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STACK AND QUEUE LAYOUTS OF POSETS*

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Abstract. The stacknumber (queuenumber) of a poset is defined as the stacknumber (queuenumber) of its Hasse diagram viewed as a directed acyclic graph. Upper bounds on the queuenumber of a poset are derived in terms of its jumpnumber, its length, its width, and the queuenumber of its covering graph. A lower bound of $\Omega(\sqrt{n})$ is shown for the queuenumber of the class of n-element planar posets. The queuenumber of a planar poset is shown to be within a small constant factor of its width. The stacknumber of n-element posets with planar covering graphs is shown to be $\Theta(n)$. These results exhibit sharp differences between the stacknumber and queuenumber of posets as well as between the stacknumber (queuenumber) of a poset and the stacknumber (queuenumber) of its covering graph.

Key words. poset, queue layout, stack layout, book embedding, Hasse diagram, jumpnumber

AMS subject classifications. 05C99, 68R10, 94C15

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1. Introduction.

Stack and queue layouts of undirected graphs appear in a variety of contexts such as VLSI, fault-tolerant processing, parallel processing, and sorting networks (Pemmaraju [16]). In a new context, Heath, Pemmaraju, and Ribbens [10] use queue layouts as the basis of an efficient scheme to perform matrix computations on a data driven network. Bernhart and Kainen [1] introduce the concept of a stack layout, which they call *book embedding*. Chung, Leighton, and Rosenberg [3] study stack layouts of undirected graphs and provide optimal stack layouts for a variety of classes of graphs. Heath and Rosenberg [13] develop the notion of queue layouts and provide optimal queue layouts for many classes of undirected graphs. Heath, Leighton, and Rosenberg [8] study relationships between queue and stack layouts of undirected graphs. In some applications of stack and queue layouts, it is more realistic to model the application domain with directed acyclic graphs (dags) or with posets, rather than with undirected graphs. Various questions that have been asked about stack and queue layouts of undirected graphs acquire a new flavor when there are directed edges (arcs). This is because the direction of the arcs imposes restrictions on the node orders that can be considered. Heath and Pemmaraju [9] and Heath, Pemmaraju, and Trenk [11, 12] initiate the study of stack and queue layouts of dags and provide optimal stack and queue layouts for several classes of dags.

In this paper, we focus on stack and queue layouts of posets. Posets are ubiquitous mathematical objects, and various measures of their structure have been defined. Some of these measures are bumpnumber, jumpnumber, length, width, dimension, and thickness [2, 7]. Nowakowski and Parker [15] define the stacknumber of a poset as the stacknumber of its Hasse diagram viewed as a dag. They derive a general lower bound on the stacknumber of a planar poset and an upper bound on the stacknumber of a lattice. Nowakowski and Parker [15] conclude by asking whether the stacknumber of the class of planar posets is unbounded. Hung [14] shows that there exists a planar

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poset with stacknumber 4; moreover, no planar poset with stacknumber 5 is known. Sysło [17] provides a lower bound on the stacknumber of a poset in terms of its bumpnumber. He also shows that, while posets with jumpnumber 1 have stacknumber at most 2, posets with jumpnumber 2 can have an arbitrarily large stacknumber.

The organization of this paper is as follows. Section 2 contains definitions. In section 3, we derive upper bounds on the queuenumber of a poset in terms of its jumpnumber, its length, its width, and the queuenumber of its covering graph. In section 4, we show that the queuenumber of the class of planar posets is unbounded. In a complementary upper bound result, we show that the queuenumber of a planar poset is within a small constant factor of its width. In section 5, we show that the stacknumber of the class of n-element posets with planar covering graphs is $\Theta(n)$. In section 6, the decision problem of recognizing a 4-queue poset is defined; Heath and Pemmaraju [9] and Heath, Pemmaraju, and Trenk [11] show that the problem is NP-complete. In section 7, we present several open questions and conjectures concerning stack and queue layouts of posets.

2. Definitions. This section contains the definitions of stack and queue layouts of undirected graphs, dags, and posets. Other measures of the structure of posets are also defined.

Let $G = (V, E)$ be an undirected graph without multiple edges or loops. A *k-stack layout* of G consists of a total order σ on V along with an assignment of each edge in E to one of k stacks, $s_1, s_2, \ldots, s_k$. Each stack s_j operates as follows. The vertices of V are scanned in left-to-right (ascending) order according to σ. When a vertex v is encountered, any edges assigned to s_j that have v as their right endpoint must be at the top of the stack and are popped. Any edges that are assigned to s_j and have left endpoint v are pushed onto s_j in descending order (according to σ) of their right endpoints. The *stacknumber* $SN(G)$ of G is the smallest k such that G has a *k-stack layout*. G is said to be a *k-stack* graph if $SN(G) = k$. The *stacknumber* of a class of graphs $\mathcal{C}$, denoted by $SN_{\mathcal{C}}(n)$, is the function of the natural numbers that equals the least upper bound of the stacknumber of all graphs in $\mathcal{C}$ with at most n vertices. We are interested in the asymptotic behavior of $SN_{\mathcal{C}}(n)$ or in whether $SN_{\mathcal{C}}(n)$ is bounded above by a constant.

A *k-queue layout* of G consists of a total order σ on V along with an assignment of each edge in E to one of k queues, $q_1, q_2, \ldots, q_k$. Each queue q_j operates as follows. The vertices of V are scanned in left-to-right (ascending) order according to σ. When a vertex v is encountered, any edges assigned to q_j that have v as their right endpoint must be at the front of the queue and are dequeued. Any edges that are assigned to q_j and have left endpoint v are enqueued into q_j in ascending order (according to σ) of their right endpoints. The *queuenumber* $QN(G)$ of G is the smallest k such that G has a *k-queue layout*. The *queuenumber* of a class of graphs $\mathcal{C}$, denoted by $QN_{\mathcal{C}}(n)$, is the function of the natural numbers that equals the least upper bound of the queuenumber of all graphs in $\mathcal{C}$ with at most n vertices. We are interested in the asymptotic behavior of $QN_{\mathcal{C}}(n)$ or in whether $QN_{\mathcal{C}}(n)$ is bounded above by a constant.

For a fixed order σ on V, we identify sets of edges that are obstacles to minimizing the number of stacks or queues. A *k-rainbow* is a set of k edges $\{(a_i, b_i) \mid 1 \leq i \leq k\}$ such that

$$a_1 <_\sigma a_2 <_\sigma \cdots <_\sigma a_{k-1} <_\sigma a_k <_\sigma b_k <_\sigma b_{k-1} <_\sigma \cdots <_\sigma b_2 <_\sigma b_1;$$

i.e., a rainbow is a *nested* matching. Any two edges in a rainbow are said to *nest*.

A *k-twist* is a set of k edges $\{(a_i, b_i) \mid 1 \le i \le k\}$ such that

$$a_1 <_\sigma a_2 <_\sigma \cdots <_\sigma a_{k-1} <_\sigma a_k <_\sigma b_1 <_\sigma b_2 <_\sigma \cdots <_\sigma b_{k-1} <_\sigma b_k,$$

i.e., a twist is a *fully crossing* matching. Any two edges in a twist are said to *cross*.

A rainbow is an obstacle for a queue layout because no two edges that nest can be assigned to the same queue, while a twist is an obstacle for a stack layout because no two edges that cross can be assigned to the same stack. Intuitively, we can think of a stack layout or a queue layout of a graph as a drawing of the graph in which the vertices are laid out on a horizontal line and the edges appear as arcs above the line. In a stack layout no two edges that intersect can be assigned to the same stack, while in a queue layout no two edges that nest can be assigned to the same queue. Clearly, the size of the largest twist (rainbow) in a layout is a lower bound on the number of stacks (queues) required for that layout. Heath and Rosenberg [13] show that the size of the largest rainbow in a layout equals the minimum queue requirement of the layout.

PROPOSITION 2.1 (Heath and Rosenberg, [Theorem 2.3, 13]). *Suppose $G = (V, E)$ is an undirected graph, and σ is a fixed total order on V. If G has no rainbow of more than k edges with respect to σ, then G has a k-queue layout with respect to σ.*

In contrast, the size of the largest twist in a layout may be strictly less than the minimum stack requirement of the layout (see [13, Proposition 2.4]).

