Priority Queues and Heaps

September 9, 2014

Motivation: Sort a List of Numbers

Sort

INSTANCE: Nonempty list $x_1, x_2, \ldots, x_n$ of integers.

SOLUTION: A permutation $y_1, y_2, \ldots, y_n$ of $x_1, x_2, \ldots, x_n$ such that $y_i \leq y_{i+1}$, for all $1 \leq i < n$.

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- Possible algorithm:
  - Store all the numbers in a data structure D.
  - Repeatedly find the smallest number in D, output it, and remove it.
- To get $O(n \log n)$ running time, each "find minimum" step and each "remove" step must take $O(\log n)$ time.

Candidate Data Structures for Sorting

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Sorted array Finding minimum takes $O(1)$ time but insertion and deletion can take $\Omega(n)$ time in the worst case.

Priority Queue

- Store a set S of elements, where each element v has a priority value $\texttt{key}(v)$.
- Smaller key values $\equiv$ higher priorities.
- Operations supported:
  - find the element with smallest key
  - remove the smallest element
  - insert an element
  - delete an element
  - update the key of an element
- Element deletion and key update require knowledge of the position of the element in the priority queue.

Heaps

- Combine benefits of both lists and sorted arrays.
- Conceptually, a heap is a balanced binary tree.
- *Heap order*: For every element v at a node i, the element w at i's parent satisfies $\texttt{key}(w) \leq \texttt{key}(v)$.

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- Alternatively, assume maximum number N of elements is known in advance.
- Store nodes of the heap in an array.
  - Node at index i has children at indices $2i$ and $2i+1$ and parent at index $\lfloor i/2 \rfloor$.
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Example of a Heap

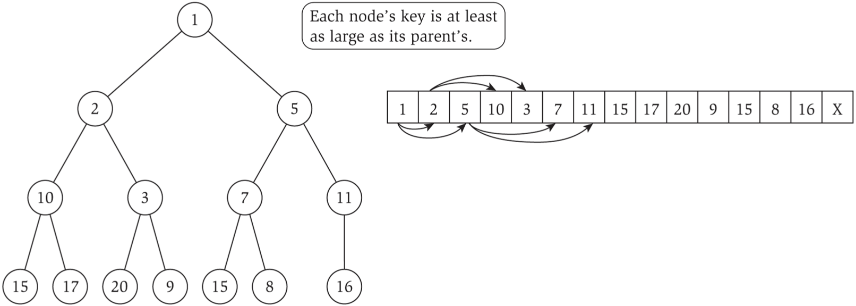

Figure 2.3 Values in a heap shown as a binary tree on the left, and represented as an array on the right. The arrows show the children for the top three nodes in the tree.

Inserting an Element: `Heapify-up`

1. Insert new element at index $n + 1$.
2. Fix heap order using `Heapify-up`$(H, n + 1)$.

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Heapify-up(H,i):
  If i > 1 then
    let j = parent(i) = ⌊i/2⌋
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Example of `Heapify-up`

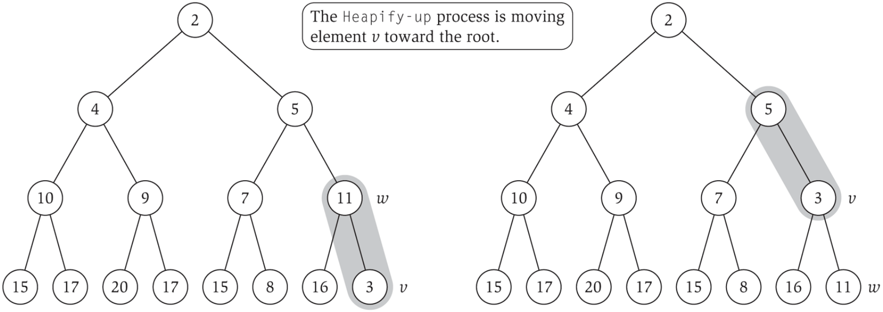
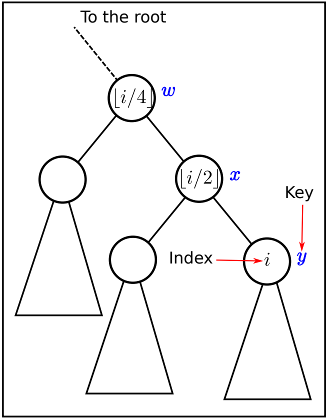


Figure 2.4 The `Heapify-up` process. Key 3 (at position 16) is too small (on the left). After swapping keys 3 and 11, the heap violation moves one step closer to the root of the tree (on the right).

