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September 16, 2014

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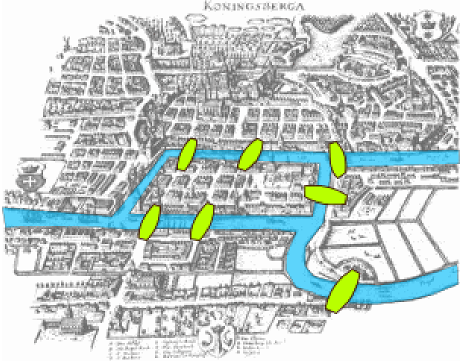

Definition of a Graph

- *Undirected graph* $G = (V, E)$: set V of nodes and set E of edges, where $E \subseteq V \times V$. Elements of E are unordered pairs.
  - Abuse of notation: write an edge e between nodes u and v as $e = (u, v)$ and not as $e = \{u, v\}$.
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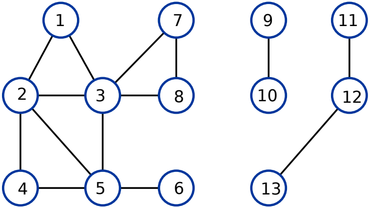
- By default, "graph" will mean an "undirected graph".

Paths and Connectivity



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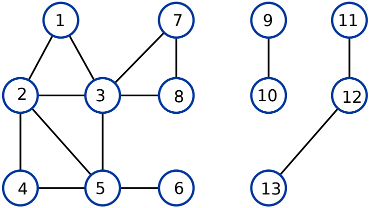

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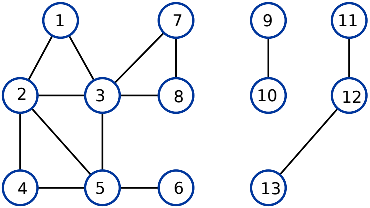

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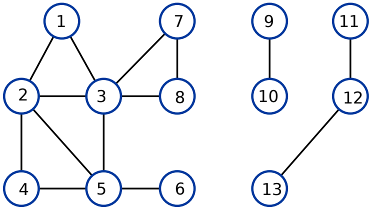

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- *Distance* between two nodes u and v is the minimum number of edges in any u-v path.

Trees

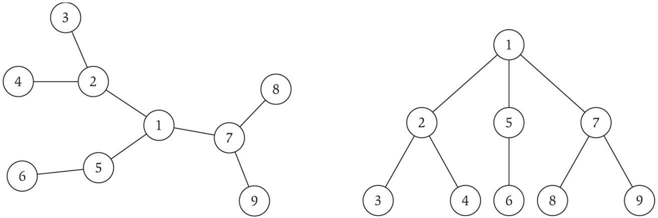

Figure 3.1 Two drawings of the same tree. On the right, the tree is rooted at node 1.

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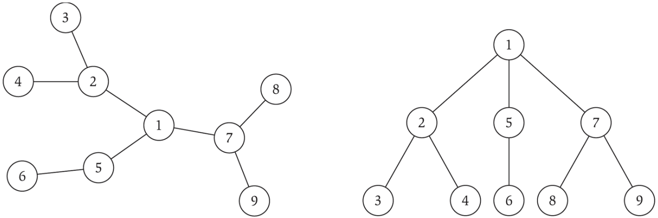

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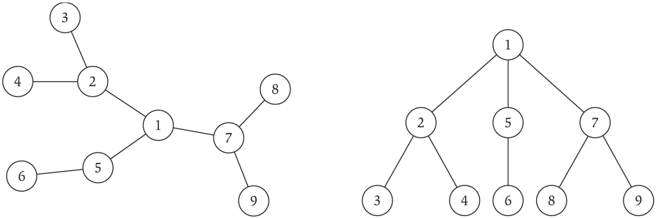

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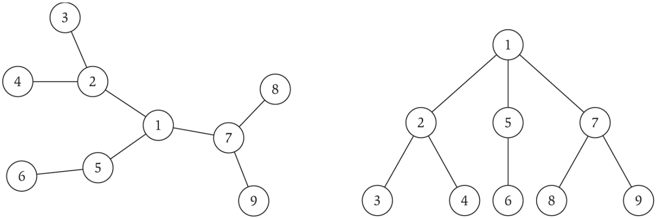

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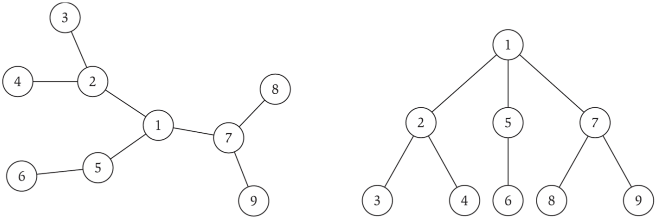

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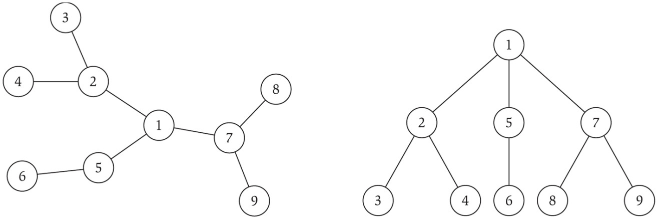

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- Examples of (rooted) trees: organisational hierarchy, class hierarchies in object-oriented languages.

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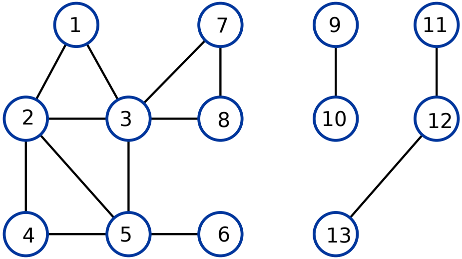
  - 3 and 1 $\Rightarrow$ 2: prove yourself.

s-t Connectivity



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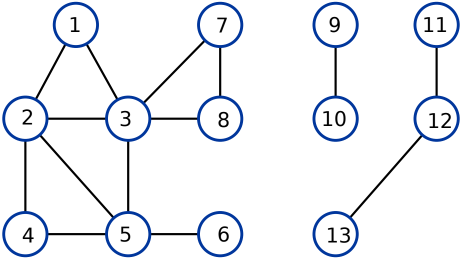

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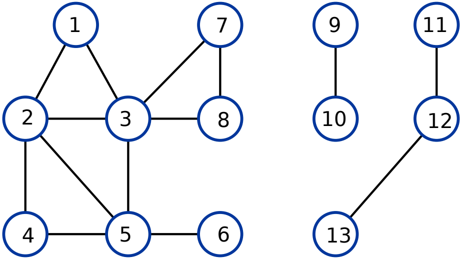

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- Algorithm for the s-t Connectivity problem: compute the connected component of G that contains s and check if t is in that component.