The definitions of stack and queue layouts are now extended to dags by requiring that the layout order be a topological order. Following a common distinction, we use *vertices* and *edges* for undirected graphs, but *nodes* and *arcs* for directed graphs. Suppose that $G = (V, E)$ is an undirected graph and that $\vec{G} = (V, \vec{E})$ is a dag whose arc set $\vec{E}$ is obtained by directing the edges in E. A *topological order* of $\vec{G}$ is a total order σ on V such that $(u, v) \in \vec{E}$ implies $u <_\sigma v$. A k-stack (k-queue) layout of the dag $\vec{G} = (V, \vec{E})$ is a k-stack (k-queue) layout of the graph G such that the total order is a *topological order* of $\vec{G}$. As before, $SN(\vec{G})$ is the smallest k such that $\vec{G}$ has a k-stack layout, and $QN(\vec{G})$ is the smallest k such that $\vec{G}$ has a k-queue layout.

A *partial order* is a reflexive, transitive, antisymmetric binary relation. A *poset* $P = (V, \le)$ is a set V with a partial order $\le$ (see Birkhoff [2] or Davey and Priestly [4]). The cardinality $|P|$ of a poset P equals $|V|$. We only consider posets with finite cardinality in this paper. We write $u < v$ if $u \le v$ and $u \ne v$. The *Hasse diagram* $\vec{H}(P) = (V, \vec{E})$ of a poset $P = (V, \le)$ is a dag with arc set

$$\vec{E} = \{(u, v) \mid u < v \text{ and there is no } w \text{ such that } u < w < v\}$$

(see Davey and Priestly [4]). A Hasse diagram is a minimal representation of a poset because it contains none of the arcs implied by transitivity of $\le$. The stacknumber $SN(P)$ of a poset P is $SN(\vec{H}(P))$, the stacknumber of its Hasse diagram. Similarly, the queuenumber $QN(P)$ of a poset P is $QN(\vec{H}(P))$, the queuenumber of its Hasse diagram. Figure 2.1 gives an example of a 2-stack poset, while Fig. 2.2 gives an example of a 2-queue poset. The underlying undirected graph, $H(P)$, of $\vec{H}(P)$ is called the *covering graph* of P. Clearly, for any poset P, we have

$$SN(H(P)) \le SN(P)$$

and

$$QN(H(P)) \le QN(P).$$

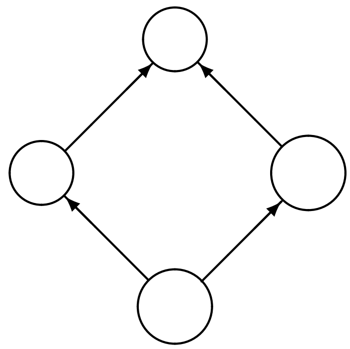

FIG. 2.1. *A 2-stack poset.*

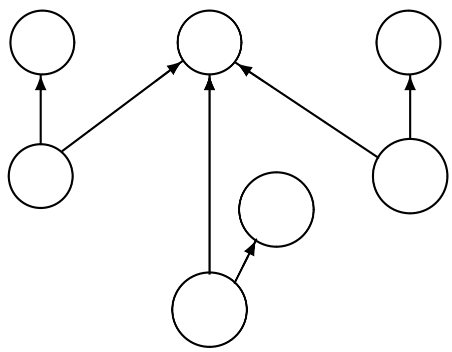

FIG. 2.2. *A 2-queue poset.*

The stacknumber and the queuenumber of the covering graphs of the posets in both Fig. 2.1 and Fig. 2.2 are 1. A poset P is *planar* if its Hasse diagram $\vec{H}(P)$ has a planar embedding in which all arcs are drawn as straight line segments with the tail of each arc strictly below its head with respect to a Cartesian coordinate system; call such an embedding of any dag an *upwards embedding.* Without loss of generality, we may always assume that no two nodes of $\vec{H}(P)$ are on the same horizontal line. (If two nodes are on the same horizontal line, a slight vertical perturbation of either of them yields another upwards embedding with the nodes on different horizontal lines.) Given an upwards embedding of a dag, the y coordinates of the nodes give a topological order on the nodes from lowest to highest called the *vertical order.* Note that the covering graph $H(P)$ may be planar even though the poset P is not. Figure 2.3 shows an example of a nonplanar poset whose covering graph is planar.

Let γ be a fixed topological order on $\vec{H}(P)$. Two elements u and v are *adjacent* in γ if there is no w such that $u <_\gamma w <_\gamma v$ or $v <_\gamma w <_\gamma u$. A *spine arc* in $\vec{H}(P)$ with respect to γ is an arc (u, v) in $\vec{H}(P)$ such that u and v are adjacent in γ. A *break* in $\vec{H}(P)$ with respect to γ is a pair (u, v) of adjacent elements such that $u <_\gamma v$ and (u, v) is not an arc in $\vec{H}(P)$. A *connection* C in $\vec{H}(P)$ with respect to γ is a maximal sequence of elements $u_1 <_\gamma u_2 <_\gamma \cdots <_\gamma u_k$ such that (u_i, u_{i+1}) is a spine arc for all $i, 1 \leq i < k$; in other words a connection is a maximal path of spine arcs without a break. Since $\vec{H}(P)$ contains no transitive arcs, there can be no nonspine arcs between nodes in a connection. The *breaknumber* $BN(\gamma, P)$ of a topological order γ of $\vec{H}(P)$

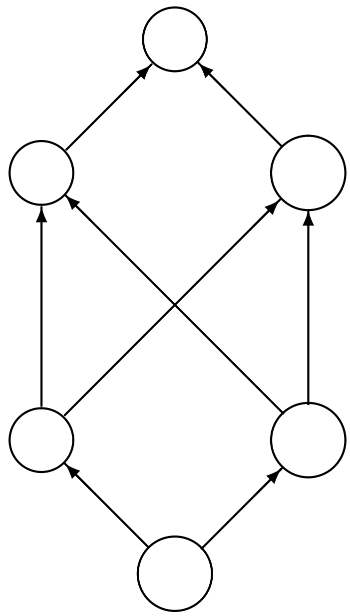

FIG. 2.3. *A nonplanar poset whose covering graph is planar.*

is the number of breaks in $\vec{H}(P)$ with respect to γ. The *jumpnumber of* P, denoted by $JN(P)$, is the minimum of $BN(\gamma, P)$ over all topological orders γ on $\vec{H}(P)$.

A *chain* in a poset P is a set of elements $\{u_1, u_2, \ldots, u_k\}$ such that $u_1 < u_2 < \cdots < u_k$. The *length* $L(P)$ of a poset P is the maximum cardinality of any chain in P. An *antichain* in a poset P is a subset of elements of S that does not contain a chain of size 2. The *width* $W(P)$ of a poset P is the maximum cardinality of any antichain in P.

3. Upper bounds on queuenumber.

In this section we derive upper bounds on the queuenumber of a poset in terms of its jumpnumber, its length, its width, and the queuenumber of its covering graph.

3.1. Jumpnumber and queuenumber.

Sysło [17] proves the following relationship between the jumpnumber and the stacknumber of posets.

PROPOSITION 3.1 (Sysło [17]). *For any poset P with $JN(P) = 1$, we have $SN(P) \leq 2$. If $\mathcal{J}_2$ is the infinite class of posets having jumpnumber 2, then $SN_{\mathcal{J}_2}(n) = \Omega(n)$.*

In contrast, we show that, for any poset P, the queuenumber of P is at most the jumpnumber of P plus 1. Moreover, we show that this bound is tight within a small constant factor.

THEOREM 3.2. *For any poset P, $QN(P) \leq JN(P) + 1$. For every $n \geq 2$, there exists a poset P such that $|P| = 2n$ and $JN(P)/2 < QN(P)$.*

Proof. For the upper bound on queuenumber, suppose that P is any poset and that $JN(P) = k$. Let γ be a topological order on $\vec{H}(P)$ that has exactly k breaks and $k + 1$ connections. Lay out $\vec{H}(P)$ according to γ and label these connections $C_0, C_1, \ldots, C_k$ from left to right. Let (u_1, v_1) and (u_2, v_2) be any two nonspine arcs such that u_1 and u_2 are in C_i and v_1 and v_2 are in C_j, where $1 \leq i < j \leq k$. If (u_1, v_1) and (u_2, v_2) nest, then one of (u_1, v_1) and (u_2, v_2) (the arc that nests over the other arc) is a transitive arc. Since $\vec{H}(P)$ contains no transitive arcs, (u_1, v_1) and (u_2, v_2) do not nest. This suggests the following assignment of arcs to queues: Assign all nonspine arcs between pairs of connections C_i and C_j, where $|i-j| = \ell, 1 \le \ell \le k$, to queue q_ℓ. Assign all the spine arcs to a queue q_0. Hence, we use k queues for nonspine arcs and one queue for spine arcs, for a total of $k+1$ queues.