Correctness of `Heapify-up`: Setup



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- `Heapify-up`$(H, n+1)$ invokes `Heapify-up`$(H, \lfloor (n+1)/2 \rfloor)$, which invokes `Heapify-up`$(H, \lfloor (n+1)/4 \rfloor)$, ... which invokes `Heapify-up`$(H, 1)$.
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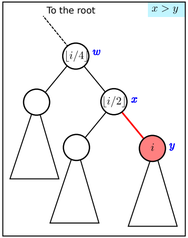

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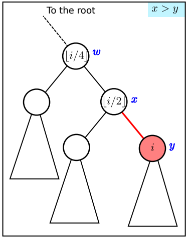

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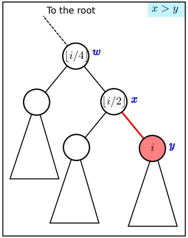

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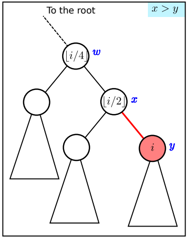

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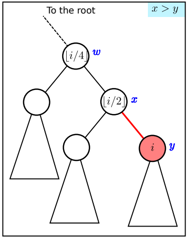

- Two elements to proof strategy:
  1. Heap violation can occur *at most* one index.
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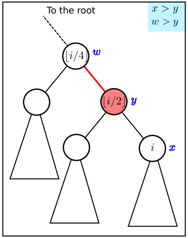

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- Base case: $i = 1$.

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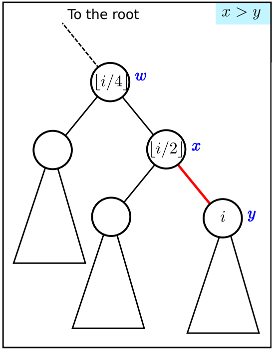
- Inductive step: What do we need to prove? If the heap violation occurs only at i, then after the swap statement in `Heapify-up`(H, i), H is a heap or the heap violation occurs only at $\lfloor i/2 \rfloor$.

Correctness of `Heapify-up`: Inductive Step

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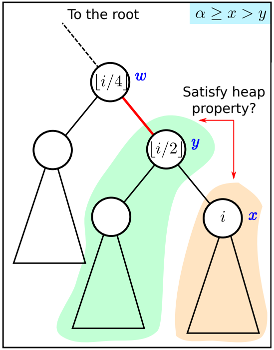


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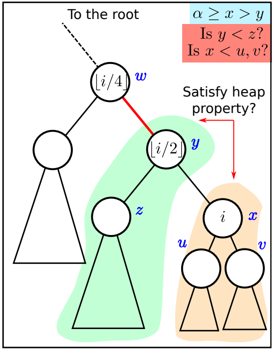


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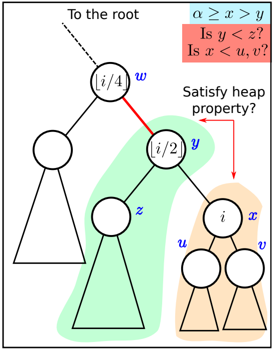


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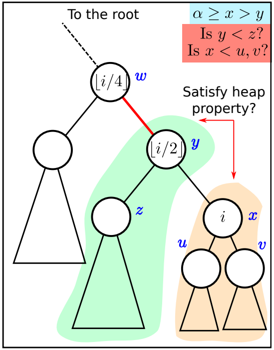


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- Let us try definition from the textbook (slightly modified).

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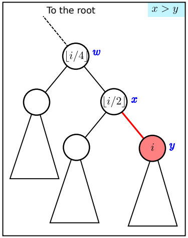

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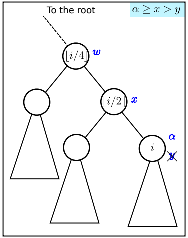

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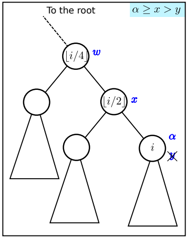

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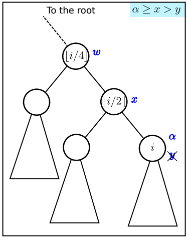

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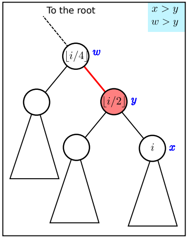

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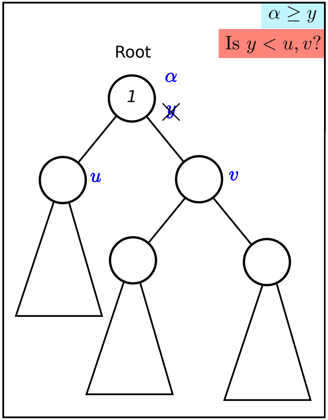
- We want to prove for every $1 \le i \le n+1$, if H is too small at i, then after $\texttt{Heapify-up}(H, i)$ returns, H is a heap.
- Two elements to proof strategy:
  1. Heap violation can occur *at most* one index and we know how to fix it.
  2. For every i, execution of $\texttt{Heapify-up}(H, i)$ pushes heap violation from index i to index $\lfloor i/2 \rfloor$.

