Hyperbolic geometric model for complex networks

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The hyperbolic model of complex networks is a novel model to explain the topological characteristics and growth mechanisms of complex networks. It assumes that a complex network is a random graph embedded into a hyperbolic space with latent geometries and robust clustering. Hyperbolic space is the kind of space whose curvature is negative and constant. As the higher dimensional Euclidean space is inconvenient for defining a hyperbolic space, various models of hyperbolic spaces have been developed, such as the Extended Poincaré disc model, Hyperboloid model, Klein model, etc. Here we take one of the most commonly used hyperbolic space models, the extended Poincaré disc model, as an example.

Suppose a random graph with n nodes is generated in hyperbolic space as follows: For a node $p = (r, \phi)$, the radial coordinate r is set to $\log(t)$, t is time step, and the angle ϕ of p is sampled uniformly from $[0, 2\pi]$. The hyperbolic distance of a pair of node $d(u, v)$ satisfies $\cosh(d(u, v)) = \cosh(r_u)\cosh(r_v) - \sinh(r_u)\sinh(r_v)\cos(\Delta\theta)$, where $\Delta\theta = \pi - \mid \pi - \mid \phi_u - \phi_v \mid\mid$ is the angle between two nodes. Nodes are added to the generated network one by one. Each node is connected to the m nodes closest to its hyperbolic distance, where m is a fixed parameter. In this manner, the network will grow until it consists of n nodes.

Since the radial coordinate is related to its birth time, the earlier the node is born, the greater the probability that a new node will link it. Like the preferential attachment model, the degree distribution of the hyperbolic model obeys the power law. On the other hand, the θ of a node describes the similarity. Those node pairs with smaller $\Delta\theta$ are the more similar ones. Therefore, in the hyperbolic model, similarity and popularity are equalized. Nodes with similar angles and radial coordinates will form more triangles in the network, making the clustering more significant.

The hyperbolic space is also better suited to describe real networks, as most real networks have hierarchical structures similar to trees. In fact, the hyperbolic space is the continuous version of a tree network. Given a complete binary tree, the number of nodes at a level n is $2^n = e^{n \ln 2}$, which is exponential growth. If we consider the coordinate origin in hyperbolic space as the root of the tree, and the distance to the origin is the number of levels n. The circumference, $C = 2\pi \sinh(n) = \pi(e^n - e^{-n})$ at radius n of the hyperbolic space, is equivalent to the number of nodes at the level n of the tree, which also approximates an exponential growth of e^n. Therefore, these two are equivalent.

Hyperbolic geometry is a significant branch of non-Euclidean geometry and has many practical engineering applications. For example, it has been found that if we place the Internet router on the hub nodes embedded in hyperbolic space, the overall navigation efficiency of the network is significantly improved. We expect to exploit the hyperbolic geometric models in the computational epidemiology area as well, for instance, to improve the efficiency of group testing by finding the hubs and clusters under the hyperbolic space.