For the lower bound on queuenumber, construct the Hasse diagram of a poset P from the complete bipartite graph $K_{n,n} = (V_1, V_2, E)$ by directing all the edges from vertices in V_1 to vertices in V_2. All topological orders on $\vec{H}(P)$ yield isomorphic layouts. Hence, $JN(P) = 2(n-1)$, $QN(P) = n$, and

$$QN(P) = \frac{n}{2(n-1)} JN(P).$$

The lower bound follows. □

Proposition 3.1 and Theorem 3.2 lead to the following corollary.

COROLLARY 3.3. *There exists a class of posets $\mathcal{P}$ for which the ratio*

$$\frac{SN_{\mathcal{P}}(n)}{QN_{\mathcal{P}}(n)} = \Omega(n).$$

Looking ahead, Theorem 4.2 shows the existence of a class of posets $\mathcal{P}$ for which the reciprocal ratio $QN_{\mathcal{P}}(n)/SN_{\mathcal{P}}(n)$ is unbounded.

3.2. Length and queuenumber. To prove the next theorem, we need the following lemma that gives a bound on the queuenumber of a layout of a graph whose vertices have been rearranged in a limited fashion.

LEMMA 3.4 (Pemmaraju [16]). *Suppose that σ is an order on the vertices of an m-partite graph $G = (V_1, V_2, \ldots, V_m, E)$ that yields a k-queue layout of G. Let σ' be an order on the vertices of G in which the vertices in each set V_i, $1 \le i \le m$, appear consecutively and in the same order as in σ. Then σ' yields a layout of G in $2(m-1)k$ queues.*

Theorem 3.5, the main result of this section, gives an upper bound on the queuenumber of a poset in terms of its length and the queuenumber of its covering graph.

THEOREM 3.5. *For any poset P,*

$$QN(P) \;\le\; 2 \cdot (L(P) - 1) \cdot QN(H(P)).$$

There exists an infinite class of posets $\mathcal{P}$ such that $L_{\mathcal{P}}(n) = 2$ and, for all $P \in \mathcal{P}$,

$$\left\lceil \frac{QN(P)}{2} \right\rceil = (L(P) - 1) \cdot QN(H(P)).$$

Proof. Suppose P is any poset, $\vec{H}(P) = (V, \vec{E})$, and $QN(H(P)) = k$. Let σ be a total order on V that yields a k-queue layout of $H(P)$. The nodes of $\vec{H}(P)$ can be labeled by a function $l : V \to \{1, \ldots, L(P)\}$ such that $l(u) < l(v)$ if $u < v$ in P, as follows. Let $\vec{H}_0 = \vec{H}(P)$. Label all the nodes with indegree 0 in $\vec{H}_0$ with the label 1. Delete all the labeled nodes in $\vec{H}_0$ to obtain $\vec{H}_1$. In general, label the nodes with indegree 0 in $\vec{H}_i$ with the label $i+1$. Delete the labeled nodes in $\vec{H}_i$ to obtain $\vec{H}_{i+1}$. By an inductive proof, it can be checked that the labeling so obtained satisfies the required conditions. Let $V_i = \{u \in V \mid l(u) = i\}$. For any arc $(u, v) \in \vec{E}$, if $u \in V_i$ and $v \in V_j$, then $i < j$. Therefore $\vec{H}(P) = (V_1, V_2, \ldots, V_{L(P)}, \vec{E})$ is an $L(P)$-partite dag. Define the total order γ on the nodes of $\vec{H}(P)$ by the following:

1. The elements in each set V_i, $1 \le i \le L(P)$, occur contiguously and in the order prescribed by σ.
2. The elements in V_i occur before the elements in V_{i+1} for all i, $1 \le i < L(P)$.

Since every arc in $\vec{H}(P)$ is from a node in V_i to a node in V_j, $1 \le i < j \le L(P)$, γ is a topological order on $\vec{H}(P)$. By Lemma 3.4 γ yields a layout that requires no more than $2 \cdot (L(P) - 1) \cdot k$ queues.

We now prove the second part of the theorem. For each $n \ge 2$, let $p = \lfloor n/2 \rfloor$ and $q = \lceil n/2 \rceil$. Let the complete bipartite graph $K_{p,q} = (V_1, V_2, E)$ be such that $|V_1| = p$ and $|V_2| = q$. We get the Hasse diagram of a poset P of size n by directing the edges in $K_{p,q}$ from V_1 to V_2. Clearly, $L(P) = 2$ and $QN(P) = p$. Heath and Rosenberg [13] and Pemmaraju [16] present different proofs of the following formula that gives the precise queuenumber of an arbitrary complete bipartite graph:

$$QN(K_{r,s}) = \min(\lceil r/2 \rceil, \lceil s/2 \rceil).$$

Since $p \le q$, $QN(K_{p,q}) = \lceil p/2 \rceil$. Therefore,

$$\left\lceil \frac{QN(P)}{2} \right\rceil = (L(P) - 1) \cdot QN(H(P)).$$

Let $\mathcal{P}$ be the class of all posets constructed in the manner described above. The second part of the theorem follows. ∎

Note that Theorem 3.5 holds for dags as well as for posets as its proof does not rely on the absence of transitive arcs. Theorem 3.5 leads to the following corollary.

COROLLARY 3.6. *For any poset P,*

$$QN(H(P)) \;\le\; QN(P) \;\le\; 2 \cdot (L(P) - 1) \cdot QN(H(P)).$$

Suppose $\mathcal{P}$ is a class of posets such that there exists a constant K with $L(P) \le K$, for all $P \in \mathcal{P}$. Then $QN_{\mathcal{P}}(n) = \Theta(QN_{H(\mathcal{P})}(n))$.

We conjecture, but have been unable to show, that the upper bound in Theorem 3.5 is tight, within constant factors, for larger values of $L(P)$ also.

3.3. Width and queuenumber. In this section, we establish an upper bound on the queuenumber of a poset in terms of its width. We need the following result of Dilworth.

LEMMA 3.7 (Dilworth [5]). *Let $P = (V, \le)$ be a poset. Then V can be partitioned into $W(P)$ chains.*

For a poset $P = (V, \le)$, let $Z_1, Z_2, \ldots, Z_{W(P)}$ be a partition of V into $W(P)$ chains. Define an *i-chain arc* as an arc in $\vec{H}(P)$, both of whose end points belong to chain Z_i, $1 \le i \le W(P)$. An (i,j)-*cross arc*, $i \ne j$, is an arc whose tail belongs to chain Z_i and whose head belongs to chain Z_j.

THEOREM 3.8. *The largest rainbow in any layout of a poset P is of size no greater than $W(P)^2$. Hence, the queuenumber of any layout of P is at most $W(P)^2$.*

Proof. Fix an arbitrary topological order of $\vec{H}(P)$. Let $Z_1, Z_2, \ldots, Z_{W(P)}$ be a partition of V into $W(P)$ chains. For any i, no two i-chain arcs nest, since $\vec{H}(P)$ contains no transitive arcs. Therefore, the largest rainbow of chain arcs has size no greater than $W(P)$. If $i \ne j$ then no two (i,j)-cross arcs can nest without one of them being a transitive arc. Therefore, the largest rainbow of cross arcs has size no greater than $W(P)(W(P) - 1)$. The size of the largest rainbow is at most $W(P) + W(P)(W(P) - 1) = W(P)^2$. By Proposition 2.1, the theorem follows. ∎

The bound established in the above theorem is not known to be tight. In fact, we believe that the queuenumber of a poset is bounded above by its width (see Conjecture 1 in Section 7).

4. The queuenumber of planar posets.

In this section, we first show that the queuenumber of the class of planar posets is unbounded. We then establish an upper bound on the queuenumber of a planar poset in terms of its width.

4.1. A lower bound on the queuenumber of planar posets.

We construct a sequence of planar posets P_n with $|P_n| = 3n + 3$ and $QN(P_n) = \Theta(\sqrt{n})$. In fact, we determine the queuenumber of P_n almost exactly. To prove the theorem, we need the following result of Erdös and Szekeres.

LEMMA 4.1 (Erdös and Szekeres [6]). *Let $(x_i)_{i=1}^n$ be a sequence of distinct elements from a set X. Let δ be a total order on X. Then $(x_i)_{i=1}^n$ either contains a monotonically increasing subsequence of size $\lceil \sqrt{n} \rceil$ or a monotonically decreasing subsequence of size $\lceil \sqrt{n} \rceil$ with respect to δ.*

The proof of Theorem 4.2 constructs the desired sequence of posets.

