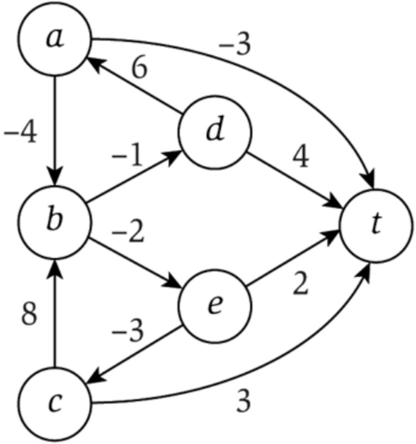
- Sub-problems defined by varying the number of edges in the shortest path and by varying the starting node in the shortest path.



Dynamic Programming Recursion

- $OPT(i, v)$: minimum cost of a v-t path that uses at most i edges.
- t is not explicitly mentioned in the sub-problems.
- Goal is to compute $\text{OPT}(n-1, s)$.

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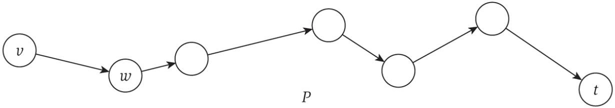

Figure 6.22 The minimum-cost path P from v to t using at most i edges.

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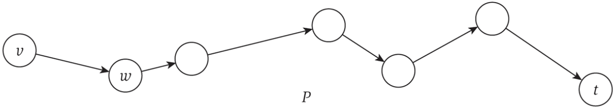

Figure 6.22 The minimum-cost path P from v to t using at most i edges.

- Let P be the optimal path whose cost is $\text{OPT}(i, v)$.
  1. If P actually uses $i-1$ edges, then $\text{OPT}(i, v) = \text{OPT}(i-1, v)$.
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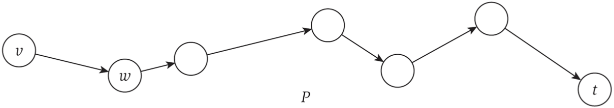

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$$\text{OPT}(i, v) = \min\left(\text{OPT}(i-1, v), \min_{w \in V}\left(c_{vw} + \text{OPT}(i-1, w)\right)\right)$$

Example of Dynamic Programming Recursion

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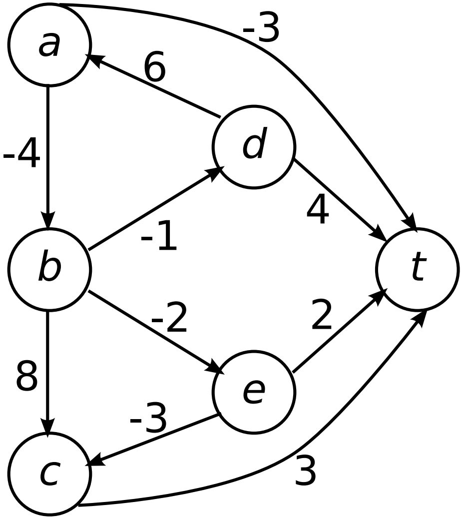

	0	1	2	3	4	5
t						
a						
b						
c						
d						
e						

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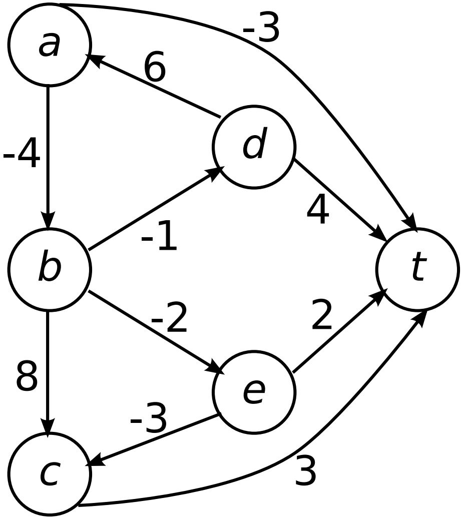