To prove:

For every $1 \le i \le n+1$, if H is too small at i, then after $\texttt{Heapify-up}(H, i)$ returns, H is a heap.

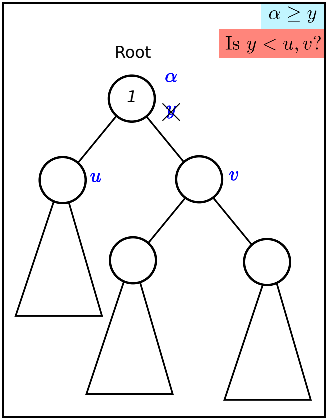
Correctness of `Heapify-up`: Base Case




```
Heapify-up(H,i):
  If i > 1 then
    let j = parent(i) = ⌊i/2⌋
    If key[H[i]]<key[H[j]] then
      swap the array entries H[i] and H[j]
      Heapify-up(H,j)
    Endif
  Endif
```

- Base case: $i = 1$.

Correctness of `Heapify-up`: Base Case

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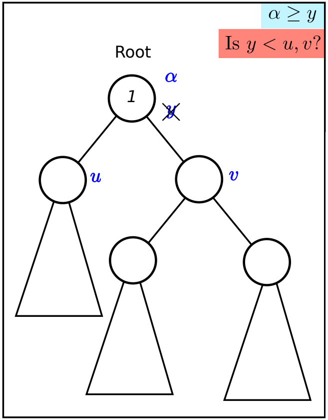
    If key[H[i]]<key[H[j]] then
      swap the array entries H[i] and H[j]
      Heapify-up(H,j)
    Endif
  Endif
```

- Base case: $i = 1$.
  - H is too small at 1.

Correctness of `Heapify-up`: Base Case

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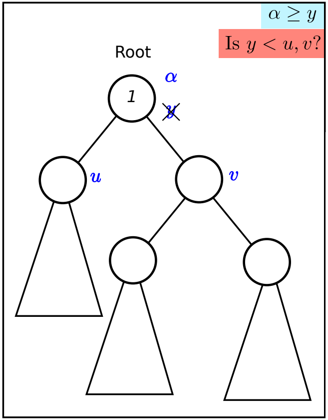
      Heapify-up(H,j)
    Endif
  Endif
```

- Base case: $i = 1$.
  - H is too small at 1.
  - There is a value $\alpha \geq \texttt{key}(H[1])$ such that increasing $\texttt{key}(H[1])$ to α makes H a heap.

Correctness of `Heapify-up`: Base Case




```
Heapify-up(H,i):
  If i > 1 then
    let j = parent(i) = ⌊i/2⌋
    If key[H[i]]<key[H[j]] then
      swap the array entries H[i] and H[j]
      Heapify-up(H,j)
    Endif
  Endif
```

- Base case: $i = 1$.
  - H is too small at 1.
  - There is a value $\alpha \geq \texttt{key}(H[1])$ such that increasing $\texttt{key}(H[1])$ to α makes H a heap.
  - $\texttt{key}(H[1]) \leq \alpha \leq \texttt{key}(H[2]), \texttt{key}(H[3]) \implies H$ is a heap.

Correctness of `Heapify-up`: Inductive Hypothesis

To prove:

For every $1 \leq i \leq n+1$, if H is too small at i, then after `Heapify-up`(H, i) returns, H is a heap.

Correctness of `Heapify-up`: Inductive Hypothesis

To prove:

For every $1 \leq i \leq n+1$, if H is too small at i, then after `Heapify-up`(H, i) returns, H is a heap.

- Inductive hypothesis: (suggested by statement we need to prove)

Correctness of `Heapify-up`: Inductive Hypothesis

To prove:

For every $1 \leq i \leq n+1$, if H is too small at i, then after `Heapify-up`(H, i) returns, H is a heap.

- Inductive hypothesis: (suggested by statement we need to prove) If H is too small at $\lfloor i/2 \rfloor$, then after `Heapify-up`$(H, \lfloor i/2 \rfloor)$ returns, H is a heap.

Correctness of `Heapify-up`: Inductive Hypothesis

To prove:

For every $1 \leq i \leq n+1$, if H is too small at i, then after `Heapify-up`(H, i) returns, H is a heap.