THEOREM 4.2. *For each $n \geq 1$, there exists a planar poset P_n with $3n + 3$ elements such that*

$$\left\lceil \sqrt{n+1} \right\rceil \leq QN(P_n) \leq \left\lceil \sqrt{n} \right\rceil + 1.$$

Proof. Suppose $n \geq 1$. Define three disjoint sets U, V, and W as follows:

$$U = \{u_i \mid 0 \leq i \leq n\},$$
$$V = \{v_i \mid 0 \leq i \leq n\},$$
$$W = \{w_i \mid 0 \leq i \leq n\}.$$

Let $S = U \cup V \cup W$. The planar poset $P_n = (S, \leq)$ is given by

$$u_i < u_{i-1},$$
$$v_{i-1} < v_i,$$

for $1 \leq i \leq n$, and

$$u_i \quad < \quad w_i \quad < \quad v_i,$$

for $0 \leq i \leq n$. Figure 4.1 shows the Hasse diagram of P_4. Let σ be an arbitrary order on the elements of S. The elements of $U \cup V \cup \{w_0\}$ appear in the order $u_n, u_{n-1}, \ldots, u_0, w_0, v_0, v_1, \ldots, v_n$ in σ, and all elements of W appear between u_n and v_n. Define a total order δ on the elements of W by $w_i <_\delta w_j$ if $i < j$. Suppose that

$$w_{i_1}, w_{i_2}, \ldots, w_{i_k}$$

is an increasing sequence of nodes in W with respect to δ. Since w_{i_1} appears after u_{i_1} in any topological order of $\vec{H}(P_n)$, the following sequence of nodes is a subsequence of σ:

$$u_{i_k}, u_{i_{k-1}}, \ldots, u_{i_1}, w_{i_1}, w_{i_2}, \ldots, w_{i_k}.$$

Therefore, the set $\{(u_{i_j}, w_{i_j}) \mid 1 \leq j \leq k\}$ is a k-rainbow in σ. Similarly, if

$$w_{i_1}, w_{i_2}, \ldots, w_{i_k}$$

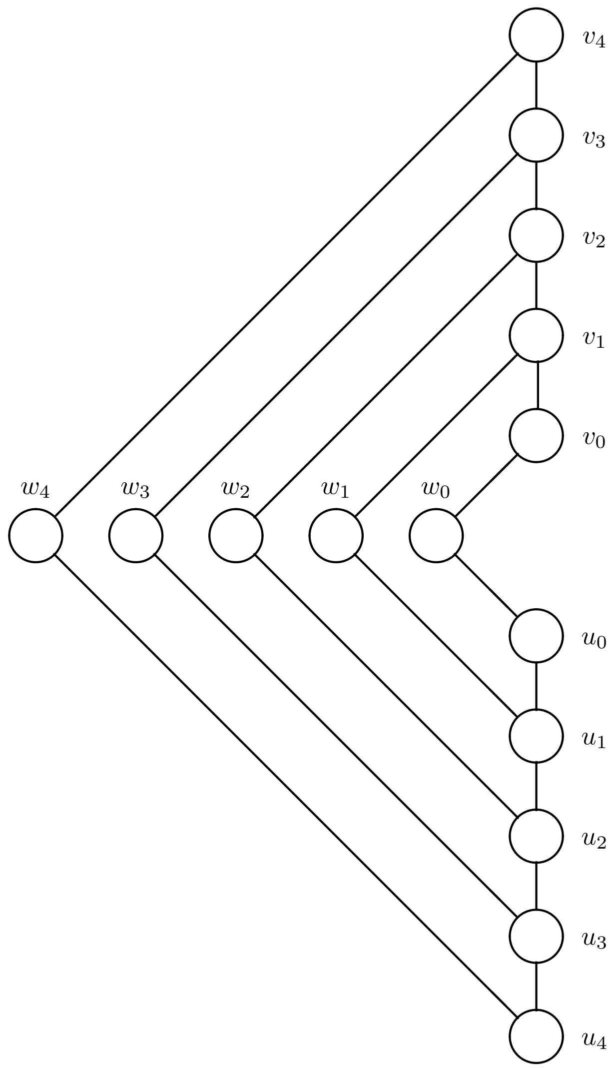

FIG. 4.1. *The planar poset P_4.*

is a decreasing sequence of nodes in W with respect to δ, then the set $\{(w_{i_j}, v_{i_j}) \mid 1 \leq j \leq k\}$ is a k-rainbow in σ. By Lemma 4.1, in σ, there is an increasing subsequence of size $\lceil\sqrt{n+1}\rceil$ or a decreasing subsequence of size $\lceil\sqrt{n+1}\rceil$ with respect to δ. Thus there is a rainbow of size $\lceil\sqrt{n+1}\rceil$ in any topological order on $\vec{H}(P_n)$. Therefore, $QN(P_n) \geq \lceil\sqrt{n+1}\rceil$. This is the desired lower bound.

To prove the upper bound, we give a layout of P_n in $\lceil\sqrt{n}\,\rceil + 1$ queues. Let $s = \lceil\sqrt{n}\,\rceil$, and let $t = \lceil n/s\rceil \leq \lceil\sqrt{n}\,\rceil$. Partition $W - \{w_0\}$ into s nearly equal-sized subsets

$$W_1, W_2, \ldots, W_s$$

as follows:

$$W_i = \begin{cases} \{w_j \mid (i-1)t + 1 \leq j \leq it\}, & 1 \leq i \leq s-1, \\ \{w_j \mid (s-1)t + 1 \leq j \leq n\}, & i = s. \end{cases}$$

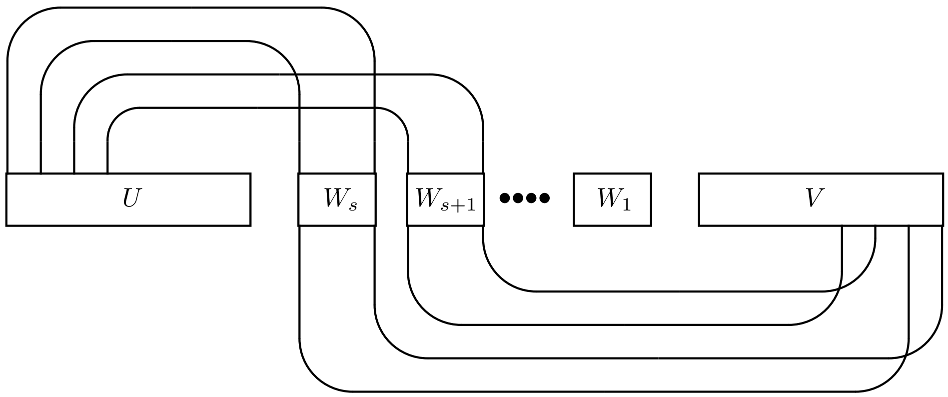

FIG. 4.2. *Schematic layout of planar poset P_n.*

Construct an order σ on the elements of S by first placing the elements in $U \cup V \cup \{w_0\}$ in the order

$$u_n, u_{n-1}, \ldots, u_0, w_0, v_0, v_1, \ldots, v_n.$$

Now place the elements of $W - \{w_0\}$ between u_0 and v_0 such that the elements belonging to each set W_i appear contiguously and the sets themselves appear in the order

$$W_s, W_{s-1}, \ldots, W_1.$$

Within each set W_i, $1 \le i \le s$, place the elements in increasing order with respect to δ. Figure 4.2 schematically represents the constructed order. The arcs from U to W form s mutually intersecting rainbows each of size at most t. Therefore, t queues suffice for these arcs. The arcs from W to V form s nested twists each of size at most t. Therefore s queues suffice for these arcs. Since no two arcs, one from U to W and the other from W to V nest, they can all be assigned to the same set of s queues. An additional queue is required for the remaining arcs. This is a layout of P_n in $\lceil \sqrt{n} \rceil + 1$ queues. Therefore, $QN(P_n) \le \lceil \sqrt{n} \rceil + 1$, as desired. □

We believe that the upper bound in the above proof can be tightened to exactly match the lower bound. In fact, we have been able to show that for $m^2 \le n \le m(m+1)$, $QN(P_n) = m + 1 = \lceil \sqrt{n+1} \rceil$.

The situation for stacknumber of planar posets is somewhat different in that there is no known example of a sequence of planar posets with unbounded stacknumber. Two observations about the sequence P_n in Theorem 4.2 are in order. The first observation is that $SN(P_n) = 2$. A 2-stack layout of $\vec{H}(P_4)$ is shown in Fig. 4.3. The second observation is that the stacknumber *and* the queuenumber of $H(P_n)$ is 2. A 2-queue layout of $H(P_4)$ is shown in Fig. 4.4. Theorem 4.2 and the above observations imply the following corollaries.