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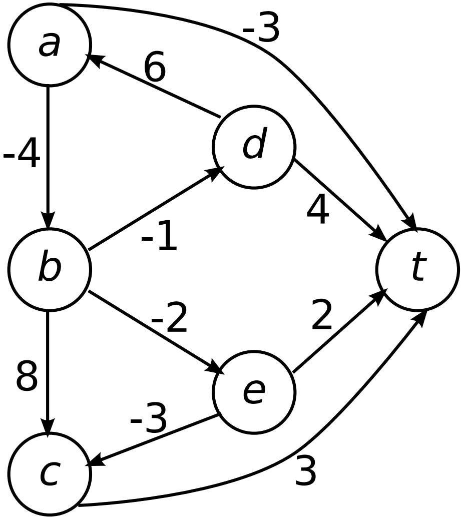

	0	1	2	3	4	5
t	0	0	0	0	0	0
a	∞					
b	∞					
c	∞					
d	∞					
e	∞					

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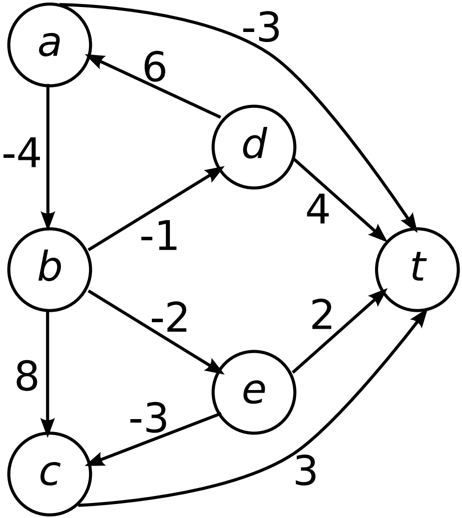

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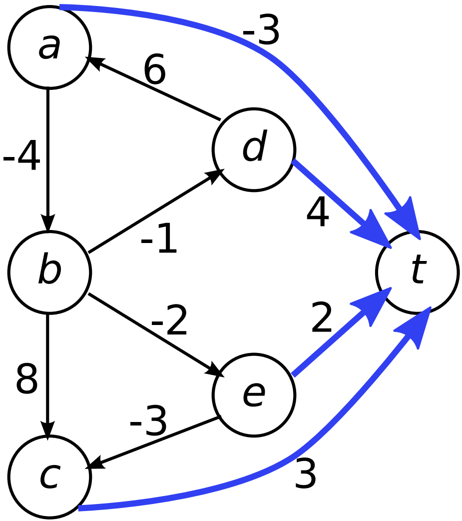

	0	1	2	3	4	5
t	0	0	0	0	0	0
a	∞	-3				
b	∞	∞				
c	∞	3				
d	∞	4				
e	∞	2				

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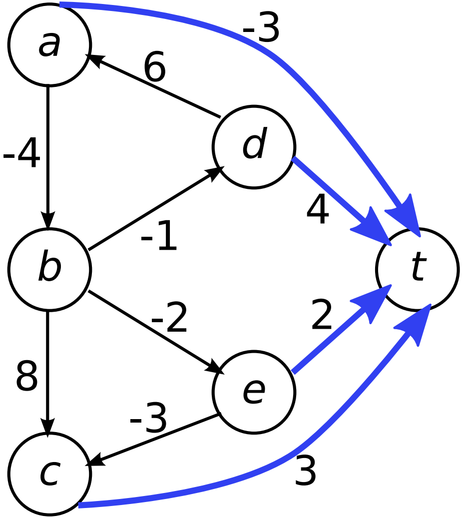

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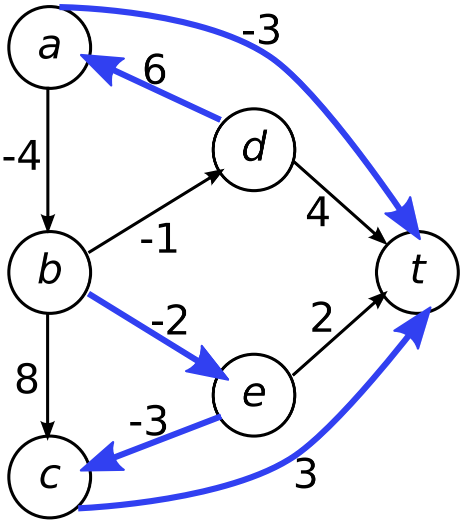