- Inductive hypothesis: (suggested by statement we need to prove) If H is too small at $\lfloor i/2 \rfloor$, then after `Heapify-up`$(H, \lfloor i/2 \rfloor)$ returns, H is a heap.
- Inductive step: What do we need to prove?

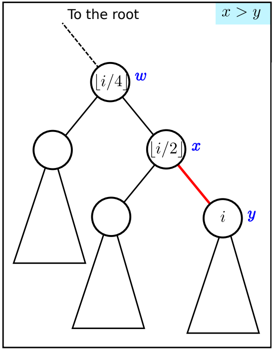
Correctness of `Heapify-up`: Inductive Hypothesis

To prove:

For every $1 \leq i \leq n+1$, if H is too small at i, then after $\texttt{Heapify-up}(H, i)$ returns, H is a heap.

- Inductive hypothesis: (suggested by statement we need to prove) If H is too small at $\lfloor i/2 \rfloor$, then after $\texttt{Heapify-up}(H, \lfloor i/2 \rfloor)$ returns, H is a heap.
- Inductive step: What do we need to prove? If H is too small at i, then after the swap statement in $\texttt{Heapify-up}(H, i)$, H is a heap or H is too small at $\lfloor i/2 \rfloor$.

Correctness of `Heapify-up`: Inductive Step

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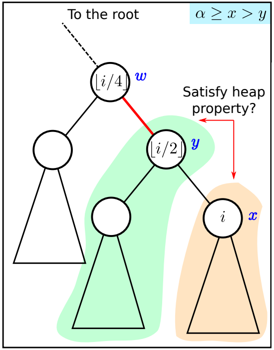
  If i > 1 then
    let j = parent(i) = ⌊i/2⌋
    If key[H[i]]<key[H[j]] then
      swap the array entries H[i] and H[j]
      Heapify-up(H,j)
    Endif
  Endif
```

- Starting point: H is too small at i.
- Goal: Before `Heapify-up`$(H, \lfloor i/2 \rfloor)$: H is a heap or H is too small at $\lfloor i/2 \rfloor$.

Correctness of `Heapify-up`: Inductive Step

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      swap the array entries H[i] and H[j]
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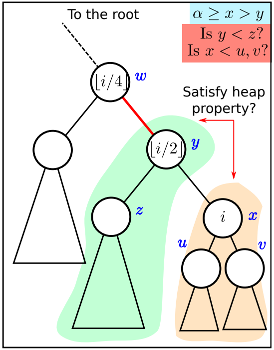
    Endif
  Endif
```

- Starting point: H is too small at i.
- Goal: Before `Heapify-up`$(H, \lfloor i/2 \rfloor)$: H is a heap or H is too small at $\lfloor i/2 \rfloor$.
- What is the situation after the swap statement in `Heapify-up`(H, i)?

Correctness of `Heapify-up`: Inductive Step

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  If i > 1 then
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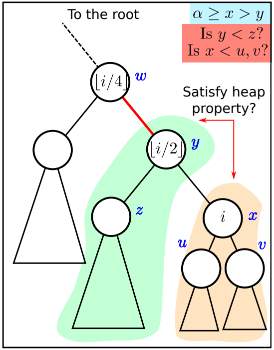
    If key[H[i]]<key[H[j]] then
      swap the array entries H[i] and H[j]
      Heapify-up(H,j)
    Endif
  Endif
```

- Starting point: H is too small at i.
- Goal: Before `Heapify-up`$(H, \lfloor i/2 \rfloor)$: H is a heap or H is too small at $\lfloor i/2 \rfloor$.
- What is the situation after the swap statement in `Heapify-up`(H, i)?

Correctness of Heapify-up: Inductive Step

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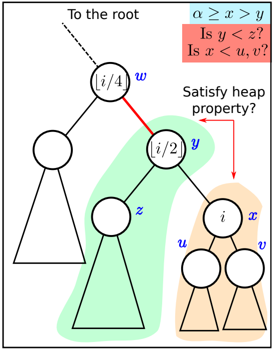
    If key[H[i]]<key[H[j]] then
      swap the array entries H[i] and H[j]
      Heapify-up(H,j)
    Endif
  Endif
```

- Starting point: H is too small at i.
- Goal: Before Heapify-up(H, $\lfloor i/2 \rfloor$): H is a heap or H is too small at $\lfloor i/2 \rfloor$.
- What is the situation after the swap statement in Heapify-up(H, i)?
- How do we show that $x < u, v$?