COROLLARY 4.3. *There exists a class $\mathcal{P}$ of planar posets such that*

$$\frac{QN_{\mathcal{P}}(n)}{SN_{\mathcal{P}}(n)} = \Omega\left(\sqrt{n}\right).$$

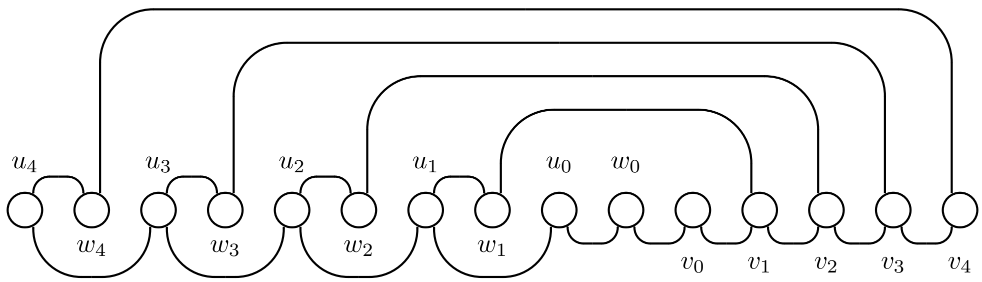

FIG. 4.3. *A 2-stack layout of the planar poset P_4.*

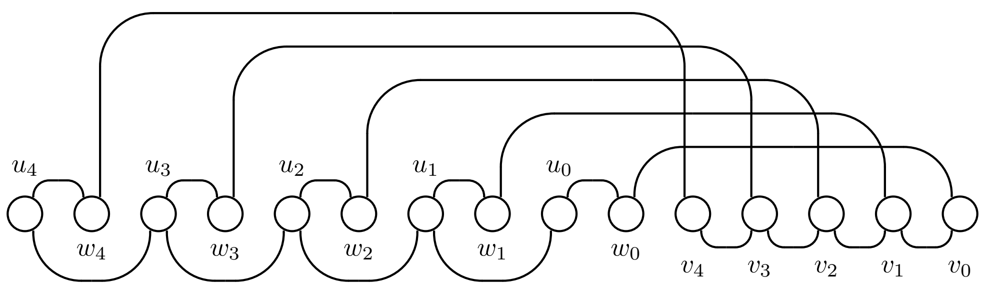

FIG. 4.4. *A 2-queue layout of the covering graph of P_4.*

COROLLARY 4.4. *There exists a class $\mathcal{P}$ of planar posets such that*

$$\frac{QN_{\mathcal{P}}(n)}{QN_{H(\mathcal{P})}(n)} = \Omega\left(\sqrt{n}\right).$$

While Theorem 4.2 establishes a lower bound of $\Omega(\sqrt{n})$ on the queuenumber of the class of n-element planar posets, a matching upper bound is not known (see Conjecture 2 in section 7).

4.2. An upper bound on the queuenumber of planar posets. In this section, we show that the queuenumber of a planar poset is bounded above by a small constant multiple of its width. The bound is a consequence of the following theorem, the proof of which occupies the remainder of the section.

THEOREM 4.5. *For any planar poset P where $\vec{H}(P)$ contains at least one arc and for any upward embedding of $\vec{H}(P)$, the layout of $\vec{H}(P)$ given by the vertical order σ has queuenumber less than $4W(P)$.*

Before the proof of Theorem 4.5, we present some definitions, some observations, and a series of three lemmas. First, we fix notation and terminology to use throughout the section. Suppose that $P = (V, \leq_P)$ is a planar poset with a given upwards embedding of $\vec{H}(P)$. Let σ be the vertical order on V. Now suppose that the size of a largest rainbow in the vertical order of $\vec{H}(P)$ is $k \geq 1$. By Proposition 2.1, the queuenumber of this layout is k. Focus on a particular k-rainbow whose arcs are $(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)$. Call these arcs the *rainbow arcs;* in particular, the arc (a_i, b_i) is the *rainbow arc of* a_i and *of* b_i. The nodes in the set $A = \{a_1, a_2, \ldots, a_k\}$ are *bottom nodes,* and the nodes in the set $B = \{b_1, b_2, \ldots, b_k\}$ are *top nodes.* Let $y(v)$ denote the y-coordinate of a node v in the upwards embedding. Suppose that (a_i, b_i) and (a_j, b_j) are distinct rainbow arcs. Since these arcs nest in the vertical order σ,

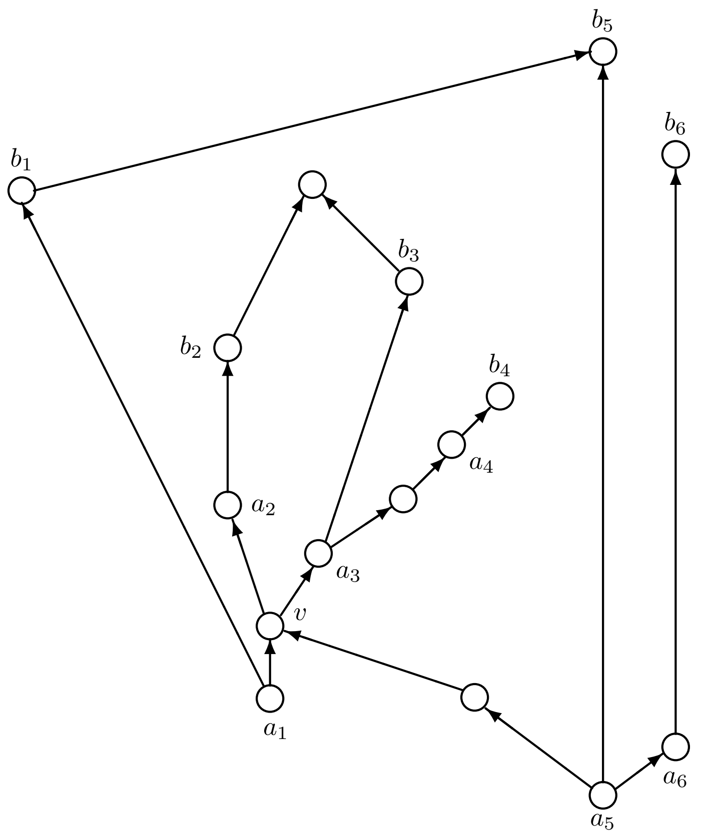

FIG. 4.5. *An example of rainbow arcs.*

we know that $\max\{y(a_i), y(a_j)\} < \min\{y(b_i), y(b_j)\}$. More generally,

$$y_1 = \max_{1 \le i \le k} y(a_i) < \min_{1 \le i \le k} y(b_i) = y_2.$$

The horizontal line defined by the equation $y = (y_1 + y_2)/2$ intersects every (a_i, b_i). In moving along this line from left to right, we encounter these intersections in a definite order. By re-indexing the rainbow arcs, we may assume that these intersections are encountered in the order $(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)$; call this the left-to-right order of the rainbow arcs. Figure 4.5 illustrates an upwards embedding of a Hasse diagram with $k = 6$. The arcs are indexed in left-to-right order.

Define the *left-to-right total order* $\le_{LR}$ on A (respectively, B) by $a_i \le_{LR} a_j$ (respectively, $b_i \le_{LR} b_j$) if $i \le j$. If $a_i \le_{LR} a_j$, we say that a_i is to the *left* of a_j and that a_j is to the *right* of a_i. These notions of left and right do not always correspond to our normal understanding of these notions when looking at an upwards embedding. For example, in Fig. 4.5, the x-coordinate of a_1 is greater than that of a_2, though $a_1 <_{LR} a_2$ and hence a_1 is to the left of a_2. We consistently use left and right with respect to the order $\le_{LR}$.

A *bottom chain* is any chain of bottom nodes, and a *top chain* is any chain of top nodes. In Fig. 4.5, the set $\{a_1, a_3, a_4\}$ is a bottom chain, while the set $\{a_2, a_3, a_5\}$ is not. If C is a chain of P and $u, v \in V$, then the *closed interval* from u to v is the subchain $C[u, v] = \{w \in C \mid u \le_P w \le_P v\}$, and the *open interval* from u to v is the subchain $C(u,v) = \{w \in C \mid u <_P w <_P v\}$. Subchains $C(u,v]$ and $C[u,v)$, the corresponding *half-open intervals*, are defined analogously. For any bottom chain C, the *extent* of C is

$$\langle C \rangle = \left(\max_{a_i \in C} i \right) - \left(\min_{a_j \in C} j \right);$$

that is, the extent is the distance from the leftmost node in C to the rightmost node in C, measured in rainbow arcs. The extent of a top chain is defined analogously. Suppose C is any chain. We say that C *covers* the nodes it contains. If D is a path in $\vec{H}(P)$ that contains every node of C, then D *covers* C. Note that there must be at least one path in $\vec{H}(P)$ that covers C.