Computing Connected Components

- "Explore" G starting from s and maintain set R of visited nodes.

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R will consist of nodes to which s has a path
Initially R = {s}
While there is an edge (u, v) where u ∈ R and v ∉ R
  Add v to R
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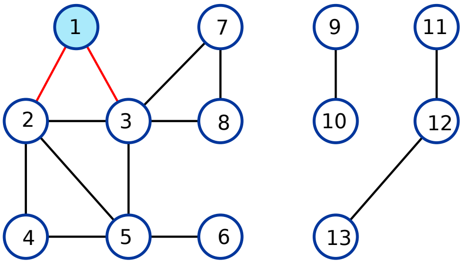

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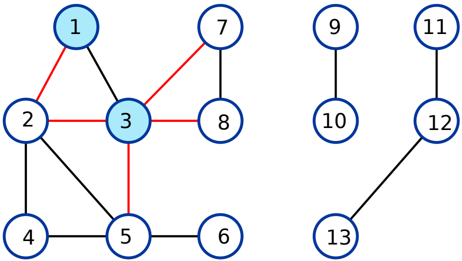

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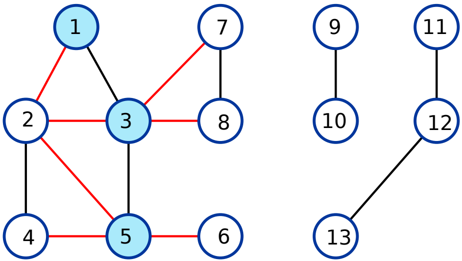

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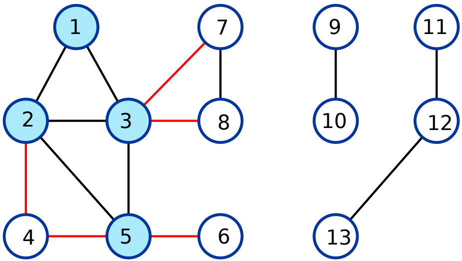

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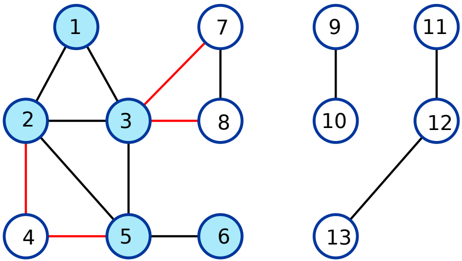

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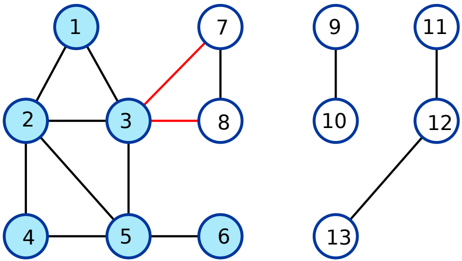

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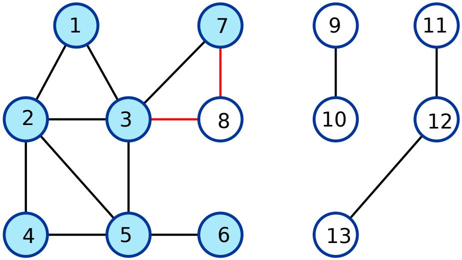

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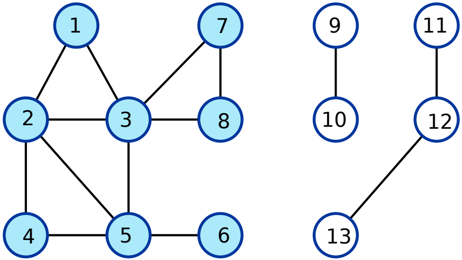

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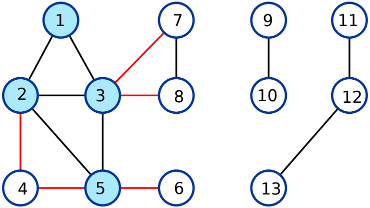

- How do we implement the `while` loop?

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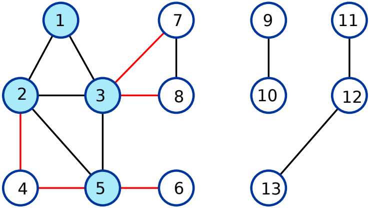

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Issues in Computing Connected Components

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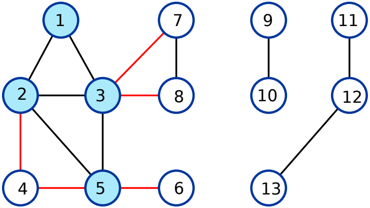

- How do we implement the `while` loop? Examine each edge in E.
- Other issues to consider:
  - Why does the algorithm terminate?
  - Does the algorithm truly compute connected component of G containing s?
  - What is the running time of the algorithm?