Correctness of `Heapify-up`: Inductive Step

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Heapify-up(H,i):
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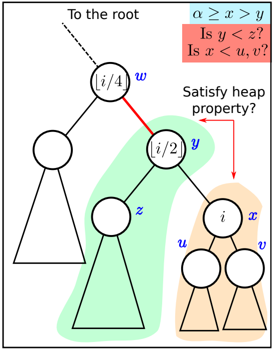
      swap the array entries H[i] and H[j]
      Heapify-up(H,j)
    Endif
  Endif
```

- Starting point: H is too small at i.
- Goal: Before `Heapify-up`$(H, \lfloor i/2 \rfloor)$: H is a heap or H is too small at $\lfloor i/2 \rfloor$.
- What is the situation after the swap statement in `Heapify-up`(H, i)?
- How do we show that $x < u, v$?
  - Use the fact that H is too small at i: there is an $\alpha \geq x$ such that increasing x to α makes H a heap.

Correctness of `Heapify-up`: Inductive Step

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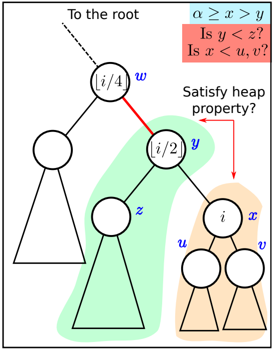
    let j = parent(i) = ⌊i/2⌋
    If key[H[i]]<key[H[j]] then
      swap the array entries H[i] and H[j]
      Heapify-up(H,j)
    Endif
  Endif
```

- Starting point: H is too small at i.
- Goal: Before `Heapify-up`$(H, \lfloor i/2 \rfloor)$: H is a heap or H is too small at $\lfloor i/2 \rfloor$.
- What is the situation after the swap statement in `Heapify-up`(H, i)?
- How do we show that $x < u, v$?
  - Use the fact that H is too small at i: there is an $\alpha \geq x$ such that increasing x to α makes H a heap.
  - Therefore, $x \leq \alpha \leq u, v$.

Correctness of Heapify-up: Inductive Step

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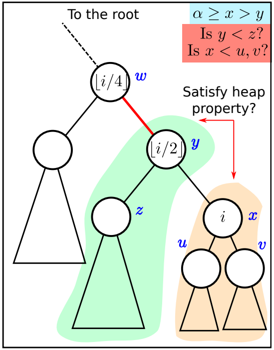
Heapify-up(H,i):
  If i > 1 then
    let j = parent(i) = ⌊i/2⌋
    If key[H[i]]<key[H[j]] then
      swap the array entries H[i] and H[j]
      Heapify-up(H,j)
    Endif
  Endif
```

- Starting point: H is too small at i.
- Goal: Before Heapify-up$(H, \lfloor i/2 \rfloor)$: H is a heap or H is too small at $\lfloor i/2 \rfloor$.
- What is the situation after the swap statement in Heapify-up(H, i)?
- How do we show that $x < u, v$?
  - Use the fact that H is too small at i: there is an $\alpha \geq x$ such that increasing x to α makes H a heap.
  - Therefore, $x \leq \alpha \leq u, v$.
- Now if H is not a heap, why is it too small at $\lfloor i/2 \rfloor$?

Correctness of `Heapify-up`: Inductive Step




```
Heapify-up(H,i):
  If i > 1 then
    let j = parent(i) = ⌊i/2⌋
    If key[H[i]]<key[H[j]] then
      swap the array entries H[i] and H[j]
      Heapify-up(H,j)
    Endif
  Endif
```

- Starting point: H is too small at i.
- Goal: Before `Heapify-up`$(H, \lfloor i/2 \rfloor)$: H is a heap or H is too small at $\lfloor i/2 \rfloor$.
- What is the situation after the swap statement in `Heapify-up`(H, i)?
- How do we show that $x < u, v$?
  - Use the fact that H is too small at i: there is an $\alpha \geq x$ such that increasing x to α makes H a heap.
  - Therefore, $x \leq \alpha \leq u, v$.
- Now if H is not a heap, why is it too small at $\lfloor i/2 \rfloor$? Increasing y to x makes H a heap!

Correctness of `Heapify-up`: Completing the Proof

- For every $1 \leq i \leq n+1$, we have shown if H is too small at i, then after `Heapify-up`(H, i) returns, H is a heap.

Correctness of `Heapify-up`: Completing the Proof

- For every $1 \leq i \leq n+1$, we have shown if H is too small at i, then after `Heapify-up`(H, i) returns, H is a heap.
- We know that before `Heapify-up`$(H, n+1)$, H is too small at $n+1$. Why?

Correctness of `Heapify-up`: Completing the Proof

- For every $1 \leq i \leq n+1$, we have shown if H is too small at i, then after `Heapify-up`(H, i) returns, H is a heap.
- We know that before `Heapify-up`$(H, n+1)$, H is too small at $n+1$. Why?
- Therefore, setting $i = n+1$, we have that `Heapify-up`$(H, n+1)$ creates a heap on all $n+1$ elements.

Running time of `Heapify-up`