In what follows, we show that more than $k/4$ chains are required to cover the set $A \cup B$. Since $W(P)$ is the minimum number of chains required to cover all the nodes in the poset, it follows that $k/4 < W(P)$ and therefore $QN(P) < 4W(P)$. As the proof is long and tedious, we give here an informal overview. Start with a partition $\mathcal{C}_A$ of A into bottom chains and a partition $\mathcal{C}_B$ of B into top chains. Because each element of $\mathcal{C}_A \cup \mathcal{C}_B$ is a chain, there is a path in $\vec{H}(P)$ covering it. Thinking of each such path as a vertex, we construct a graph G that contains an edge connecting a pair of vertices if the corresponding paths in $\vec{H}(P)$ are connected by a rainbow arc. It is easy to see that G is planar if the paths in $\vec{H}(P)$ covering the chains in $\mathcal{C}_\mathcal{A} \cup \mathcal{A}_\mathcal{B}$ are pairwise nonintersecting. The construction of a collection of pairwise nonintersecting paths that cover the chains of $\mathcal{C}_A \cup \mathcal{C}_B$ is not always possible. This leads us to the weaker notion of a crossing of two chains and to the construction of G from chains rather than paths. Since the final step of the proof requires G to be planar, we first show (Lemmas 4.7 and 4.8) that all crossings between pairs of chains can be eliminated. Applying Euler's formula to the resulting planar G finally yields the bound in Theorem 4.5.

At this point, we restrict our argument to bottom nodes, as the corresponding argument for top nodes is similar. If C is any bottom chain, the order in which its elements appear with respect to $\leq_P$ is constrained by the rainbow arcs. In particular, we make the following observation.

Observation 1. Suppose that C is a bottom chain whose nodes occur in the following order with respect to $\leq_P$:

$$c_1 \leq_P c_2 \leq_P \cdots \leq_P c_t.$$

For any i with $1 \leq i \leq t-1$, if $c_i <_{LR} c_{i+1}$, then $c_i <_{LR} c_j$ for all $j \geq i+1$. Similarly, for any i with $1 \leq i \leq t-1$, if $c_i >_{LR} c_{i+1}$, then $c_i >_{LR} c_j$ for all $j \geq i+1$.

Intuitively, if the chain starts going to the right after c_i, then the remainder of the chain must be to the right of the rainbow arc of c_i. The rainbow arc of c_i is a barrier to the chain reaching a bottom node to the left of c_i. For example, in Fig. 4.5, the rainbow arc (a_5, b_5) is a barrier to any path originating at a_6. Since $a_5 <_P a_6$ and $a_5 <_{LR} a_6$, no bottom chain containing both a_5 and a_6 has a node $a_i >_P a_6$ to the left of a_5.

By Lemma 3.7, there is a partition of A into at most $W(P)$ chains. Let $\mathcal{C}_A$ be such a partition. Let $C_1 \in \mathcal{C}_A$ have the order

$$c_1 \leq_P c_2 \leq_P \cdots \leq_P c_m,$$

and let $C_2 \in \mathcal{C}_A$, $C_1 \neq C_2$, have the order

$$d_1 \leq_P d_2 \leq_P \cdots \leq_P d_n.$$

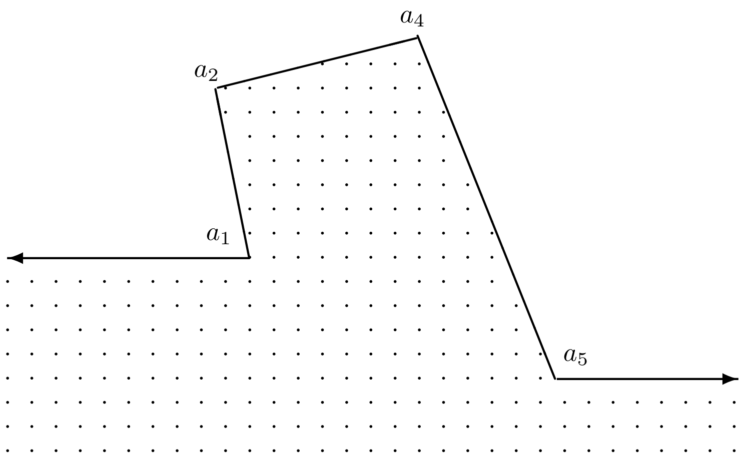

FIG. 4.6. *The region R.*

These two bottom chains *cross* if there exist $c_p, c_q \in C_1$ and $d_r, d_s \in C_2$ such that $c_p <_{LR} d_r <_{LR} c_q <_{LR} d_s$ or $c_p >_{LR} d_r >_{LR} c_q >_{LR} d_s$; in such a case, the 4-tuple (c_p, c_q, d_r, d_s) is a *crossing* of C_1 and C_2. Since c_p and c_q are related by $\leq_P$, there is a directed path D_1 in $\vec{H}(P)$ between c_p and c_q that covers $C_1[c_p, c_q]$. Similarly, there is a directed path D_2 in $\vec{H}(P)$ between d_r and d_s that covers $C_2[d_r, d_s]$.

LEMMA 4.6. *D_1 and D_2 have at least one node in common.*

Proof. Without loss of generality, assume that $c_p <_{LR} d_r <_{LR} c_q <_{LR} d_s$. Consider the polygonal path consisting of the horizontal ray from c_p to $-\infty$, followed by the line segments (c_p, d_r), (d_r, c_q), and (c_q, d_s), and completed by the horizontal ray from d_s to ∞. Let R be the region of the plane consisting of this polygonal path and all points below it. (Figure 4.6 illustrates the region R derived from Fig. 4.5 with crossing (a_1, a_4, a_2, a_5).) Topologically, R is a 2-dimensional ball with a single boundary point removed. Topologically, D_1 and D_2 are paths in the plane with endpoints on the boundary of R. By Observation 1, neither path can cross either of the two infinite rays. Also, neither path can pass above the rainbow arc of c_q or d_r, because every top node is higher than any bottom nodes in the upwards embedding of $\vec{H}(P)$. Hence, if either path crosses one of the three line segments of the polygonal path and proceeds outside of R, then that path must return to the polygonal path at a higher point on the *same line segment.* In essence, we can disregard any excursions outside of R and assume, from a topological viewpoint, that both paths remain within R. The nodes of D_1 and D_2 alternate along the polygonal path. Hence, these paths must intersect topologically, and D_1 and D_2 must have at least one node in common. ∎

A node that D_1 and D_2 have in common is an *intersection* of C_1 and C_2. Note that an intersection need not be a bottom node. In Fig. 4.5, the chains $\{a_1, a_3, a_4\}$ and $\{a_5, a_2\}$ cross and have the intersection v, which is not a bottom node.

Observation 2. Since, with respect to $\leq_P$, an intersection associated with the crossing (c_p, c_q, d_r, d_s) is between c_p and c_q and between d_r and d_s, we have these relations:

$$\min_P\{c_p, c_q\} <_P \max_P\{d_r, d_s\},$$
$$\min_P\{d_r, d_s\} <_P \max_P\{c_p, c_q\}.$$

The following observation is helpful in constructing pairs of noncrossing chains.

Observation 3. Suppose that $C_1 - \{c_i\}$ and C_2 do not cross. If no $d_j \in C_2$ is between c_{i-1} and c_i with respect to $\leq_{LR}$ or if no $d_j \in C_2$ is between c_i and c_{i+1} with respect to $\leq_{LR}$, then C_1 and C_2 do not cross.

We wish to be able to assume that $\mathcal{C}_A$ does not contain a pair of crossing chains. The first of two steps in justifying that assumption is to show that we can replace two crossing chains with two noncrossing chains according to the following lemma. The replacing pair is further constrained to satisfy the five properties in the lemma. The need for properties 1, 2, and 3 is clear. Property 4 states that, if the original pair crosses, then the replacing pair is smaller, in a precise technical sense, than the original pair; hence the process of replacement of a crossing pair by a noncrossing pair cannot be repeated forever. Property 5 allows us to identify the minima in the replacing pair; this property is a technical condition useful only within the inductive proof of the lemma.

