Priority Queues

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August 31, 2009

Motivation: Sort a List of Numbers

Sort

INSTANCE: Nonempty list $x_1, x_2, \ldots, x_n$ of integers.

SOLUTION: A permutation $y_1, y_2, \ldots, y_n$ of $x_1, x_2, \ldots, x_n$ such that $y_i \leq y_{i+1},$ for all $1 \leq i < n$.

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  - Store all the numbers in a data structure D.
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- Possible algorithm:
  - Store all the numbers in a data structure D.
  - Repeatedly find the smallest number in D, output it, and remove it.
- To get $O(n \log n)$ running time, each "find minimum" step must take $O(\log n)$ time.

Candidate Data Structures for Sorting

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Sorted array Finding minimum takes $O(1)$ time but insertion and deletion can take $\Omega(n)$ time in the worst case.

Priority Queue

- Store a set S of elements, where each element v has a priority value $\texttt{key}(v)$.
- Smaller key values $\equiv$ higher priorities.
- Operations supported: find the element with smallest key, remove the smallest element, update the key of an element, insert an element, delete an element.
- Key update and element deletion require knowledge of the position of the element in the priority queue.

Heaps

- Combine benefits of both lists and sorted arrays.
- Conceptually, a heap is a balanced binary tree.
- *Heap order*: For every element v at a node i, the element w at i's parent satisfies $\texttt{key}(w) \leq \texttt{key}(v)$.

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- Alternatively, assume maximum number N of elements is known in advance.
- Store nodes of the heap in an array.
  - Node at index i has children at indices $2i$ and $2i+1$ and parent at index $\lfloor i/2 \rfloor$.
  - Index 1 is the root.
  - How do you know that a node at index i is a leaf?

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  - How do you know that a node at index i is a leaf? If $2i > n$, the number of elements in the heap.

Example of a Heap

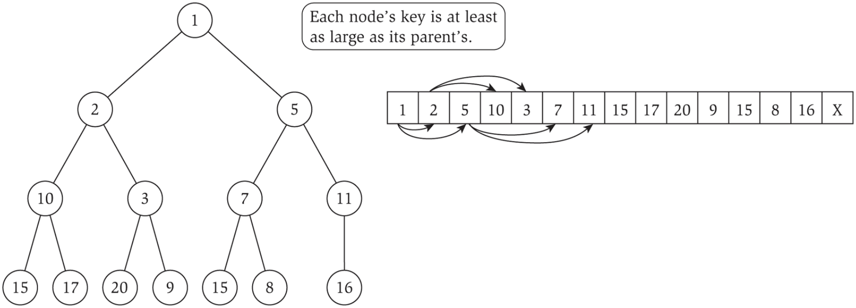


Figure 2.3 Values in a heap shown as a binary tree on the left, and represented as an array on the right. The arrows show the children for the top three nodes in the tree.

Inserting an Element: `Heapify-up`

1. Insert new element at index $n+1$.
2. Fix heap order using `Heapify-up`$(H, n+1)$.

```
Heapify-up(H,i):
  If i > 1 then
    let j = parent(i) = ⌊i/2⌋
    If key[H[i]]<key[H[j]] then
      swap the array entries H[i] and H[j]
      Heapify-up(H,j)
    Endif
  Endif
```

