Graphs

T. M. Murali

September 2, 7, 9 2009

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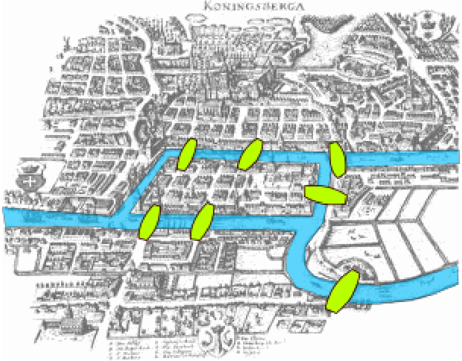

Definition of a Graph

- *Undirected graph* $G = (V, E)$: set V of nodes and set E of edges, where $E \subseteq V \times V$. Elements of E are unordered pairs.
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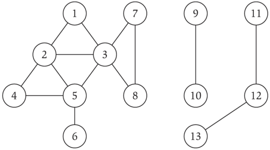
- By default, "graph" will mean an "undirected graph".

Paths and Connectivity



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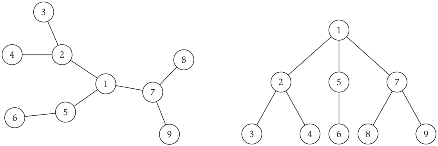

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  - P is called a path *from v_1 to v_K* or a *v_1-v_k* path.
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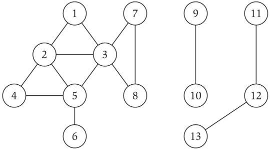

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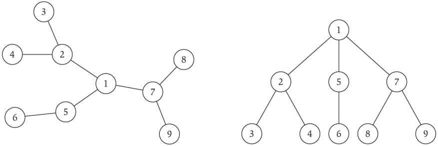

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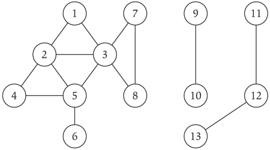

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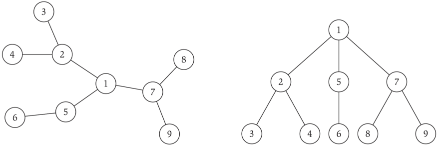

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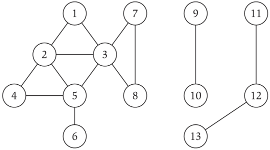

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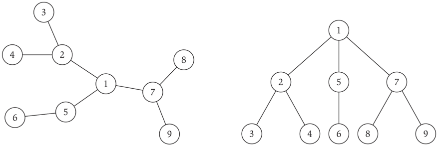

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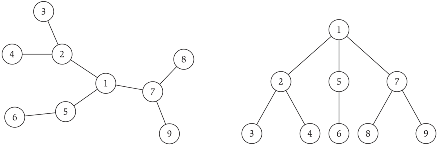
- The *distance* between two nodes u and v is the minimum number of edges in a u-v path.

Trees



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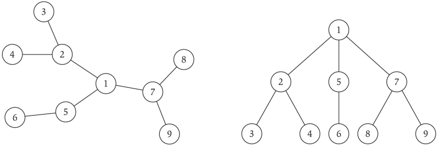

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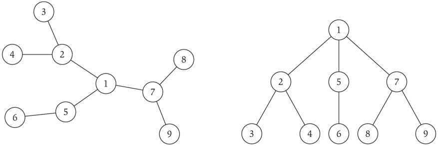

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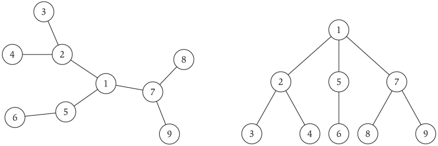

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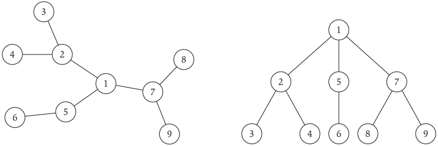

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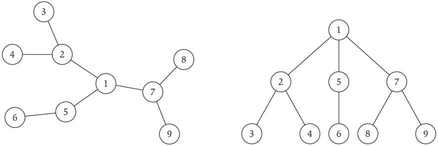

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- Examples of (rooted) trees: organisational hierarchy, a department's web pages, class hierarchies in object-oriented languages.