	0	1	2	3	4	5
t	0	0	0	0	0	0
a	∞	-3	-3			
b	∞	∞	0			
c	∞	3	3			
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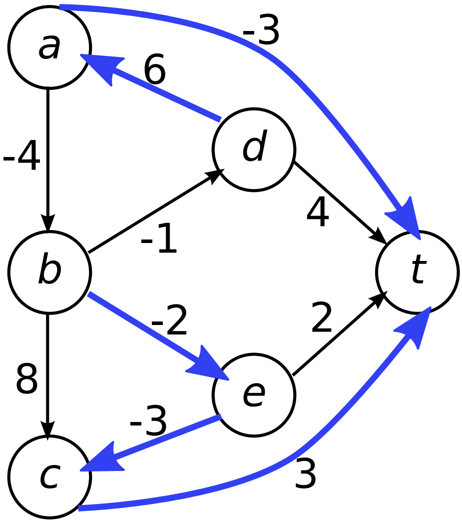

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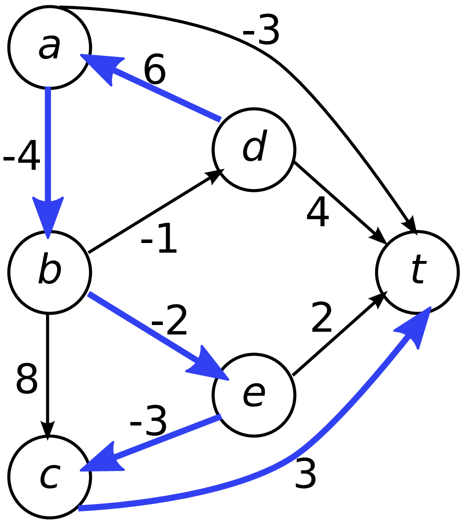

	0	1	2	3	4	5
t	0	0	0	0	0	0
a	∞	-3	-3	-4		
b	∞	∞	0	-2		
c	∞	3	3	3		
d	∞	4	3	3		
e	∞	2	0	0		

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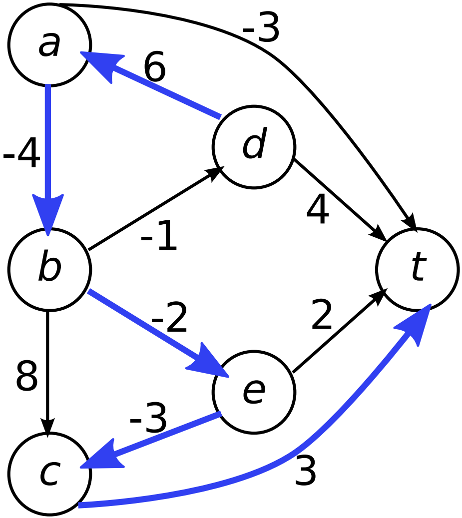

	0	1	2	3	4	5
t	0	0	0	0	0	0
a	∞	-3	-3	-4		
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c	∞	3	3	3		
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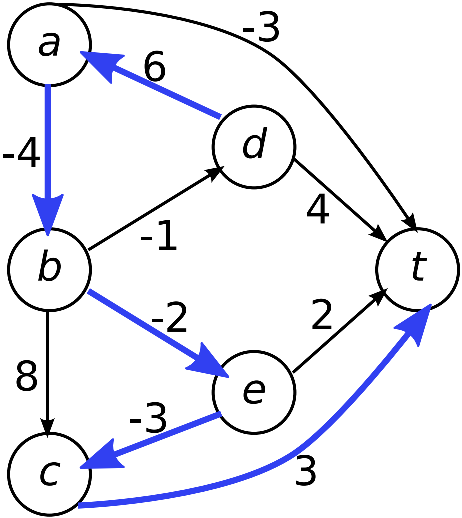

	0	1	2	3	4	5
t	0	0	0	0	0	0
a	∞	-3	-3	-4	-6	
b	∞	∞	0	-2	-2	
c	∞	3	3	3	3	
d	∞	4	3	3	2	
e	∞	2	0	0	0	

