# Summation Formulas

Let N $\geq$ 0, let A, B, and C be constants, and let *f* and *g* be any functions. Then:

$$\sum_{k=1}^{N} Cf(k) = C\sum_{k=1}^{N} f(k)$$

**S1: factor out constant**

$$\sum_{k=1}^{N} (f(k) \pm g(k)) = \sum_{k=1}^{N} f(k) \pm \sum_{k=1}^{N} g(k)$$

**S2: separate summed terms**

$$\sum_{k=1}^{N} C = NC$$

**S3: sum of constant**

$$\sum_{k=1}^{N} k = \frac{N(N+1)}{2}$$

**S4: sum of *k***

$$\sum_{k=1}^{N} k^2 = \frac{N(N+1)(2N+1)}{6}$$

**S5: sum of *k* squared**

$$\sum_{k=0}^{N} 2^k = 2^{N+1} - 1$$

**S6: sum of 2^*k***

$$\sum_{k=1}^{N} k2^{k-1} = (N-1)2^N + 1$$

**S7: sum of k2^(*k-1)***

# Logarithms

Let b be a real number, $b > 0$ and $b \neq 1$. Then, for any real number $x > 0$, the *logarithm of x to base b* is the power to which b must be raised to yield x. That is:

$$\log_b(x) = y \text{ if and only if } b^y = x$$

For example:

$$\log_2(64) = 6 \text{ because } 2^6 = 64$$

$$\log_2(1/8) = -3 \text{ because } 2^{-3} = 1/8$$

$$\log_2(1) = 0 \text{ because } 2^0 = 1$$

If the base is omitted, the standard convention in mathematics is that log base 10 is intended; in computer science the standard convention is that log base 2 is intended.

# Logarithms

Let a and b be real numbers, both positive and neither equal to 1. Let x > 0 and y > 0 be real numbers.

**L1:** $\log_b(1) = 0$

**L2:** $\log_b(b) = 1$

**L3:** $\log_b(x) < 0 \text{ for all } 0 < x < 1$

**L4:** $\log_b(x) > 0 \text{ for all x} > 1$

**L5:** $\log_b(b^y) = y$

**L6:** $b^{\log_b(x)} = x$

**L7:** $\log_b(xy) = \log_b(x) + \log_b(y)$

**L8:** $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$

**L9:** $\log_b(x^y) = y\log_b(x)$

**L10:** $\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$

# Limit of a Function

<u>Definition:</u>

Let f(x) be a function with domain (a, b) and let a < c < b. The *limit of f(x) as x approaches c is L* if, for every positive real number ε, there is a positive real number δ such that whenever |x-c| < δ then |f(x) – L| < ε.

The definition being cumbersome, the following theorems on limits are useful. We assume f(x) is a function with domain as described above and that K is a constant.

**C1:** $$\lim_{x \to c} K = K$$

**C2:** $$\lim_{x \to c} x = c$$

**C3:** $$\lim_{x \to c} x^r = c^r \text{ for all } r > 0$$

# Limit of a Function

Here assume f(x) and g(x) are functions with domain as described above and that K is a constant, and that both the following limits exist (and are finite):

$$\lim_{x \to c} f(x) = A \qquad \lim_{x \to c} g(x) = B$$

Then:

**C4:** $$\lim_{x \to c} Kf(x) = K \lim_{x \to c} f(x)$$

**C5:** $$\lim_{x \to c} \left( f(x) \pm g(x) \right) = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x)$$

**C6:** $$\lim_{x \to c} \left( f(x) * g(x) \right) = \lim_{x \to c} f(x) * \lim_{x \to c} g(x)$$

**C7:** $$\lim_{x \to c} \left( f(x) / g(x) \right) = \lim_{x \to c} f(x) / \lim_{x \to c} g(x) \text{ provided } B \neq 0$$

# Limit as x Approaches Infinity

Definition:

Let f(x) be a function with domain $[0, \infty)$. The *limit of f(x) as x approaches $\infty$ is L*, where L is finite, if, for every positive real number ε, there is a positive real number N such that whenever x > N then $|f(x) - L| < \varepsilon$.

The definition being cumbersome, the following theorems on limits are useful. We assume f(x) is a function with domain $[0, \infty)$ and that K is a constant.