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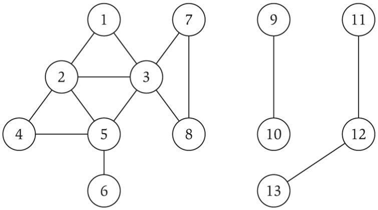
  - 2 and 3 $\Rightarrow$ 1: prove by contradiction.
  - 3 and 1 $\Rightarrow$ 2: prove yourself.

s-t Connectivity



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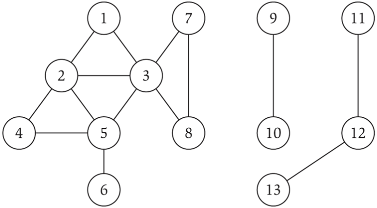

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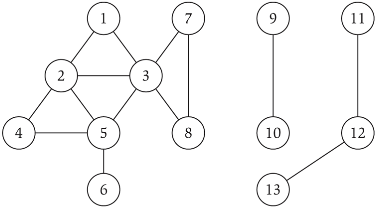

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- Algorithm for the s-t Connectivity problem: compute the connected component of G that contains s and check if t is in that component.

Computing Connected Components

- "Explore" G starting from s and maintain set R of visited nodes.

R will consist of nodes to which s has a path
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While there is an edge (u, v) where $u \in R$ and $v \notin R$
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- Issues to consider:
  - Why does the algorithm terminate?
  - Does the algorithm truly compute connected component of G containing s?
  - What is the running time of the algorithm?

Termination of the Connected Components Algorithm

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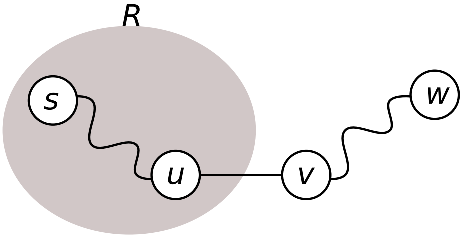
  - either $R = V$ at the end or
  - in the last iteration, every edge either has both nodes in R or both nodes not in R.

Correctness of the Connected Components Algorithm



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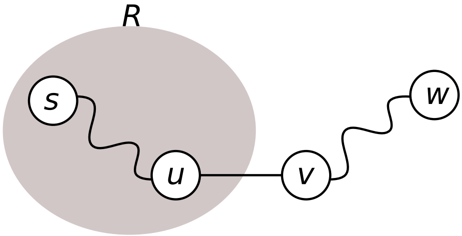

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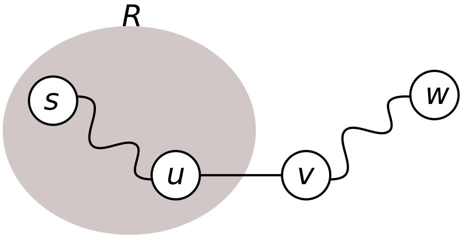

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  - (u, v) is an edge with $u \in R$ but $v \notin R$, contradicting the stopping rule.

Recovering Paths

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- To recover the s-t path, trace these edges backwards from t until we reach s.

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- How many while loops does the algorithm execute?

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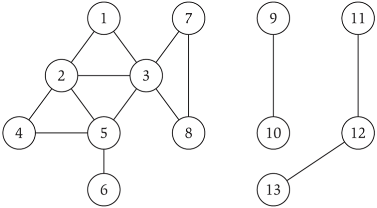
Breadth-First Search (BFS)



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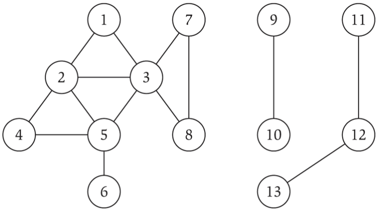

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- Idea: explore G starting at s and going "outward" in all directions, adding nodes one layer at a time.
- Layer L_0 contains only s.
- Layer L_1 contains all neighbours of s.
- Given layers $L_0, L_1, \ldots, L_j$, layer L_{j+1} contains all nodes that
  1. do not belong to an earlier layer and
  2. are connected by an edge to a node in layer L_j.