Termination of the Algorithm

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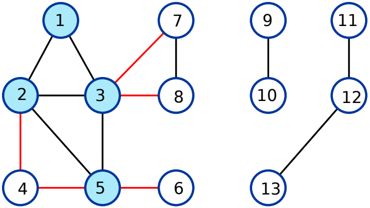

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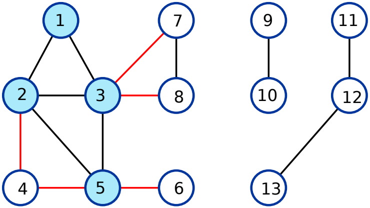

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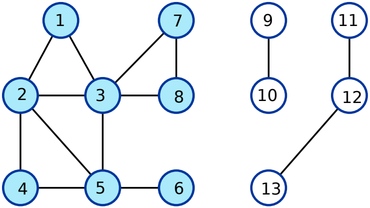

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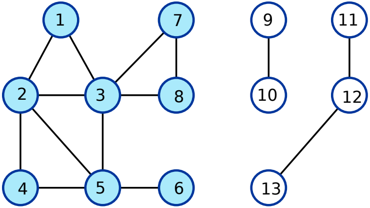

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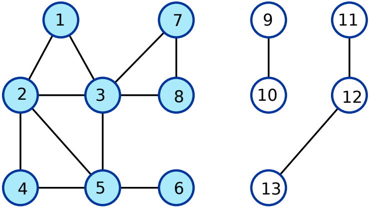

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- How many times is the while loop executed? At most n times.
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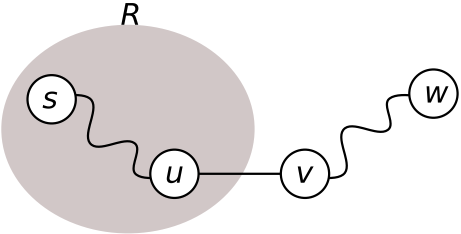
  - either $R = V$ at the end or
  - in the last iteration, every edge either has both nodes in R or both nodes not in R.

Correctness of the Algorithm



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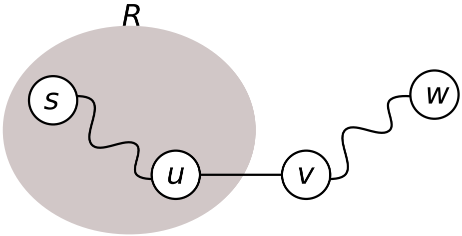

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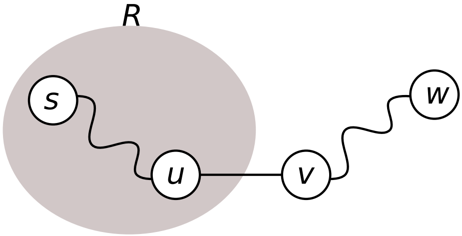

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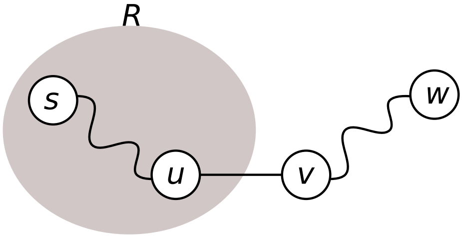

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  - Note: wrong to assume that predecessor of w in P is not in R.

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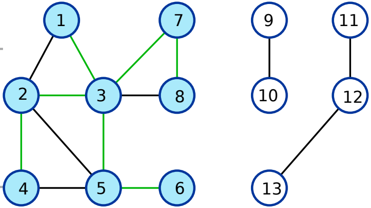

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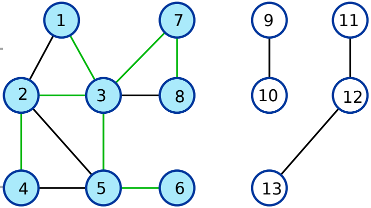

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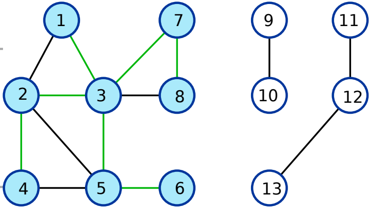

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- To recover the s-t path, trace these edges backwards from t until we reach s.

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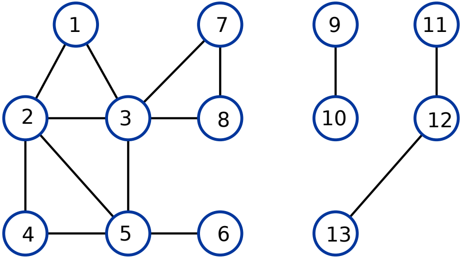
- Can we improve the running time by processing edges more carefully?

Breadth-First Search (BFS)



- Idea: explore G starting at s and going "outward" in all directions, adding nodes one layer at a time.

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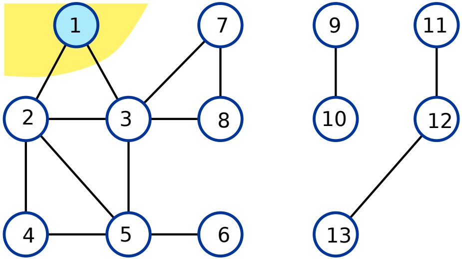

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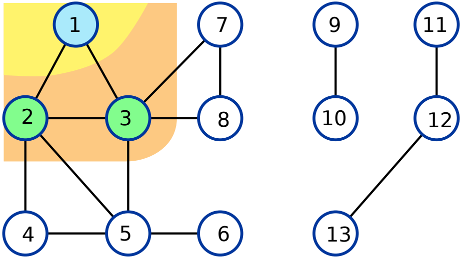

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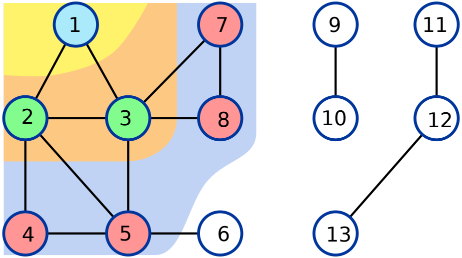

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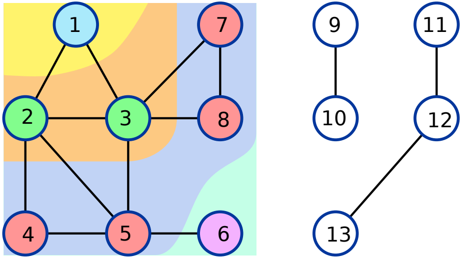