LEMMA 4.7. *Suppose C_1 and C_2 are disjoint bottom chains. Then there exists a function $\mathcal{NC}$ that yields a pair of bottom chains $(C_1', C_2') = \mathcal{NC}(C_1, C_2)$ with these properties*:

1. $C_1' \cup C_2' = C_1 \cup C_2$;
2. C_1' *and* C_2' *are disjoint;*
3. C_1' *and* C_2' *do not cross;*
4. *the sum of extents does not increase:*

$$\langle C_1' \rangle + \langle C_2' \rangle \leq \langle C_1 \rangle + \langle C_2 \rangle;$$

if equality holds and if C_1 and C_2 cross, then the minimum extent decreases:

$$\min\{\langle C_1' \rangle, \langle C_2' \rangle\} < \min\{\langle C_1 \rangle, \langle C_2 \rangle\};$$

and

5. *chain minima are preserved:*

$$\begin{array}{ccccc} c_1 & = & \min_P C_1' & = & \min_P C_1, \\ d_1 & = & \min_P C_2' & = & \min_P C_2. \end{array}$$

Proof. In addition to our previous notation for C_1 and C_2, we define

$$\alpha = \min_{LR} C_1,$$
$$\beta = \max_{LR} C_1,$$
$$\gamma = \min_{LR} C_2,$$
$$\delta = \max_{LR} C_2.$$

By Observation 1, either $c_1 = \alpha$ or $c_1 = \beta$, and either $d_1 = \gamma$ or $d_1 = \delta$. If $c_1 = \alpha$, choose a path D_1 from c_1 to β that covers the subchain $C_1[c_1, \beta]$; if $c_1 = \beta$, choose a path D_1 from c_1 to α that covers the subchain $C_1[c_1, \alpha]$. Similarly, if $d_1 = \gamma$, choose a path D_2 from d_1 to δ that covers the subchain $C_2[d_1, \delta]$; if $d_1 = \delta$, choose a path D_2 from d_1 to γ that covers the subchain $C_2[d_1, \gamma]$. By Observation 1, both paths are monotonic with respect to $\leq_{LR}$.

We proceed to show the lemma by induction on the pair (m, n). Recall that m is the cardinality of C_1 and n is the cardinality of C_2. The base cases are all pairs (m, n) with either $m = 1$ or $n = 1$. In these cases, C_1 and C_2 do not cross, and setting $\mathcal{NC}(C_1, C_2) = (C_1, C_2)$ yields the desired pair of bottom chains.

For the inductive case, we assume that $m \geq 2$, that $n \geq 2$, and that the lemma holds for (m', n') whenever $m' < m$ and $n' \leq n$ or whenever $m' \leq m$ and $n' < n$. We show that the lemma then holds for C_1 and C_2. Without loss of generality, we assume $\alpha <_{LR} \gamma$. There are now three main cases depending on the relative order of α, β, γ, and δ with respect to $<_{LR}$.

Case 1. $\alpha <_{LR} \beta <_{LR} \gamma <_{LR} \delta$. In this case, C_1 and C_2 do not cross and the lemma trivially holds.

Case 2. $\alpha <_{LR} \gamma <_{LR} \beta <_{LR} \delta$. In this case, C_1 and C_2 necessarily cross. There are four subcases.

Case 2.1. $c_1 = \alpha$ *and* $d_1 = \gamma$. Paths D_1 and D_2 necessarily contain at least one intersection. Let v be the intersection that occurs first in going from α to β on D_1. The subpath D_1' of D_1 from c_1 to v does not meet the subpath D_2' of D_2 from d_1 to v until v. Hence, unless D_2' consists only of d_1 (that is, $d_1 = v$), one of D_1' and D_2' is above the other in the upwards embedding. D_1' cannot be above D_2', because the rainbow arc of d_1 is a barrier to D_1' going above d_1. Hence, either D_2' consists only of d_1 or D_2' is above D_1'. There are two subcases, depending on the relative order of c_2 and v according to P.

Case 2.1.1. $c_2 <_P v$. Since c_2 is on D_1' and the rainbow arc of c_2 must not be a barrier for D_2', we have $c_2 <_{LR} d_1$. Let $(C_1', C_2') = \mathcal{NC}(C_1 - \{c_2\}, C_2)$. Since $c_1 \in C_1'$, we set $\mathcal{NC}(C_1, C_2) = (C_1' \cup \{c_2\}, C_2')$. For this case only, we provide a full proof that the lemma holds for $\mathcal{NC}(C_1, C_2)$, leaving the details for the remaining cases to the reader. We employ the properties that hold for (C_1', C_2') by the inductive hypothesis. By property 5 of the inductive hypothesis, C_1' and C_2' are bottom chains with $c_1 \in C_1'$ and $d_1 \in C_2'$. We must show that $C_1' \cup \{c_2\}$ is a bottom chain. If $d_j \in C_2[v, d_n]$, we have $c_2 <_P d_j$. If $d_j \in C_2[d_1, v)$ and $c_1 <_P d_j$, then $c_2 <_P d_j$, since any path in $\vec{H}(P)$ between c_1 and d_j must cross D_1' between c_2 and v. In any case, for any $d_j \in C_2'$, if $c_1 <_P d_j$, then $c_2 <_P d_j$. Hence, $(C_1' \cup \{c_2\}, C_2')$ is a pair of bottom chains, as required. We now establish that $\mathcal{NC}(C_1, C_2)$ satisfies the 5 properties.

1. By property 1 of the inductive hypothesis, $C_1' \cup C_2' = (C_1 - \{c_2\}) \cup C_2$. Hence, $(C_1' \cup \{c_2\}) \cup C_2' = C_1 \cup C_2$.
2. By property 2 of the inductive hypothesis, C_1' and C_2' are disjoint. Since $c_2 \notin C_1' \cup C_2'$, $C_1' \cup \{c_2\}$ and C_2' are disjoint.
3. By property 3 of the inductive hypothesis, C_1' and C_2' do not cross. Since $c_2 <_{LR} d_1$, there is no node of C_2 between c_1 and c_2. Also, by Observation 1 there is no node in C_1 that is between c_1 and c_2. Therefore there is no node in C_2' between c_1 and c_2 and hence by Observation 3, $C_1' \cup c_2$ and C_2' do not cross. Since there is no node of C_2' between c_1 and c_2 with respect to $\leq_{LR}$, $C_1' \cup \{c_2\}$ and C_2' do not cross by Observation 3.
4. To be definite, let $\alpha = a_p$, $\beta = a_q$, $\gamma = a_r$, and $\delta = a_s$. Then, by property 4 of the induction hypothesis and the fact that $c_1 <_{LR} c_2 <_{LR} \beta$, we have

$$\begin{aligned} \langle C_1' \rangle + \langle C_2' \rangle &\leq \langle C_1 - \{c_2\} \rangle + \langle C_2 \rangle \\ &= (q - p) + (s - r) \\ &= \langle C_1 \rangle + \langle C_2 \rangle, \end{aligned}$$

and, if equality holds and if $C_1 - \{c_2\}$ and C_2 cross,

$$\min\{\langle C_1' \rangle, \langle C_2' \rangle\} < \min\{\langle C_1 - \{c_2\} \rangle, \langle C_2 \rangle\}.$$

If $\langle C_1' \rangle + \langle C_2' \rangle < \langle C_1 \rangle + \langle C_2 \rangle$, then we are done. So assume that $\langle C_1' \rangle + \langle C_2' \rangle = \langle C_1 \rangle + \langle C_2 \rangle$. Calculate $\langle C_1 \rangle = q - p$ and $\langle C_2 \rangle = s - r$. We obtain

$\langle C_1' \rangle + \langle C_2' \rangle = (q-p) + (s-r)$. If $\delta \in C_2'$, then $\langle C_2' \rangle = s - r = \langle C_2 \rangle$, $\langle C_1' \rangle = \langle C_1 \rangle = q - p$, and $a_q = \beta \in C_1'$, a contradiction to C_1' and C_2' not crossing. Hence, $\delta \in C_1'$, $\langle C_1' \rangle = s - p$, $\langle C_2' \rangle = q - r$, and $\langle C_1' \cup \{c_2\} \rangle = s - p$. Then we have

$$\begin{aligned}\min\{\langle C_1' \cup \{c_2\} \rangle, \langle C_2' \rangle\} &= \min\{s-p, q-r\} \\ &= q - r \\ &< \min\{q-p, s-r\} \\ &= \min\{\langle C_1 \rangle, \langle C_2 \rangle\}.\end{aligned}$$

Hence, property 4 holds for $(C_1' \cup \{c_2\}, C_2')$.

5. By property 5 of the inductive hypothesis, $c_1 = \min_P C_1' = \min_P C_1 - \{c_2\}$ and $d_1 = \min_P C_2' = \min_P C_2$. Since $c_1 <_P c_2$, we obtain $c_1 = \min_P C_1' \cup \{c_2\} = \min_P C_1$ and $d_1 = \min_P C_2' = \min_P C_2$, as required.