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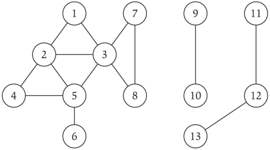
  - Why is T a tree? It is connected. The number of edges in T is the number of nodes in all the layers minus 1.
  - T is called the *breadth-first search tree*.

BFS Trees



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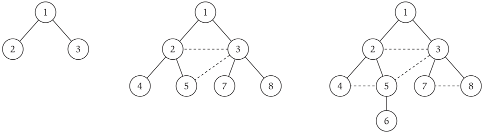

Figure 3.3 The construction of a breadth-first search tree T for the graph in Figure 3.2, with (a), (b), and (c) depicting the successive layers that are added. The solid edges are the edges of T; the dotted edges are in the connected component of G containing node 1, but do not belong to T.

- *Non-tree edge*: an edge of G that does not belong to the BFS tree T.
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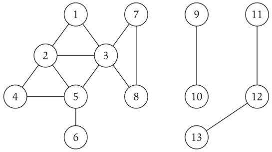

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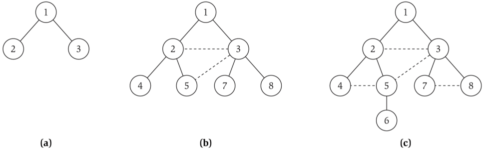

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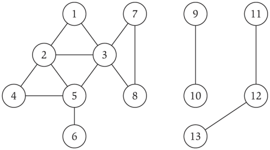

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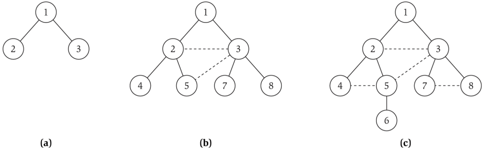

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- Still unresolved: an efficient implementation of BFS.

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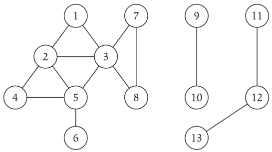
- *Depth-first search tree* is a tree T: when `DFS(`v`)` is invoked directly during the call to `DFS(`v`)`, add edge (u, v) to T.

Example of DFS



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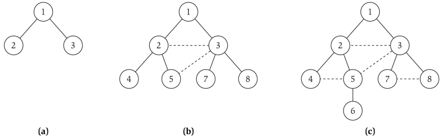

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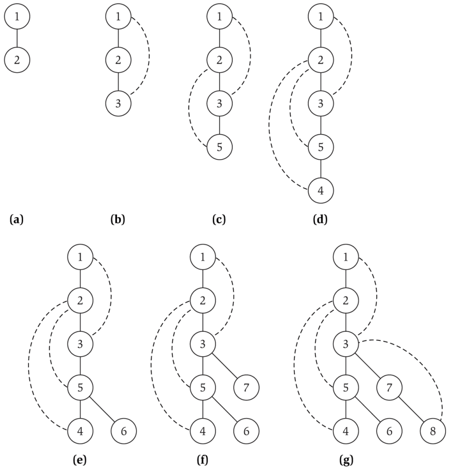

Figure 3.5 The construction of a depth-first search tree T for the graph in Figure 3.2, with (a) through (g) depicting the nodes as they are discovered in sequence. The solid edges are the edges of T; the dotted edges are edges of G that do not belong to T.