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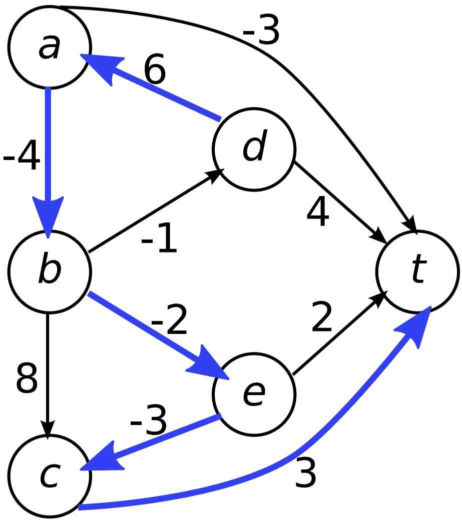

	0	1	2	3	4	5
t	0	0	0	0	0	0
a	∞	-3	-3	-4	-6	
b	∞	∞	0	-2	-2	
c	∞	3	3	3	3	
d	∞	4	3	3	2	
e	∞	2	0	0	0	

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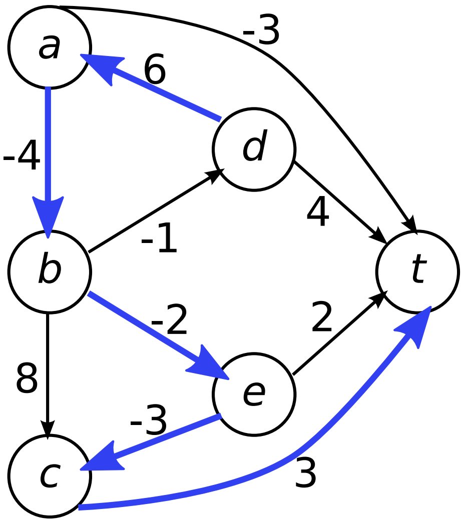

	0	1	2	3	4	5
t	0	0	0	0	0	0
a	∞	-3	-3	-4	-6	-6
b	∞	∞	0	-2	-2	-2
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d	∞	4	3	3	2	0
e	∞	2	0	0	0	0

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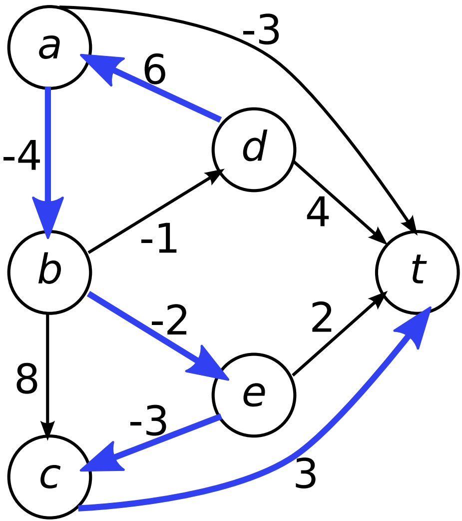

	0	1	2	3	4	5
t	0	0	0	0	0	0
a	∞	-3	-3	-4	-6	-6
b	∞	∞	0	-2	-2	-2
c	∞	3	3	3	3	3
d	∞	4	3	3	2	0
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Alternate Dynamic Programming Formulation

- $OPT_=(i, v)$: minimum cost of a v-t path that uses exactly i edges. Goal is to compute

$$\min_{i=1}^{n-1} \text{OPT}_=(\text{i, s}).$$

- Let P be the optimal path whose cost is $\text{OPT}_=(\text{i, v})$.

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- Compare the two desired solutions:

$$\min_{i=1}^{n-1} \text{OPT}_=(\text{i, s}) = \min_{i=1}^{n-1} \left(\min_{w \in V} \left(c_{sw} + \text{OPT}_=(\text{i - 1, w})\right)\right)$$

$$\text{OPT}(n-1, s) = \min\left(\text{OPT}(n-2, s), \min_{w \in V} \left(c_{sw} + \text{OPT}(n-2, w)\right)\right)$$