**C8:** $$\lim_{x \to \infty} K = K$$

**C9:** $$\lim_{x \to \infty} \frac{1}{x} = 0$$

**C10:** $$\lim_{x \to \infty} \frac{1}{x^r} = 0 \text{ for all } r > 0$$

# Limit of a Rational Function

Given a rational function the last two rules are sufficient if a little algebra is employed:

$$\lim_{x\to\infty}\frac{7x^2+5x+10}{3x^2+2x+5} = \lim_{x\to\infty}\frac{7+\frac{5}{x}+\frac{10}{x^2}}{3+\frac{2}{x}+\frac{5}{x^2}}$$

**Divide by highest power of x from the denominator.**

$$= \frac{\lim_{x\to\infty}7+\lim_{x\to\infty}\frac{5}{x}+\lim_{x\to\infty}\frac{10}{x^2}}{\lim_{x\to\infty}3+\lim_{x\to\infty}\frac{2}{x}+\lim_{x\to\infty}\frac{5}{x^2}}$$

**Take limits term by term.**

$$= \frac{7+0+0}{3+0+0}$$

**Apply theorem C3.**

$$= \frac{7}{3}$$

# Infinite Limits

In some cases, the limit may be infinite. Mathematically, this means that the limit does not exist.

**C11:** $$\lim_{x\to\infty} x^r = \infty \text{ for all } r > 0$$

**C12:** $$\lim_{x\to\infty}\left(\log_b x\right) = \infty$$

**C13:** $$\lim_{x\to\infty}\left(e^x\right) = \infty$$

Example:

$$\lim_{x\to\infty}\frac{7x^2+5x+10}{2x+5} = \lim_{x\to\infty}\frac{7x+5+\frac{10}{x}}{2+\frac{5}{x}}$$

$$= \frac{\lim_{x\to\infty} 7x + \lim_{x\to\infty} 5 + \lim_{x\to\infty}\frac{10}{x}}{\lim_{x\to\infty} 2 + \lim_{x\to\infty}\frac{5}{x}} = \infty$$

# l'Hôpital's Rule

In some cases, the reduction trick shown for rational functions does not apply:

$$\lim_{x \to \infty} \frac{7x + 5\log(x) + 10}{2x + 5} = ??$$

In such cases, l'Hôpital's Rule is often useful. If f(x) and g(x) are differentiable functions such that

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = \infty$$

**This also applies if the limit is 0.**

then:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

 

# l'Hôpital's Rule Examples

Applying l'Hôpital's Rule:

$$\lim_{x\to\infty}\frac{7x+5\log(x)+10}{2x+5}=\lim_{x\to\infty}\frac{7+\frac{5}{x}}{2}=\frac{7}{2}$$

Another example:

$$\lim_{x\to\infty}\frac{x^3+10}{e^x}=\lim_{x\to\infty}\frac{3x^2}{e^x}=\lim_{x\to\infty}\frac{6x}{e^x}=\lim_{x\to\infty}\frac{6}{e^x}=0$$

Recall that: $D\left[e^{f(x)}\right]=e^{f(x)}D\left[f(x)\right]$

# Mathematical Induction

Mathematical induction is a technique for proving that a statement is true for all integers in the range from $N_0$ to $\infty$, where $N_0$ is typically 0 or 1.

**First (or Weak) Principle of Mathematical Induction**

Let P(N) be a proposition regarding the integer N, and let S be the set of all integers k for which P(k) is true. If

1) $N_0$ is in S, and
2) whenever N is in S then N+1 is also in S,

then S contains all integers in the range $[N_0, \infty)$.

To apply the PMI, we must first establish that a specific integer, $N_0$, is in S (establishing the basis) and then we must establish that if a arbitrary integer, $N \geq N_0$, is in S then its successor, N+1, is also in S.

# Induction Example

Theorem: For all integers $n \geq 1$, $n^2+n$ is a multiple of 2.

*proof:* Let S be the set of all integers for which $n^2+n$ is a multiple of 2.

If $n = 1$, then $n^2+n = 2$, which is obviously a multiple of 2. This establishes the basis, that 1 is in S.