Example of `Heapify-up`

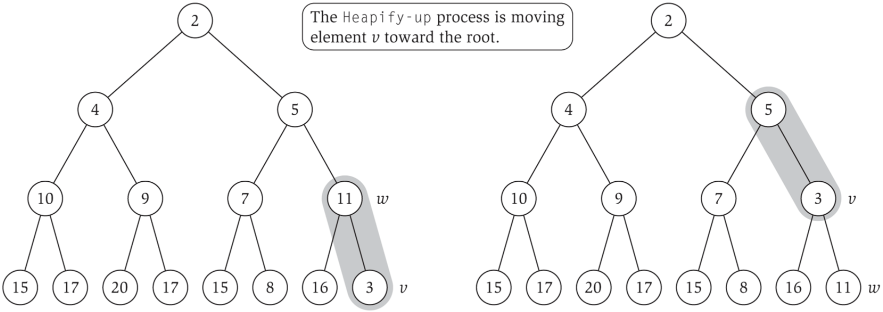
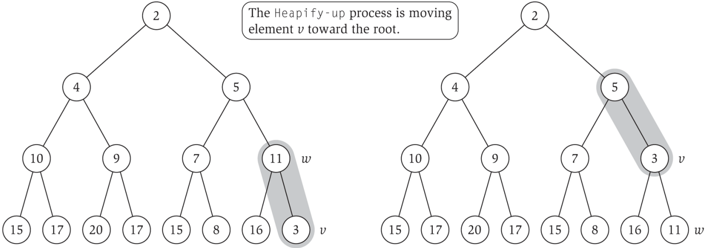


Figure 2.4 The `Heapify-up` process. Key 3 (at position 16) is too small (on the left). After swapping keys 3 and 11, the heap violation moves one step closer to the root of the tree (on the right).

Correctness of `Heapify-up`



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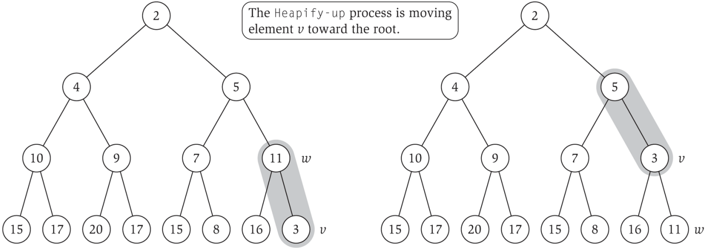

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- H is *almost a heap with key of $H[i]$ too small* if there is a value $\alpha \geq \texttt{key}(H[i])$ such that increasing $\texttt{key}(H[i])$ to α makes H a heap.
- Prove by induction on i.

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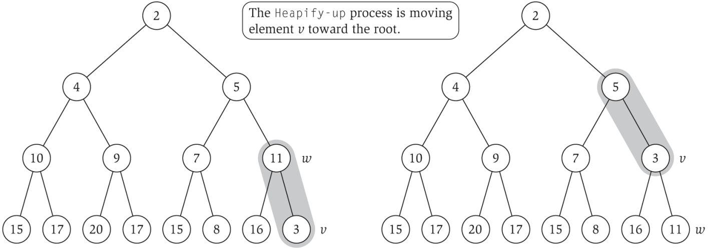

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- Prove by induction on i.
- Proof base case: $i = 1$.
- Proof inductive step: If H is almost a heap with key of $H[i]$ too small, after `Heapify-up`(H, i), H is a heap or almost a heap with the key of $H[j]$ too small.

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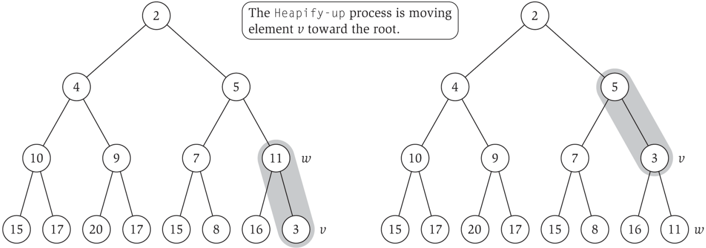

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- Running time is $O(\log i)$.

Deleting an Element: `Heapify-down`

- Suppose H has $n + 1$ elements.