Bellman-Ford Algorithm

$$\mathrm{OPT}(i, v) = \min\left(\mathrm{OPT}(i-1, v), \min_{w \in V}\left(c_{vw} + \mathrm{OPT}(i-1, w)\right)\right)$$

```
Shortest-Path(G, s, t)
  n = number of nodes in G
  Array M[0 ... n-1, V]
  Define M[0, t] = 0 and M[0, v] = ∞ for all other v ∈ V
  For i = 1, ..., n-1
    For v ∈ V in any order
      Compute M[i, v] using the recurrence (6.23)
    Endfor
  Endfor
  Return M[n-1, s]
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    Endfor
  Endfor
  Return M[n-1,s]
```

- Space used is $O(n^2)$. Running time is $O(n^3)$.
- If shortest path uses k edges, we can recover it in $O(kn)$ time by tracing back through smaller sub-problems.

An Improved Bound on the Running Time

- Suppose G has n nodes and $m \ll \binom{n}{2}$ edges. Can we demonstrate a better upper bound on the running time?

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- If $n_v = |N_v|$ is the number of outgoing neighbours of v, then in each round, we spend time equal to

$$\sum_{v \in V} n_v =$$

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- If $n_v = |N_v|$ is the number of outgoing neighbours of v, then in each round, we spend time equal to

$$\sum_{v \in V} n_v = m.$$

- The total running time is $O(mn)$.

Improving the Memory Requirements

$$M[i, v] = \min\left(M[i-1, v], \min_{w \in N_v}\left(c_{vw} + M[i-1, w]\right)\right)$$

- The algorithm uses $O(n^2)$ space to store the array M.

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- Modified algorithm:
  1. Maintain two arrays M and M' indexed over V.
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$$M[v] = \min \left(M'[v], \min_{w \in N_v} \left(c_{vw} + M'[w] \right) \right)$$

- Claim: at the beginning of iteration i, M stores values of $\mathrm{OPT}(i-1, v)$ for all nodes $v \in V$.
- Space used is $O(n)$.

Computing the Shortest Path: Algorithm

$$M[v] = \min\left(M'[v], \min_{w \in N_v}\left(c_{vw} + M'[w]\right)\right)$$

- How can we recover the shortest path that has cost $M[v]$?

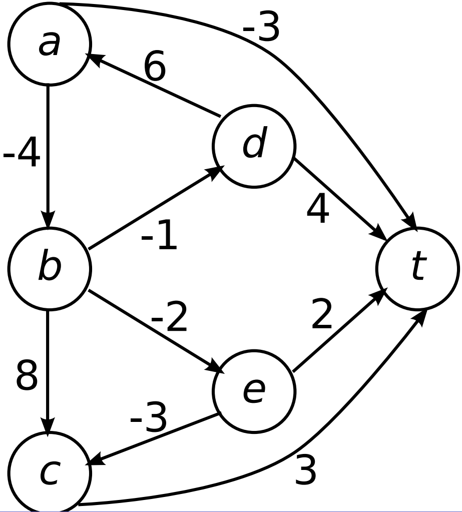
Computing the Shortest Path: Algorithm

$$M[v] = \min \left(M'[v], \min_{w \in N_v} \left(c_{vw} + M'[w] \right) \right)$$

- How can we recover the shortest path that has cost $M[v]$?
- For each node v, maintain $f(v)$, the first node after v in the current shortest path from v to t.
- To update $f(v)$, if we ever set $M[v]$ to $\min_{w \in N_v} \left(c_{vw} + M'[w] \right)$, set $f(v)$ to be the node w that attains this minimum.
- At the end, follow $f(v)$ pointers from s to t.