Now suppose that some integer $k \geq 1$ is an element of S. Then $k^2+k$ is a multiple of 2. We need to show that k+1 is an element of S; in other words, we must show that $(k+1)^2+(k+1)$ is a multiple of 2. Performing simple algebra:

$$(k+1)^2+(k+1) = (k^2 + 2k + 1) + (k + 1) = k^2 + 3k + 2$$

Now we know $k^2+k$ is a multiple of 2, and the expression above can be grouped to show:

$$(k+1)^2+(k+1) = (k^2 + k) + (2k + 2) = (k^2 + k) + 2(k + 1)$$

The last expression is the sum of two multiples of 2, so it's also a multiple of 2. Therefore, k+1 is an element of S.

Therefore, by PMI, S contains all integers $[1, \infty)$.

QED

# Inadequacy of the First Form of Induction

Theorem: Every integer greater than 3 can be written as a sum of 2's and 5's.

(That is, if $N > 3$, then there are nonnegative integers x and y such that $N = 2x + 5y$.)

This is not (easily) provable using the First Principle of Induction. The problem is that the way to write N+1 in terms of 2's and 5's has little to do with the way N is written in terms of 2's and 5's. For example, if we know that

$$N = 2x + 5y$$

we can say that

$$N + 1 = 2x + 5y + 1 = 2x + 5(y - 1) + 5 + 1 = 2(x + 3) + 5(y - 1)$$

but we have no reason to believe that $y - 1$ is nonnegative. (Suppose for example that N is 9.)

# "Strong" Form of Induction

There is a second statement of induction, sometimes called the "strong" form, that is adequate to prove the result on the preceding slide:

**Second (or Strong) Principle of Mathematical Induction**

Let P(N) be a proposition regarding the integer N, and let S be the set of all integers k for which P(k) is true. If

1) $N_0$ is in S, and

2) whenever $N_0$ through N are in S then N+1 is also in S,

then S contains all integers in the range $[N_0, \infty)$.

Interestingly, the "strong" form of induction is logically equivalent to the "weak" form stated earlier; so in principle, anything that can be proved using the "strong" form can also be proved using the "weak" form.

# Using the Second Form of Induction



Theorem: Every integer greater than 3 can be written as a sum of 2's and 5's.

*proof:* Let S be the set of all integers $n > 3$ for which $n = 2x + 5y$ for some nonnegative integers x and y.

If $n = 4$, then $n = 2*2 + 5*0$. If $n = 5$, then $n = 2*0 + 5*1$. This establishes the basis, that 4 and 5 are in S.

Now suppose that all integers from 4 through k are elements of S, where $k \geq 5$. We need to show that k+1 is an element of S; in other words, we must show that $k+1 = 2r + 5s$ for some nonnegative integers r and s.

Now $k+1 \geq 6$, so $k-1 \geq 4$. Therefore by our assumption, $k-1 = 2x + 5y$ for some nonnegative integers x and y. Then, simple algebra yields that:

$$k+1 = k-1 + 2 = 2x + 5y + 2 = 2(x+1) + 5y,$$

whence k+1 is an element of S.

Therefore, by the Second PMI, S contains all integers $[4, \infty)$.

QED

# Induction Example

Consider the sequences $\{a_k\}$ and $\{d_k\}$ :

$$a_k = 2^k - 1 \text{ for } k \geq 0$$

$$\begin{cases} d_0 = 0, d_1 = 1 \\ d_k = 3d_{k-1} - 2d_{k-2} \text{ for } k \geq 2 \end{cases}$$

Then for all $k \geq 0$. $a_k = d_k$.

*proof*: Let $S = \{k \geq 0 \mid a_k = d_k\}$.

Trivial calculations show that $a_0 = d_0$ and $a_1 = d_1$, so 0 and 1 are in S.

Suppose that there is some $N \geq 1$ such that 0, 1, …, N are in S. In other words, assume that there is some $N \geq 1$ such that $a_i = d_i$ for all i from 0 to N.

Now, $N + 1 \geq 2$, so from the definition of $\{d_k\}$and the inductive assumption we have:

$$\begin{aligned} d_{N+1} &= 3d_N - 2d_{N-1} \\ &= 3(2^N - 1) - 2(2^{N-1} - 1) \\ &= 3 \cdot 2^N - 3 - 2^N + 2 \\ &= 2 \cdot 2^N - 1 \\ &= 2^{N+1} - 1 \end{aligned}$$

Therefore $N + 1$ is in S, and so by the principle of induction, $S = \{k \geq 0\}$. QED