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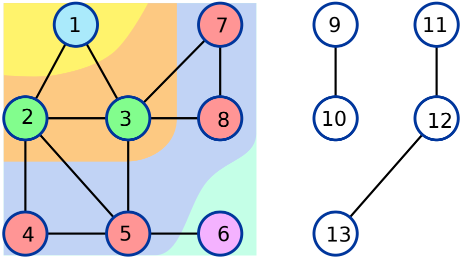

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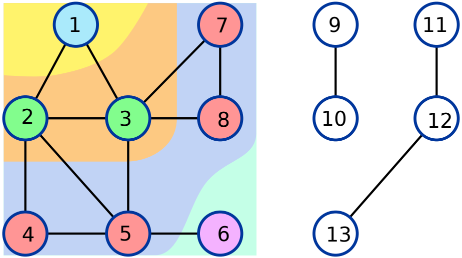

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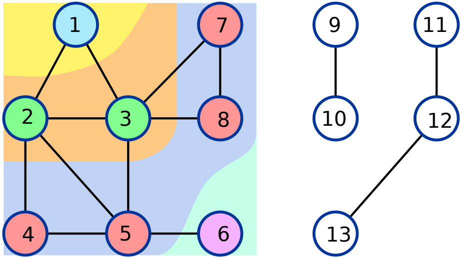

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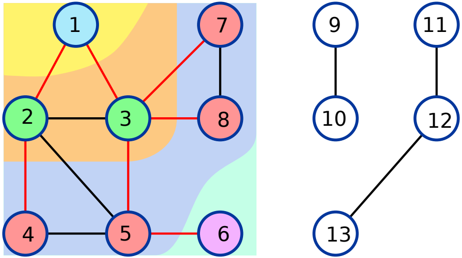

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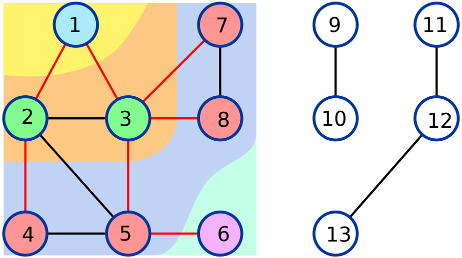

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  - Why is T a tree?

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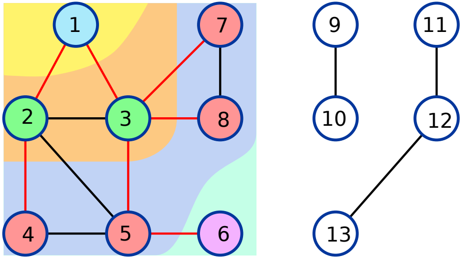

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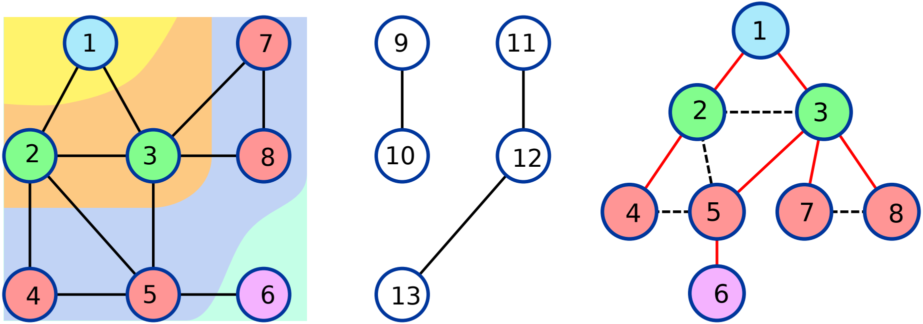
  - Why is T a tree? It is connected. The number of edges in T is the number of nodes in all the layers minus 1.
  - T is called the *breadth-first search tree*.

BFS Trees



- *Non-tree edge*: an edge of G that does not belong to the BFS tree T.
- Claim: Let T be a BFS tree, let x and y be nodes in T belonging to layers L_i and L_j, respectively, and let (x, y) be an edge of G. Then $|i - j| \leq 1$.

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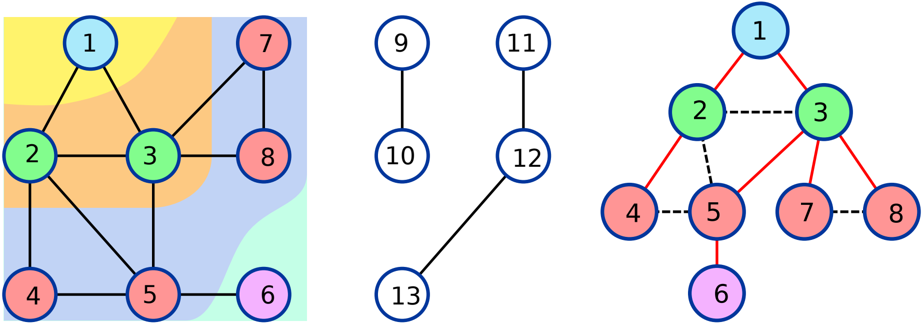

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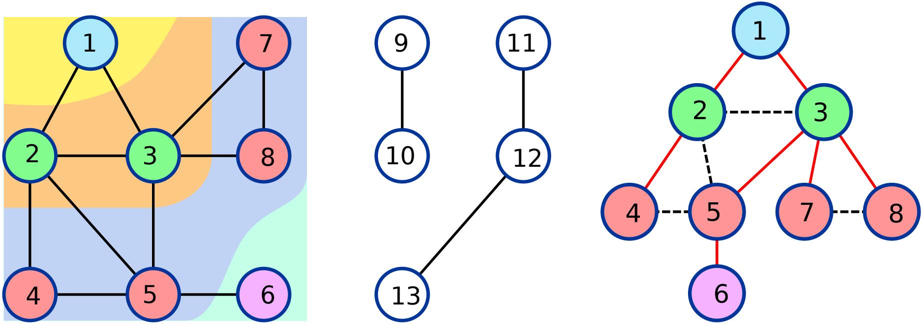

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- Still unresolved: an efficient implementation of BFS.