Example of Maintaining Pointers

$$M[v] = \min\left(M'[v], \min_{w \in N_v}\left(c_{vw} + M'[w]\right)\right)$$



	0	1	2	3	4	5
t	0	0	0	0	0	0
a	∞					
b	∞					
c	∞					
d	∞					
e	∞					

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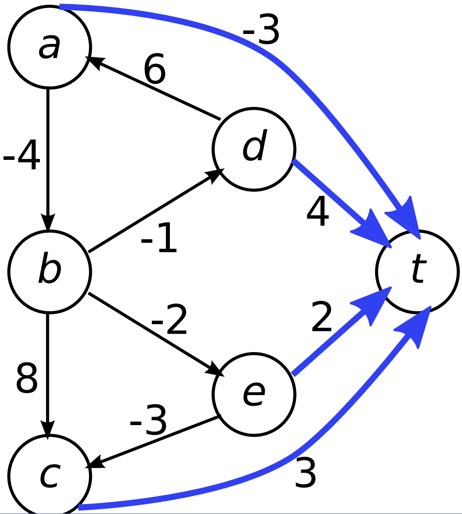
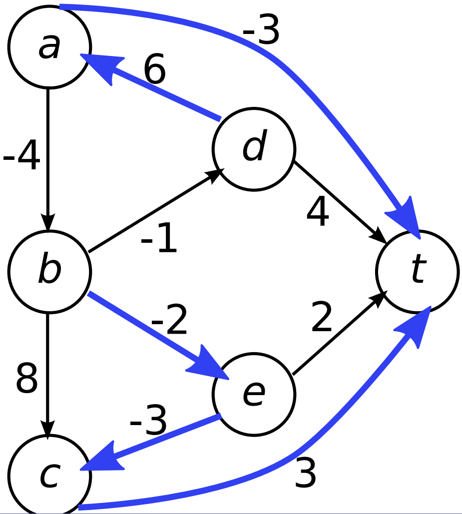
$$M[v] = \min\left(M'[v], \min_{w \in N_v}\left(c_{vw} + M'[w]\right)\right)$$



	0	1	2	3	4	5
t	0	0	0	0	0	0
a	∞	-3				
b	∞	∞				
c	∞	3				
d	∞	4				
e	∞	2				

Example of Maintaining Pointers

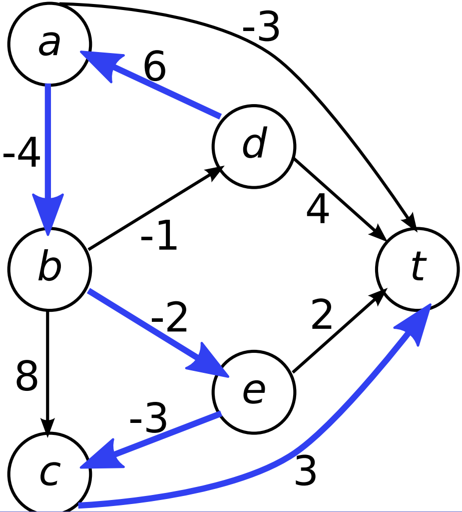
$$M[v] = \min\left(M'[v], \min_{w \in N_v}\left(c_{vw} + M'[w]\right)\right)$$



	0	1	2	3	4	5
t	0	0	0	0	0	0
a	∞	-3	-3			
b	∞	∞	0			
c	∞	3	3			
d	∞	4	3			
e	∞	2	0			

Example of Maintaining Pointers

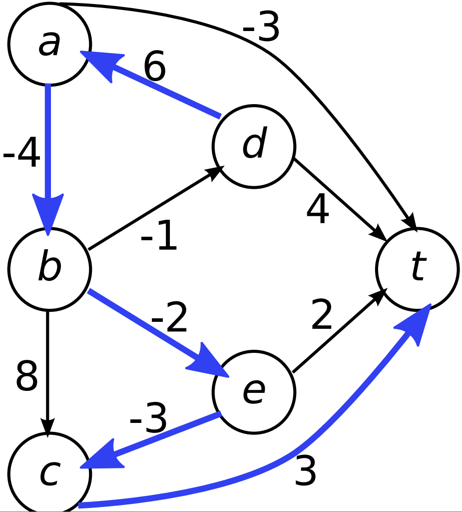
$$M[v] = \min\left(M'[v], \min_{w \in N_v}\left(c_{vw} + M'[w]\right)\right)$$



	0	1	2	3	4	5
t	0	0	0	0	0	0
a	∞	-3	-3	-4		
b	∞	∞	0	-2		
c	∞	3	3	3		
d	∞	4	3	3		
e	∞	2	0	0		