1. Delete element at $H[i]$ by moving element at $H[n+1]$ to $H[i]$.
2. If element at $H[i]$ is too small, fix heap order using `Heapify-up`(H, i).
3. If element at $H[i]$ is too large, fix heap order using `Heapify-down`(H, i).

```
Heapify-down(H,i):
  Let n = length(H)
  If 2i > n then
    Terminate with H unchanged
  Else if 2i < n then
    Let left = 2i, and right = 2i + 1
    Let j be the index that minimizes key[H[left]] and key[H[right]]
  Else if 2i = n then
    Let j = 2i
  Endif
  If key[H[j]] < key[H[i]] then
     swap the array entries H[i] and H[j]
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  Endif
```
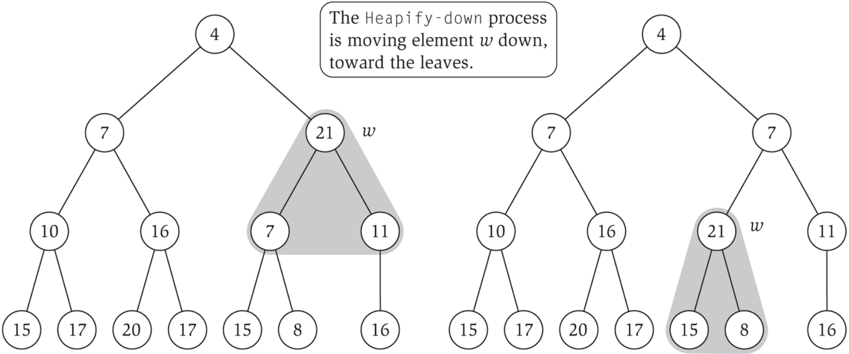

Example of `Heapify-down`



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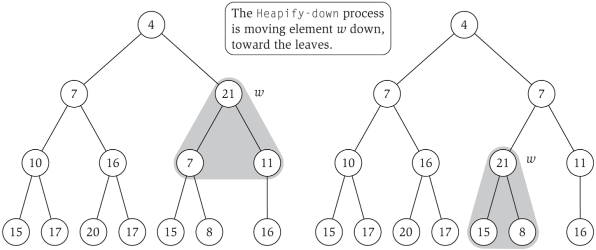

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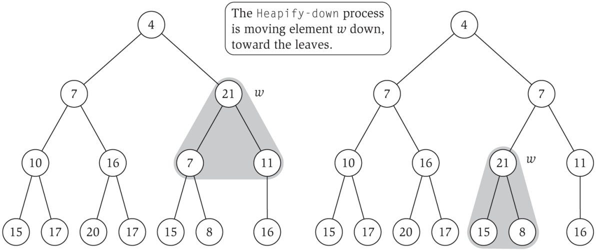

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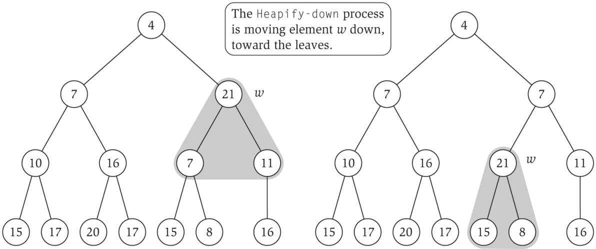

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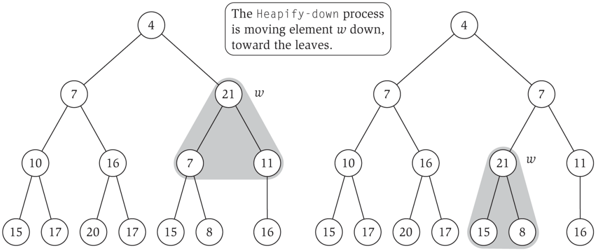

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- Proof inductive step: after `Heapify-down`(H, i), H is a heap or almost a heap with the key of $H[j]$ too big.
- Running time of `Heapify-down`(H, i) is $O(\log n)$.

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- Final algorithm:
  - Insert each number in a priority queue H.
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- Each insertion and deletion takes $O(\log n)$ time for a total running time of $O(n \log n)$.