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- Explore G as if it were a maze: start from s, traverse first edge out (to node v), traverse first edge out of v, ..., reach a dead-end, backtrack, ...

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2. Invoke `DFS(`s`)`.

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DFS(u):
  Mark u as "Explored" and add u to R
  For each edge (u,v) incident to u
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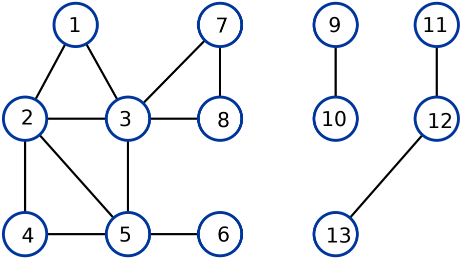
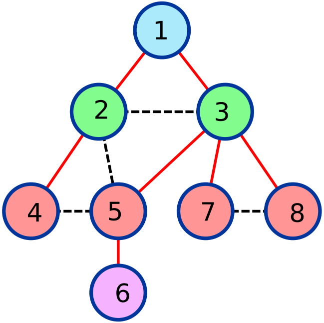
- *Depth-first search tree* is a tree T: when `DFS(`v`)` is invoked directly during the call to `DFS(`v`)`, add edge (u, v) to T.

Example of DFS





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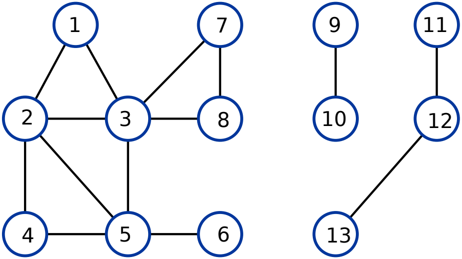

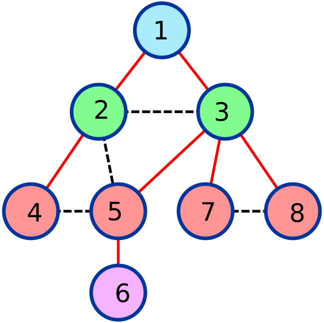

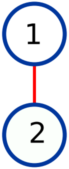

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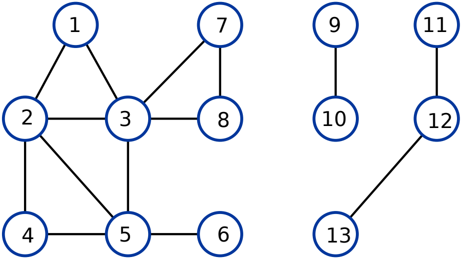

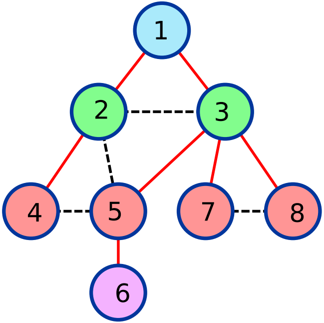

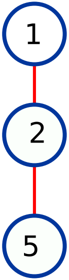

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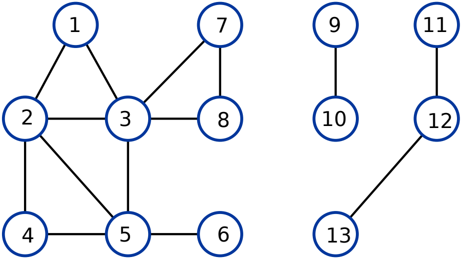

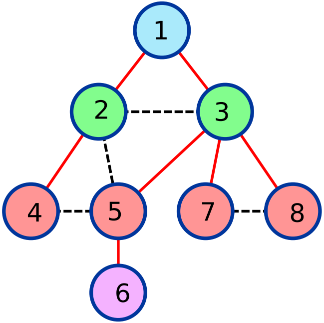

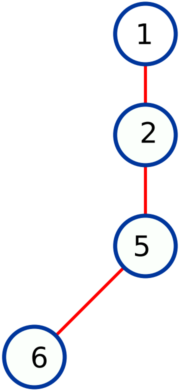

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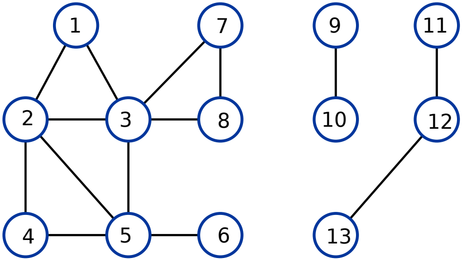

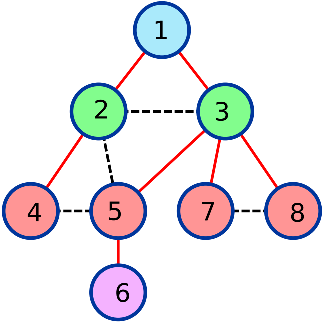

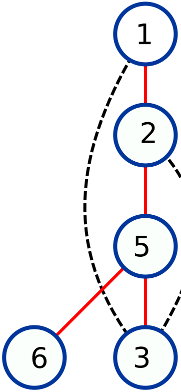

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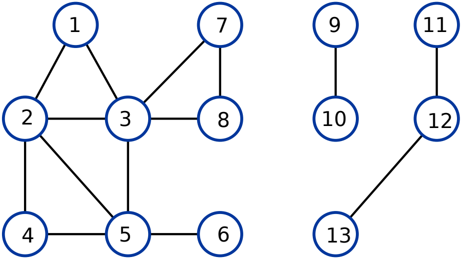