Example of Maintaining Pointers

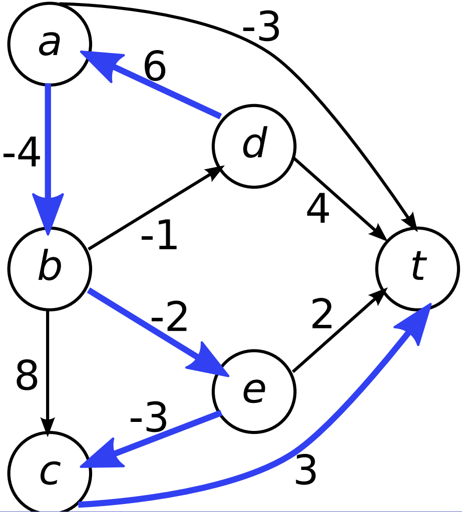
$$M[v] = \min\left(M'[v], \min_{w \in N_v}\left(c_{vw} + M'[w]\right)\right)$$



	0	1	2	3	4	5
t	0	0	0	0	0	0
a	∞	-3	-3	-4	-6	
b	∞	∞	0	-2	-2	
c	∞	3	3	3	3	
d	∞	4	3	3	2	
e	∞	2	0	0	0	

Example of Maintaining Pointers

$$M[v] = \min\left(M'[v], \min_{w \in N_v}\left(c_{vw} + M'[w]\right)\right)$$



	0	1	2	3	4	5
t	0	0	0	0	0	0
a	∞	-3	-3	-4	-6	-6
b	∞	∞	0	-2	-2	-2
c	∞	3	3	3	3	3
d	∞	4	3	3	2	0
e	∞	2	0	0	0	0

Example of Maintaining Pointers

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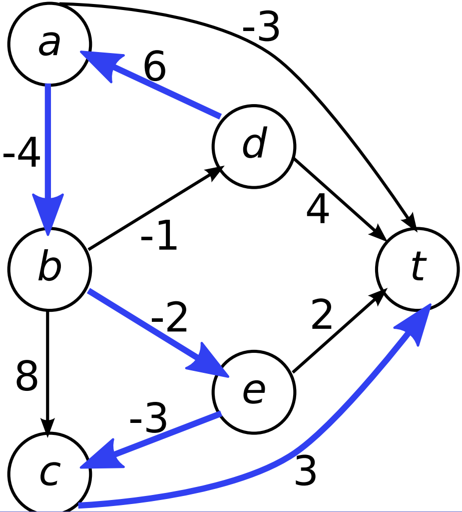
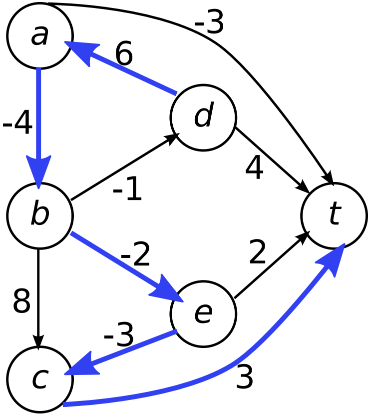


	0	1	2	3	4	5
t	0	0	0	0	0	0
a	∞	-3	-3	-4	-6	-6
b	∞	∞	0	-2	-2	-2
c	∞	3	3	3	3	3
d	∞	4	3	3	2	0
e	∞	2	0	0	0	0

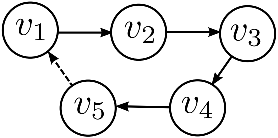
Computing the Shortest Path: Correctness

- *Pointer graph* $P(V, F)$: each edge in F is $(v, f(v))$.
  - Can P have cycles?
  - Is there a path from s to t in P?
  - Can there be multiple paths s to t in P?
  - Which of these is the shortest path?



	0	1	2	3	4	5
t	0	0	0	0	0	0
a	∞	-3	-3	-4	-6	-6
b	∞	∞	0	-2	-2	-2
c	∞	3	3	3	3	3
d	∞	4	3	3	2	0
e	∞	2	0	0	0	0

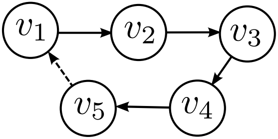
Computing the Shortest Path: Cycles in P



$$M[v] = \min\left(M'[v], \min_{w \in V}\left(c_{vw} + M'[w]\right)\right)$$

- Claim: If P has a cycle C, then C has negative cost.