```
Heapify-up(H,i):
  If i > 1 then
    let j = parent(i) = ⌊i/2⌋
    If key[H[i]]<key[H[j]] then
      swap the array entries H[i] and H[j]
      Heapify-up(H,j)
    Endif
  Endif
```

- Running time of `Heapify-up`(i)

Running time of `Heapify-up`

```
Heapify-up(H,i):
  If i > 1 then
    let j = parent(i) = ⌊i/2⌋
    If key[H[i]]<key[H[j]] then
      swap the array entries H[i] and H[j]
      Heapify-up(H,j)
    Endif
  Endif
```

- Running time of `Heapify-up`(i) is $O(\log i)$.

$$T(i) \leq \begin{cases} T(\lfloor \frac{i}{2} \rfloor) + O(1) & \text{if } i > 1 \\ O(1) & \text{if } i = 1 \end{cases}$$

Deleting an Element: `Heapify-down`

- Suppose H has $n+1$ elements.

1. Delete element at $H[i]$ by moving element at $H[n+1]$ to $H[i]$.
2. If element at $H[i]$ is too small, fix heap order using `Heapify-up`(H, i).
3. If element at $H[i]$ is too large, fix heap order using `Heapify-down`(H, i).

```
Heapify-down(H,i):
  Let n = length(H)
  If 2i > n then
    Terminate with H unchanged
  Else if 2i < n then
    Let left = 2i, and right = 2i + 1
    Let j be the index that minimizes key[H[left]] and key[H[right]]
  Else if 2i = n then
    Let j = 2i
  Endif
  If key[H[j]] < key[H[i]] then
     swap the array entries H[i] and H[j]
     Heapify-down(H,j)
  Endif
```

Example of `Heapify-down`

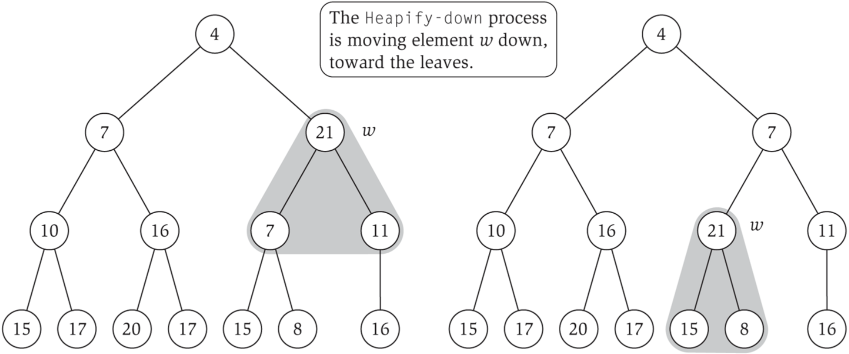
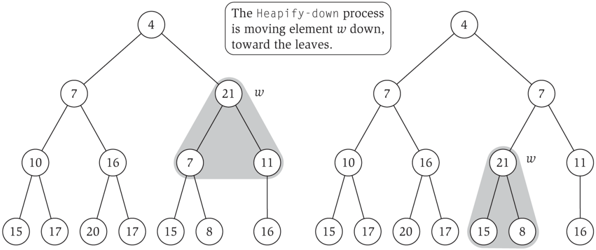


Figure 2.5 The `Heapify-down` process:. Key 21 (at position 3) is too big (on the left). After swapping keys 21 and 7, the heap violation moves one step closer to the bottom of the tree (on the right).

Correctness of `Heapify-down`



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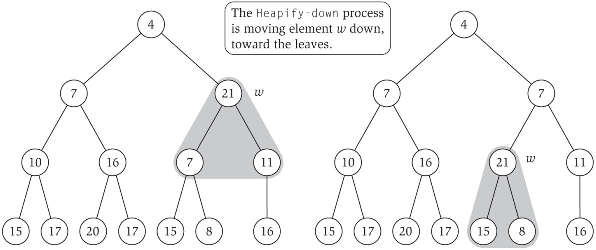


Figure 2.5 The `Heapify-down` process:. Key 21 (at position 3) is too big (on the left). After swapping keys 21 and 7, the heap violation moves one step closer to the bottom of the tree (on the right).

- H is *too big at j* if there is a value $\alpha \leq \texttt{key}(H[j])$ such that decreasing $\texttt{key}(H[j])$ to α makes H a heap. (Note: at start, H is indeed too big at i.)

Correctness of `Heapify-down`

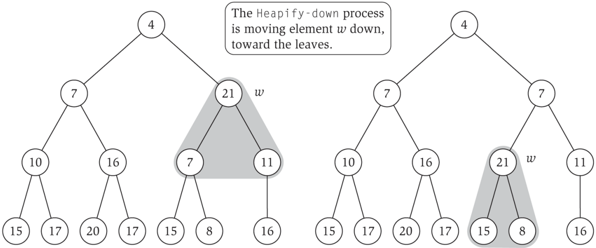
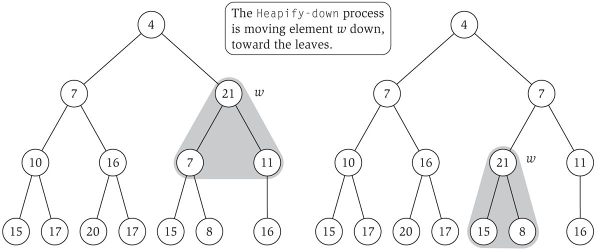


Figure 2.5 The `Heapify-down` process:. Key 21 (at position 3) is too big (on the left). After swapping keys 21 and 7, the heap violation moves one step closer to the bottom of the tree (on the right).