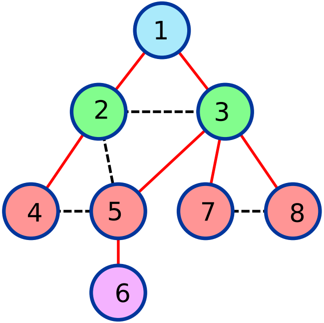

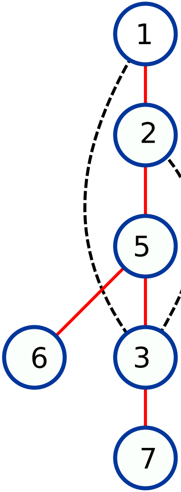

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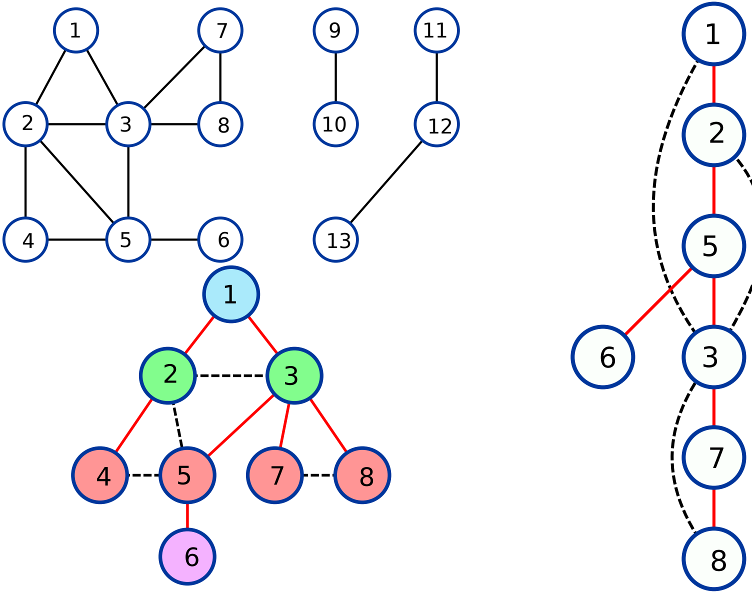

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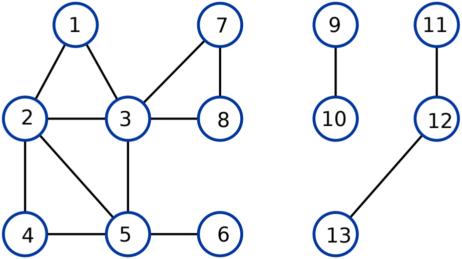
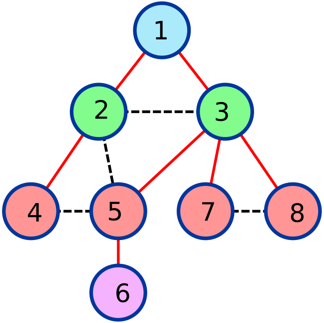
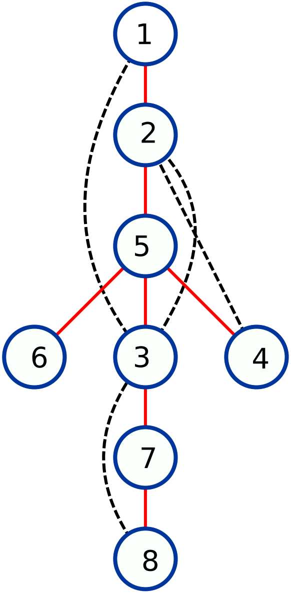
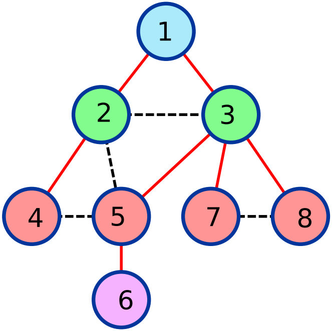
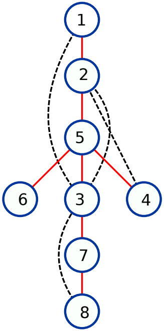






BFS vs. DFS





- Both visit the same set of nodes but in a different order.
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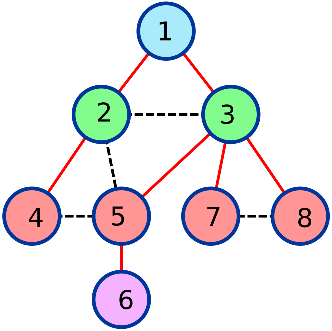

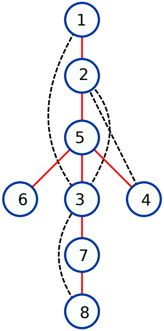

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  DFS connect ancestors to descendants.

Properties of DFS Trees

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DFS(u):
  Mark u as "Explored" and add u to R
  For each edge (u,v) incident to u
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      Recursively invoke DFS(v)
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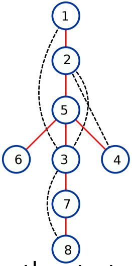

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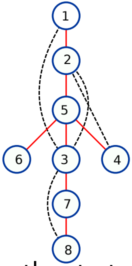

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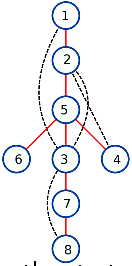

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- Proof: Assume, without loss of generality, that DFS(u) reached x first.
  - Since (x, y) is an edge in G, it is examined during DFS(x).
  - Since $(x, y) \notin T$, y must be marked as "Explored" during DFS(x) but before (x, y) is examined.
  - Since y was not marked as "Explored" before DFS(x) was invoked, it must be marked as "Explored" between the end of DFS(x).
  - Therefore, y must be a descendant of x in T.

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Computing All Connected Components

1. Pick an arbitrary node s in G.
2. Compute its connected component using BFS (or DFS).
3. Find a node (say v, not already visited) and repeat the BFS from v.
4. Repeat this process until all nodes are visited.

Representing Graphs

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Data Structures for Implementation

- "Implementation" of BFS and DFS: fully specify the algorithms and data structures so that we can obtain provably efficient times.
- Inner loop of both BFS and DFS: process the set of edges incident on a given node and the set of visited nodes.
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- How do we store the set of visited nodes? Order in which we process the nodes is crucial.
  - BFS: store visited nodes in a queue (first-in, first-out).
  - DFS: store visited nodes in a stack (last-in, first-out)

Implementing BFS

- Maintain an array `Discovered` and set `Discovered`$[v] =$ *true* as soon as the algorithm sees v.