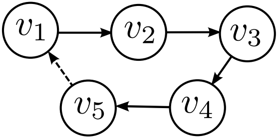
Computing the Shortest Path: Cycles in P



$$M[v] = \min\left(M'[v], \min_{w \in V}\left(c_{vw} + M'[w]\right)\right)$$

- Claim: If P has a cycle C, then C has negative cost.
  - Suppose we set $f(v) = w$. Between this assignment and the assignment of $f(v)$ to some other node, $M[v] \geq c_{vw} + M[w]$ (because $M[w]$ may itself decrease).

Computing the Shortest Path: Cycles in P



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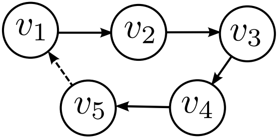
  - Let $v_1, v_2, \ldots v_k$ be the nodes in C and assume that (v_k, v_1) is the last edge to have been added.
  - What is the situation just before this addition?

Computing the Shortest Path: Cycles in P



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  - Let $v_1, v_2, \ldots v_k$ be the nodes in C and assume that (v_k, v_1) is the last edge to have been added.
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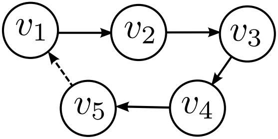
  - $M[v_i] - M[v_{i+1}] \geq c_{v_i v_{i+1}}$, for all $1 \leq i < k - 1$.
  - $M[v_k] - M[v_1] > c_{v_k v_1}$.

Computing the Shortest Path: Cycles in P



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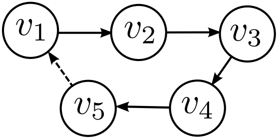
  - Adding all these inequalities, $0 > \sum_{i=1}^{k-1} c_{v_i v_{i+1}} + c_{v_k v_1} =$ cost of C.

Computing the Shortest Path: Cycles in P



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  - Adding all these inequalities, $0 > \sum_{i=1}^{k-1} c_{v_i v_{i+1}} + c_{v_k v_1} =$ cost of C.
- Corollary: if G has no negative cycles that P does not either.

Computing the Shortest Path: Paths in P

- Let P be the pointer graph upon termination of the algorithm.
- Consider the path P_v in P obtained by following the pointers from v to $f(v) = v_1$, to $f(v_1) = v_2$, and so on.

Computing the Shortest Path: Paths in P

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- Claim: P_v terminates at t.

Computing the Shortest Path: Paths in P

- Let P be the pointer graph upon termination of the algorithm.
- Consider the path P_v in P obtained by following the pointers from v to $f(v) = v_1$, to $f(v_1) = v_2$, and so on.
- Claim: P_v terminates at t.
- Claim: P_v is the shortest path in G from v to t.

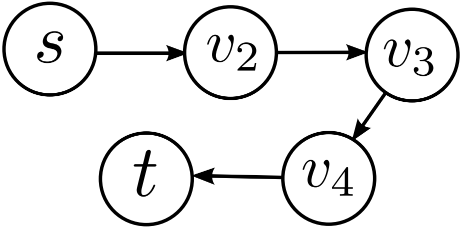
Bellman-Ford Algorithm: One Array

$$M[v] = \min\left(M[v], \min_{w \in N_v} \left(c_{vw} + M[w]\right)\right)$$

- We can prove algorithm's correctness in this case as well.

Bellman-Ford Algorithm: Early Termination

$$M[v] = \min\left(M[v], \min_{w \in V}\left(c_{vw} + M[w]\right)\right)$$



- In general, after i iterations, the path whose length is $M[v]$ may have many more than i edges.

Bellman-Ford Algorithm: Early Termination

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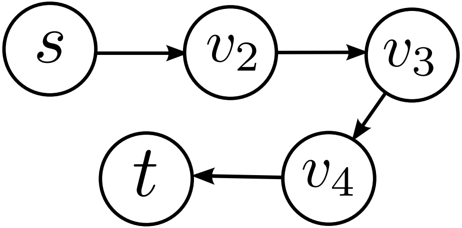

- In general, after i iterations, the path whose length is $M[v]$ may have many more than i edges.
- Early termination: If M does not change after processing all the nodes, we have computed all the shortest paths to t.