You will submit your solution to this assignment to the Curator System (as `HW02`). Your solution must be either a plain text file (e.g., NotePad) or a typed MS Word document; submissions in other formats will not be graded.

Except as noted, credit will only be given if you show relevant work.

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1. [25 points] Using the rules given in the course notes, perform an exact count complexity analysis, for the worst case, of the body of the following function.

```
void Mystery(int M[N][N], const int N) {

   for (int R = 0; R < N; R++) {      // 1 before, 2 per pass, 1 to exit
      for (int C = 1; C < N; C++) {   // 1 before, 2 per pass, 1 to exit
         if (M[R][C-1] < M[R][C])     // 6: 4 index, one arith, one comp
            M[R][C-1] = M[R][C];      // 6: 4 index, one arith, 1 assign
         else {
            M[R][C]   = M[R][C-1];    // 6: 4 index, one arith, 1 assign
            M[R][C-1] = 0;            // 4: 2 index, one arith, 1 assign
         }
      }
   }
}
```

State both a complexity function T(N) and the Θ-complexity of T(N).

According to the counts in the comments added above, and the rules for iteration and selection from the course notes, the operation count is given by:

$$
\begin{aligned}
T(N) &= 1 + \sum_{R=0}^{N-1}\left(2 + 1 + \sum_{C=1}^{N-1}\left(2 + 6 + \max(6,10)\right) + 1\right) + 1 \\
&= \sum_{R=0}^{N-1}\left(\sum_{C=1}^{N-1}(14) + 4\right) + 2 \\
&= \sum_{R=0}^{N-1}(14N + 4) + 2 \\
&= \sum_{R=1}^{N-1}(14N + 4) + 4 + 2 \\
&= 14\sum_{R=1}^{N-1} N + \sum_{R=1}^{N-1} 4 + 6 \\
&= 14N(N-1) + 4(N-1) + 6 \\
&= 14N^2 - 10N + 2
\end{aligned}
$$

From the theorems in the course notes, it's immediately apparent that $T(N)$ is $\Theta\left(N^2\right)$.

2. [25 points] Consider the following function, where α is an unknown positive constant:

$$f(n) = n^{\alpha} + \log n$$

Use Theorem 8 from the course notes on asymptotics to prove each of the following facts:

a) $f(n)$ is $\Theta(n^{\alpha})$ if $\alpha > 0$

We can begin by setting up the appropriate limit to apply Theorem 8:

$$\lim_{n\to\infty} \frac{n^{\alpha} + \log n}{n^{\alpha}} = \lim_{n\to\infty}\left(1 + \frac{\log n}{n^{\alpha}}\right) = 1 + \lim_{n\to\infty}\left(\frac{\log n}{n^{\alpha}}\right)$$

From here, we need to know that the last limit is indeterminate, since the numerator and denominator are both going to infinity. Hence, we can apply l'Hopital's rule:

$$\lim_{n\to\infty} \frac{n^{\alpha} + \log n}{n^{\alpha}} = 1 + \lim_{n\to\infty}\left(\frac{\log n}{n^{\alpha}}\right) = 1 + \lim_{n\to\infty}\left(\frac{1/n\ln 2}{\alpha n^{\alpha-1}}\right) = 1 + \lim_{n\to\infty}\left(\frac{\ln 2}{\alpha n^{\alpha}}\right) = 1 + 0 = 1$$

So, by Theorem 8, $n^{\alpha} + \log n$ is $\Theta\left(n^{\alpha}\right)$.

b) $f(n)$ is $\Theta(\log n)$ if $\alpha < 0$

We can begin by setting up the appropriate limit to apply Theorem 8:

$$\lim_{n\to\infty} \frac{n^{\alpha} + \log n}{\log n} = \lim_{n\to\infty}\left(\frac{n^{\alpha}}{\log n} + 1\right) = 1 + \lim_{n\to\infty}\left(\frac{n^{\alpha}}{\log n}\right)$$

Now, since $\alpha < 0$**, we can rewrite the quotient in the limit as shown below, and since we know that** ***n*** **is raised to a positive value in that quotient, we can immediately see that the limit follows:**

$$\lim_{n\to\infty} \frac{n^{\alpha} + \log n}{\log n} = 1 + \lim_{n\to\infty}\left(\frac{n^{\alpha}}{\log n}\right) = 1 + \lim_{n\to\infty}\left(\frac{1}{n^{-\alpha}\log n}\right) = 1 + 0 = 1$$

So, by Theorem 8, $n^{\alpha} + \log n$ is $\Theta\left(\log n\right)$.

3. [25 points] For each part, determine the simplest possible function g(n) such that the given function is $\Theta(g)$. No justification is necessary, but you might have to do some analysis using the theorems from the notes.

a) $a(n) = 14n^2 + 3n\log n$

By the theorems, the dominant term here is $14n^2$, and so $a(n)$ is $\Theta\left(n^2\right)$.

b) $b(n) = 3n^2 \log n$

By the theorems, $b(n)$ is $\Theta\left(n^2 \log n\right)$..

c) $c(n) = 3n\log^2 n + 3n^2 \log n$

The theorems don't cover this directly; since *n* dominates log n, from the list of common terms, we might guess that the dominant term here is $n^2 \log n$. We can validate that guess by computing the relevant limit and applying Theorem 8:

$$\lim_{n\to\infty} \frac{3n\log^2 n + 3n^2 \log n}{n^2 \log n} = \lim_{n\to\infty}\left(\frac{3n\log^2 n}{n^2 \log n} + 1\right)$$
$$= \lim_{n\to\infty}\left(\frac{3\log n}{n} + 1\right) = \lim_{n\to\infty}\left(\frac{3/n\ln 2}{1} + 1\right)$$
$$= \lim_{n\to\infty}\left(\frac{3}{n\ln 2} + 1\right) = 0 + 1 = 1$$

d) $d(n) = 10n^2 + 2^n$

By the theorems from the notes, $d(n)$ is $\Theta\left(2^n\right)$.

e) $e(n) = \frac{1}{n}$

This is so easy it may have been confusing. There's only one term, and Θ is reflexive, so $e(n)$ is $\frac{1}{n}$.

4. [25 points] Suppose that executing an algorithm on input of size N requires executing T(N) = 3N log N + 16N instructions. How long would it take to execute this algorithm on hardware capable of carrying out 2^{25} instructions per second if N = 2^{30}? (Give your answer in hours, minutes and seconds, to the nearest second.)

The total number of instructions executed would be:

$$T(2^{30}) = 3 \cdot 2^{30} \log 2^{30} + 16 \cdot 2^{30} = 3 \cdot 2^{30} \cdot 30 + 16 \cdot 2^{30} = 106 \cdot 2^{30}$$

The time required would b e:

$$\frac{T(2^{30})}{2^{25}} = \frac{106 \cdot 2^{30}}{2^{25}} = 106 \cdot 2^{5} = 3392 \text{ seconds}$$

And that is 56 minutes 32 seconds.