- H is *too big at j* if there is a value $\alpha \leq \texttt{key}(H[j])$ such that decreasing $\texttt{key}(H[j])$ to α makes H a heap. (Note: at start, H is indeed too big at i.)
- Statement to prove: for every $i \leq j \leq n$, if H is too big at j then $\texttt{Heapify-down}(H, j)$ creates a heap.
- Proof by reverse induction on j from n down to i.

Correctness of `Heapify-down`



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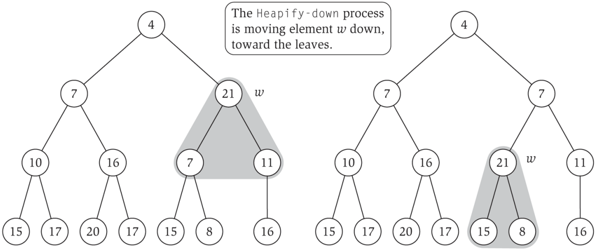
- Statement to prove: for every $i \leq j \leq n$, if H is too big at j then $\texttt{Heapify-down}(H, j)$ creates a heap.
- Proof by reverse induction on j from n down to i.
- Base case:

Correctness of `Heapify-down`



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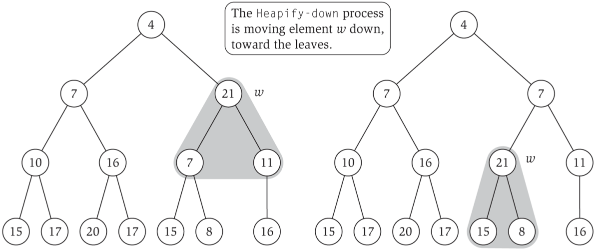
- Statement to prove: for every $i \le j \le n$, if H is too big at j then $\texttt{Heapify-down}(H, j)$ creates a heap.
- Proof by reverse induction on j from n down to i.
- Base case: $2j > n$. If decreasing $\texttt{key}(H[j])$ to α makes H a heap, then $\texttt{key}(H[j]) \le \texttt{key}(H[\lfloor j/2 \rfloor])$.

Correctness of `Heapify-down`: Inductive Step



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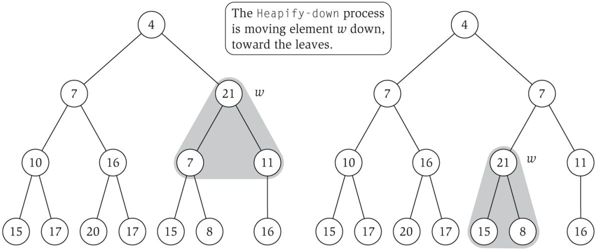
Correctness of `Heapify-down`: Inductive Step



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Correctness of `Heapify-down`: Inductive Step

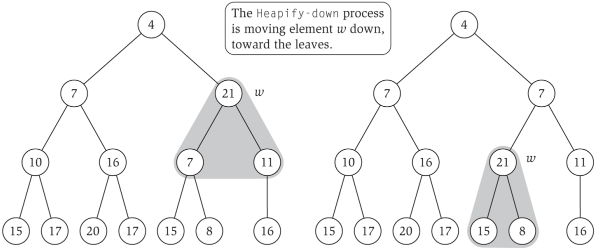
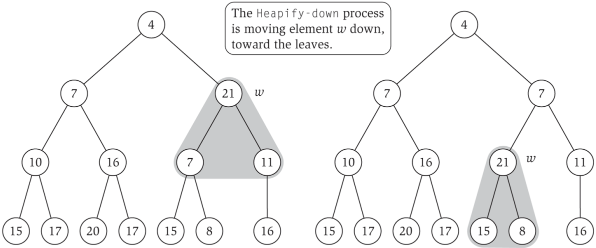


Figure 2.5 The `Heapify-down` process:. Key 21 (at position 3) is too big (on the left). After swapping keys 21 and 7, the heap violation moves one step closer to the bottom of the tree (on the right).

- H is *too big at j* if there is a value $\alpha \leq \texttt{key}(H[j])$ such that decreasing $\texttt{key}(H[j])$ to α makes H a heap.
- Inductive hypothesis (two parts):
  - If H too big at $2j$, then `Heapify-down`$(H, 2j)$ creates a heap.
  - If H is too big at $2j+1$, then `Heapify-down`$(H, 2j+1)$ creates a heap.

Correctness of `Heapify-down`: Inductive Step



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- H is *too big at j* if there is a value $\alpha \leq \texttt{key}(H[j])$ such that decreasing $\texttt{key}(H[j])$ to α makes H a heap.
- Inductive hypothesis (two parts):
  - If H too big at $2j$, then $\texttt{Heapify-down}(H, 2j)$ creates a heap.
  - If H is too big at $2j+1$, then $\texttt{Heapify-down}(H, 2j+1)$ creates a heap.