BFS(s):

  Set Discovered[s] = true and Discovered[v] = false for all other v

  Initialize $L[0]$ to consist of the single element s

  Set the layer counter $i = 0$

  Set the current BFS tree $T = \emptyset$

  While $L[i]$ is not empty

    Initialize an empty list $L[i+1]$

    For each node $u \in L[i]$

      Consider each edge (u, v) incident to u

      If Discovered[v] = false then

        Set Discovered[v] = true

        Add edge (u, v) to the tree T

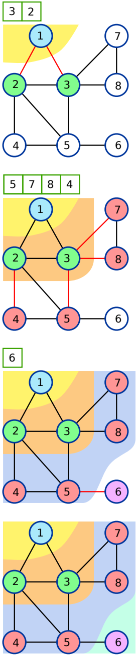
        Add v to the list $L[i+1]$

      Endif

    Endfor

    Increment the layer counter i by one

  Endwhile



Using a Queue in BFS

- Instead of storing each layer in a different list, maintain all the layers in a single queue L.

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BFS(s):
   Set Discovered[s] = true
   Set Discovered[v] = false, for all other nodes v
   Initialize L to consist of the single element s
   While L is not empty
       Pop the node u at the head of L
       Consider each edge (u, v) incident on u
       If Discovered[v] = false then
          Set Discovered[v] = true
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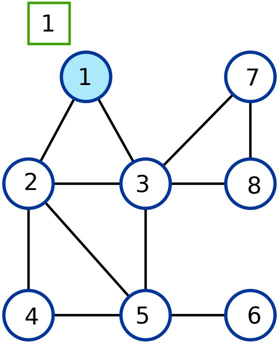

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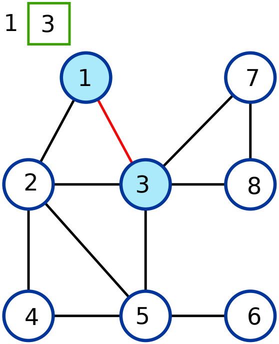

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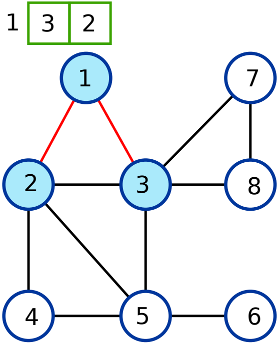

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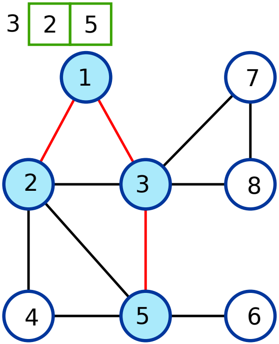

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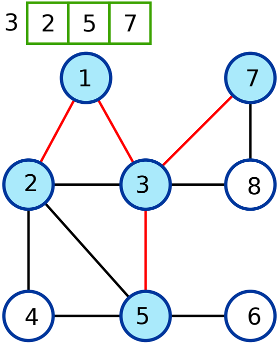

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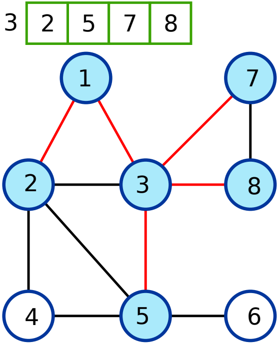

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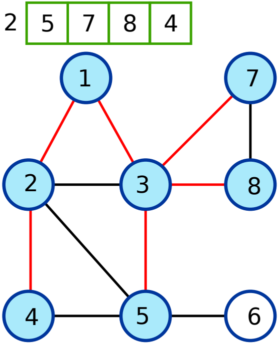

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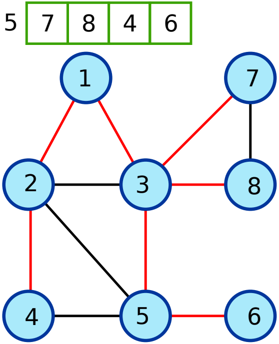

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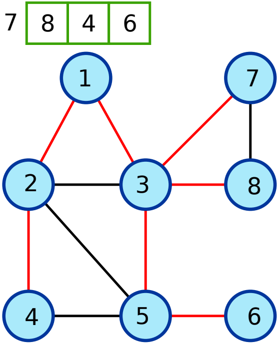

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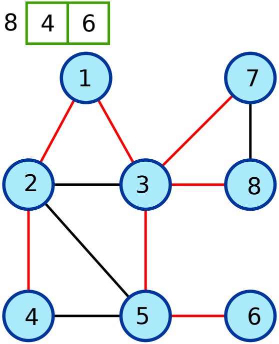

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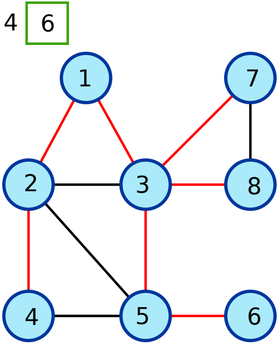

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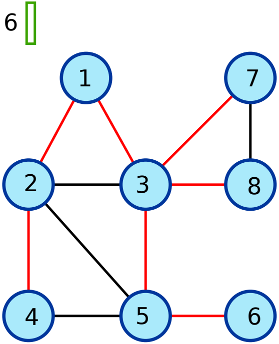

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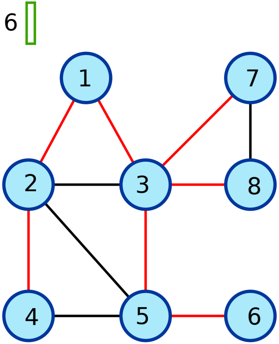

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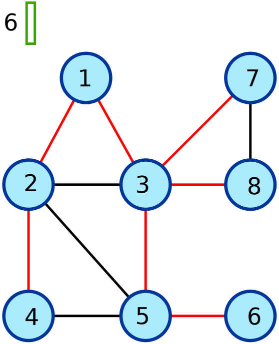

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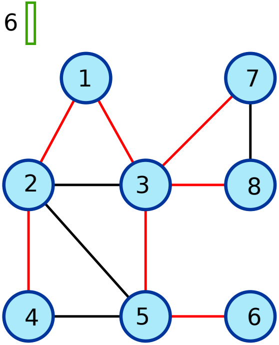

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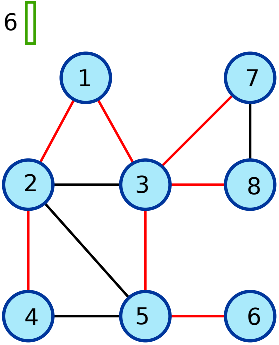

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- Claim: Nodes in layer $i+1$ will appear in L immediately after nodes in layer i. More formally: If BFS(s) pops (v, l_v) from L immediately after it pops (u, l_u), then either $l_v = l_u$ or $l_v = l_u + 1$.