BFS vs. DFS

- Both visit the same set of nodes but in a different order.
- Both traverse all the edges in the connected component but in a different order.
- BFS trees have root-to-leaf paths that look as short as possible while paths in DFS trees tend to be long and deep.
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- Proof: Assume, without loss of generality that the DFS algorithm reached x first.
  - Since (x, y) is an edge in G, it is examined during `DFS(x)`.
  - Since $(x, y) \notin T$, y must be marked as "`Explored`" during `DFS(x)` but before (x, y) is examined.
  - Since y was not marked as "`Explored`" before `DFS(x)` was invoked, it must be marked as "`Explored`" between the invocation of `DFS(x)`. and the end of this recursive call.
  - Therefore, y must be a descendant of x in T.

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Computing All Connected Components

1. Pick an arbitrary node s in G.
2. Compute its connected component using BFS (or DFS).
3. Find a node (say v, not already visited) and repeat the BFS from v.
4. Repeat this process until all nodes are visited.

Representing Graphs

- Graph $G = (V, E)$ has two input parameters: $|V| = n, |E| = m$.
  - Size of the graph is defined to be $m + n$.
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- "Implementation" of BFS and DFS: fully specify the algorithms and data structures so that we can obtain provably efficient times.
- Inner loop of both BFS and DFS: process the set of edges incident on a given node and the set of visited nodes.
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- How do we store the set of visited nodes? Order in which we process the nodes is crucial.
    - BFS: store visited nodes in a queue (first-in, first-out).
    - DFS: store visited nodes in a stack (last-in, first-out)

Implementing BFS

- Maintain an array Discovered and set Discovered$[v] = true$ as soon as the algorithm sees v.

```
BFS(s):
  Set Discovered[s] = true and Discovered[v] = false for all other v
  Initialize L[0] to consist of the single element s
  Set the layer counter i = 0
  Set the current BFS tree T = ∅
  While L[i] is not empty
    Initialize an empty list L[i + 1]
    For each node u ∈ L[i]
      Consider each edge (u, v) incident to u
      If Discovered[v] = false then
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        Add edge (u, v) to the tree T
        Add v to the list L[i + 1]
      Endif
    Endfor
    Increment the layer counter i by one
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```

Using a Queue in BFS

- Instead of storing each layer in a different list, maintain all the layers in a single queue L.

```
BFS(s):
  Set Discovered[s] = true and Discovered[v] = false for all other v
  Initialize L[0] to consist of the single element s
  Set the layer counter i = 0
  Set the current BFS tree T = ∅
  While L[i] is not empty
    Initialize an empty list L[i+1]
    For each node u ∈ L[i]
      Consider each edge (u,v) incident to u
      If Discovered[v] = false then
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- Naive bound on running time is $O(n^2)$.
- Improved bound:
  - Maintaining layers: $O(n)$ time.
  - `for` loop for a node u: $O(n_u)$ time.
  - Total time for all `for` loops: $\sum_{u \in G} O(n_u) = O(m)$ time.
  - Total time is $O(n + m)$.

Recursive DFS

```
DFS(u):
  Mark u as "Explored" and add u to R
  For each edge (u,v) incident to u
    If v is not marked "Explored" then
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```

- Procedure has "tail recursion": recursive call is the last step.

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- Procedure has "tail recursion": recursive call is the last step.
- Can replace the recursion by an iteration: use a stack to explicitly implement the recursion.

Implementing DFS

- Maintain a stack S to store nodes to be explored.
- Maintain an array Explored and set Explored$[v]$ = *true* when the algorithm pops v from the stack.
- Read textbook on how to construct the DFS tree.

```
DFS(s):
  Initialize S to be a stack with one element s
  While S is not empty
    Take a node u from S
    If Explored[u] = false then
       Set Explored[u] = true
       For each edge (u,v) incident to u
         Add v to the stack S
       Endfor
    Endif
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```

Comparing Recursion and Iteration

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- How many times is a node's adjacency list scanned?

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- How many times is a node's adjacency list scanned? Exactly once.
- The total amount of time to process edges incident on node u's is $O(n_u)$.
- The total running time of the algorithm is $O(n + m)$.