- Start of inductive step: H is too big at j.
- Inductive step:

Correctness of `Heapify-down`: Inductive Step

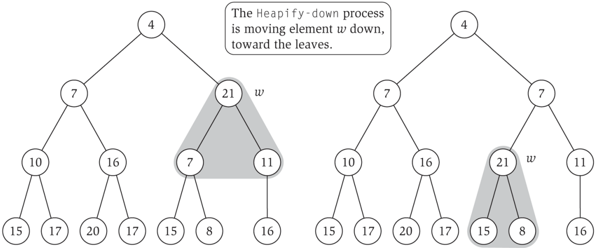

Figure 2.5 The `Heapify-down` process:. Key 21 (at position 3) is too big (on the left). After swapping keys 21 and 7, the heap violation moves one step closer to the bottom of the tree (on the right).

- H is *too big at j* if there is a value $\alpha \leq \texttt{key}(H[j])$ such that decreasing $\texttt{key}(H[j])$ to α makes H a heap.
- Inductive hypothesis (two parts):
  - If H too big at $2j$, then `Heapify-down`$(H, 2j)$ creates a heap.
  - If H is too big at $2j+1$, then `Heapify-down`$(H, 2j+1)$ creates a heap.
- Start of inductive step: H is too big at j.
- Inductive step: After the swap statement in `Heapify-down`(H, j), (a) H is a heap, (b) H is too big at $2j$, or (c) H is too big at $2j+1$.

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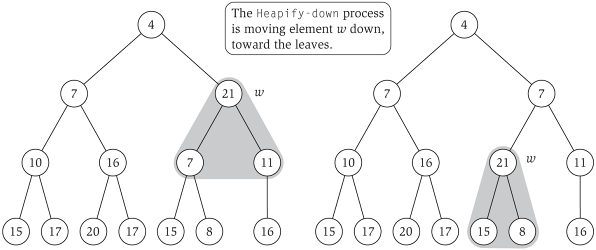

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Running time of `Heapify-down`

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Heapify-down(H,i):
  Let n = length(H)
  If 2i > n then
    Terminate with H unchanged
  Else if 2i < n then
    Let left = 2i, and right = 2i + 1
    Let j be the index that minimizes key[H[left]] and key[H[right]]
  Else if 2i = n then
    Let j = 2i
  Endif
  If key[H[j]] < key[H[i]] then
     swap the array entries H[i] and H[j]
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- Recurrence for running time of `Heapify-down`(H, i)

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- Alternative proof since the recurrence is ugly.
- Every invocation of `Heapify-down` increases its second argument by a factor of at least two.
- After k invocations argument must be at least $i2^k \leq n$, which implies that $k \leq \log_2 n/i$. Therefore running time is $O(\log_2 n/i)$.

Sorting Numbers with the Priority Queue

Sort

INSTANCE: Nonempty list $x_1, x_2, \ldots, x_n$ of integers.

SOLUTION: A permutation $y_1, y_2, \ldots, y_n$ of $x_1, x_2, \ldots, x_n$ such that $y_i \leq y_{i+1}$, for all $1 \leq i < n$.

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- Final algorithm:
  - Insert each number in a priority queue H.
  - Repeatedly find the smallest number in H, output it, and delete it from H.
- Each insertion and deletion takes $O(\log n)$ time for a total running time of $O(n \log n)$.