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  - Maintaining layer information: $O(1)$ time per node.
  - Total time is $O(n + m)$.

Recursive DFS

```
DFS(u):
  Mark u as "Explored" and add u to R
  For each edge (u,v) incident to u
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      Recursively invoke DFS(v)
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- Procedure has "tail recursion": recursive call is the last step.

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- Procedure has "tail recursion": recursive call is the last step.
- Can replace the recursion by an iteration: use a stack to explicitly implement the recursion.

Implementing DFS

- Maintain a stack S to store nodes to be explored.
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- Read textbook on how to construct the DFS tree.

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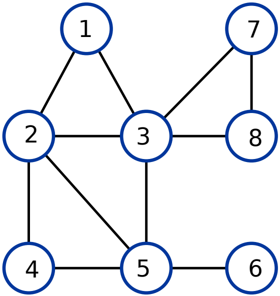

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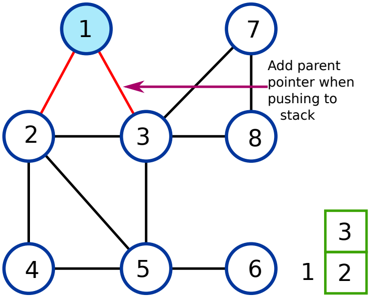

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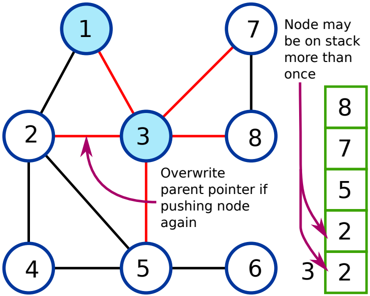

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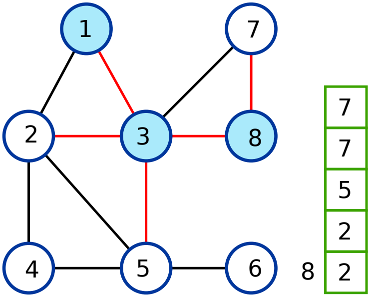

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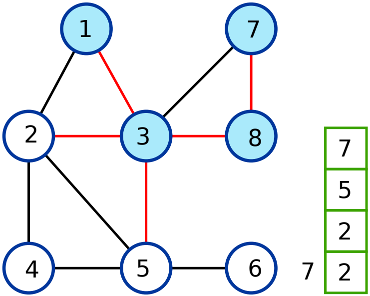

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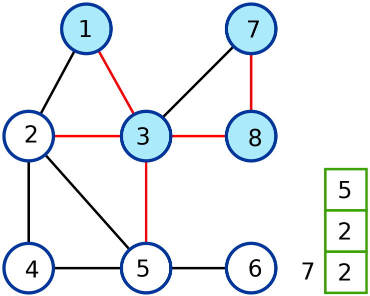

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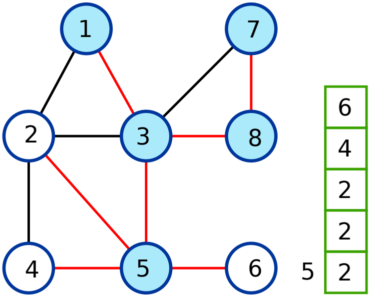

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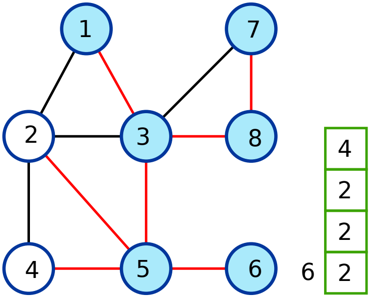

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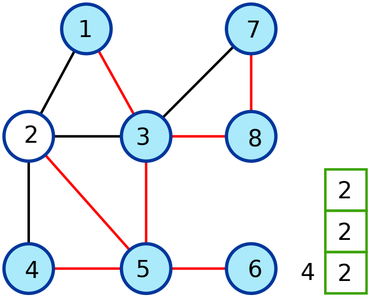
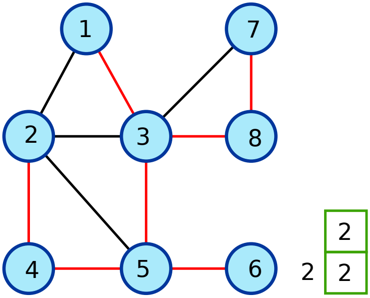


Implementing DFS

- Maintain a stack S to store nodes to be explored.
- Maintain an array `Explored` and set `Explored`$[v]$ = *true* when the algorithm pops v from the stack.
- Read textbook on how to construct the DFS tree.

```
DFS(s):
  Initialize S to be a stack with one element s
  While S is not empty
    Take a node u from S
    If Explored[u] = false then
      Set Explored[u] = true
      For each edge (u,v) incident to u
        Add v to the stack S
      Endfor
    Endif
  Endwhile
```



Comparing Recursion and Iteration

```
DFS(u):
  Mark u as "Explored" and add u to R
  For each edge (u,v) incident to u
    If v is not marked "Explored" then
      Recursively invoke DFS(v)
    Endif
  Endfor
```

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- How many times is a node's adjacency list scanned? Exactly once.
- The total amount of time to process edges incident on node u's is $O(n_u)$.
- The total running time of the algorithm is $O(n + m)$.