This completes the full proof for the case $c_2 <_P v$.

Case 2.1.2. $v \leq_P c_2$. Hence, $d_1 <_{LR} c_2$. For this case, let $c_1 = a_p$, $c_2 = a_q$, $\beta = a_x$, $d_1 = a_r$, $d_2 = a_s$, and $\delta = a_y$. We have $p < r < q \leq x < y$ and $r < s \leq y$. Consider the relative left-to-right positions of c_2 and d_2.

First suppose that $d_2 <_{LR} c_2$. Since $d_2 <_{LR} c_2 <_{LR} \delta$, no node in $C_2(d_2, d_n]$ is between d_1 and d_2. Since the subpath of D_2 from d_2 to δ must go below or through c_2, c_2 must be above d_2 in the vertical order. Hence, no node of C_1 is between d_1 and d_2. Let $(C_1', C_2') = \mathcal{NC}(C_1, C_2[d_2, d_n])$. By Observation 3, $(C_1', C_2' \cup \{d_1\})$ is a pair of noncrossing chains. Set $\mathcal{NC}(C_1, C_2) = (C_1', C_2' \cup \{d_1\})$. We need to show that $(C_1', C_2' \cup \{d_1\})$ satisfies property 4. By property 4 of the inductive hypothesis,

$$\langle C_1' \rangle + \langle C_2' \rangle \leq \langle C_1 \rangle + \langle C_2[d_2, d_n] \rangle.$$

Calculate $\langle C_1 \rangle = x - p$, $\langle C_2 \rangle = y - r$, $\langle C_2[d_2, d_n] \rangle = y - s$, and

$$\begin{aligned}\langle C_1' \rangle + \langle C_2' \cup \{d_1\} \rangle &= \langle C_1' \rangle + \langle C_2' \rangle + (s-r) \\ &\leq \langle C_1 \rangle + \langle C_2[d_2, d_n] \rangle + (s-r) \\ &= (x-p) + (y-s) + (s-r) \\ &= (x-p) + (y-r) \\ &= \langle C_1 \rangle + \langle C_2 \rangle.\end{aligned}$$

If $\langle C_1' \rangle + \langle C_2' \cup \{d_1\} \rangle < \langle C_1 \rangle + \langle C_2 \rangle$, then property 4 holds. So assume $\langle C_1' \rangle + \langle C_2' \cup \{d_1\} \rangle = \langle C_1 \rangle + \langle C_2 \rangle$. If $|C_1'| = 1$ (that is $C_1' = \{c_1\}$), then $1 = \min\{\langle C_1' \rangle, \langle C_2' \cup \{d_1\} \rangle\} < 2 \leq \min\{\langle C_1 \rangle, \langle C_2 \rangle\}$, and again property 4 holds. Otherwise, $|C_1'| \geq 2$. Since $d_2 \in C_2'$ and C_1' and C_2' do not cross, $\delta \in C_1'$. Hence, $\langle C_1' \rangle = y - p$, $\langle C_2' \rangle = x - s$, and $\langle C_2' \cup \{d_1\} \rangle = x - r$. We have

$$\begin{aligned}\min\{\langle C_1' \rangle, \langle C_2' \cup \{d_1\} \rangle\} &= x - r \\ &< \min\{x-p, y-r\} \\ &= \min\{\langle C_1 \rangle, \langle C_2 \rangle\}.\end{aligned}$$

Hence, property 4 holds.

Now suppose that $c_2 <_{LR} d_2$. There are finally three subcases to consider.

Case 2.1.2.1. *$d_2 <_{LR} \delta$ and $c_2 <_{LR} \beta$.* Let $(C_1', C_2') = \mathcal{NC}(\{c_1\} \cup C_2[d_2, d_n], C_1[c_2, c_m])$. There are no nodes of $C_1 \cup C_2$ between d_1 and c_2. So set

$\mathcal{NC}(C_1, C_2) = (C_1', C_2' \cup \{d_1\})$. By Observation 3, $\{d_1\} \cup C_2'$ is a chain that does not cross C_1'. By property 4 of the inductive hypothesis,

$$\langle C_1' \rangle + \langle C_2' \rangle \le \langle \{c_1\} \cup C_2[d_2, d_n] \rangle + \langle C_1[c_2, c_m] \rangle,$$

and, if equality holds, then either $\{c_1\} \cup C_2[d_2, d_n]$ and $C_1[c_2, c_m]$ do not cross, or

$$\min\{\langle C_1' \rangle, \langle C_2' \rangle\} < \min\{\langle \{c_1\} \cup C_2[d_2, d_n] \rangle, \langle C_1[c_2, c_m] \rangle\}.$$

We proceed to show that property 4 holds for C_1' and $\{d_1\} \cup C_2'$. Since $\langle \{d_1\} \cup C_2' \rangle = \langle C_2' \rangle + (q - r)$, we have

$$\begin{aligned} \langle C_1' \rangle + \langle \{d_1\} \cup C_2' \rangle &= \langle C_1' \rangle + \langle C_2' \rangle + (q - r) \\ &\le \langle \{c_1\} \cup C_2[d_2, d_n] \rangle + \langle C_1[c_2, c_m] \rangle + (q - r) \\ &= (\langle C_2 \rangle - (s - r) + (s - p)) + (\langle C_1 \rangle - (q - p)) + (q - r) \\ &= \langle C_1 \rangle + \langle C_2 \rangle, \end{aligned}$$

and hence

$$\langle C_1' \rangle + \langle \{d_1\} \cup C_2' \rangle \le \langle C_1 \rangle + \langle C_2 \rangle.$$

If this inequality is strict, then property 4 holds. If equality holds, then one of two possibilities holds. First suppose that $\{c_1\} \cup C_2[d_2, d_n]$ and $C_1[c_2, c_m]$ do not cross. In that case, we have $\beta <_{LR} d_2 <_{LR} \delta$ and

$$\begin{aligned} \min\{\langle C_1' \rangle, \{d_1\} \cup \langle C_2' \rangle\} &= \min\{y - p, x - r\} \\ &= x - r \\ &< \min\{x - p, y - r\} \\ &= \min\{\langle C_1 \rangle, \langle C_2 \rangle\}. \end{aligned}$$

Second suppose that

$$\begin{aligned} \min\{\langle C_1' \rangle, \langle C_2' \rangle\} &< \min\{\langle \{c_1\} \cup C_2[d_2, d_n] \rangle, \langle C_1[c_2, c_m] \rangle\} \\ &= x - q. \end{aligned}$$

Then

$$\begin{aligned} \min\{\langle C_1' \rangle, \langle \{d_1\} \cup C_2' \rangle\} &\le \min\{\langle C_1' \rangle, \langle C_2' \rangle\} + (q - r) \\ &< (x - q) + (q - r) \\ &= x - r \\ &< \min\{\langle C_1 \rangle, \langle C_2 \rangle\}. \end{aligned}$$

For both possibilities, property 4 holds. We conclude that $\mathcal{NC}(C_1, C_2) = (C_1', \{d_1\} \cup C_2')$ gives the desired pair of chains.

Case 2.1.2.2. *$d_2 <_{LR} \delta$ and $c_2 = \beta$.* Since $c_1 <_{LR} d_2$ and $d_1 <_{LR} c_2$, both $\{c_1\} \cup C_2[d_2, d_n]$ and $\{d_1\} \cup C_1[c_2, c_m]$ are chains, and they do not cross. Setting $\mathcal{NC}(C_1, C_2) = (\{c_1\} \cup C_2[d_2, d_n], \{d_1\} \cup C_1[c_2, c_m])$ gives the desired pair of chains. Since

$$\begin{aligned} \langle \{c_1\} \cup C_2[d_2, d_n] \rangle + \langle \{d_1\} \cup C_1[c_2, c_m] \rangle &= (y - p) + (q - r) \\ &= (x - p) + (y - r) \\ &= \langle C_1 \rangle + \langle C_2 \rangle, \end{aligned}$$

and

$$\begin{aligned}\min\{\langle\{c_1\}\cup C_2[d_2,d_n]\rangle,\langle\{d_1\}\cup C_1[c_2,c_m]\rangle\} &= \min\{y-p,q-r\}\\ &= q-r\\ &< \min\{q-p,y-r\}\\ &= \min\{\langle C_1\rangle,\langle C_2\rangle\},\end{aligned}$$

property 4 holds.

Case 2.1.2.3. $d_2=\delta$. Let $(C_1',C_2')=\mathcal{NC}(C_2[d_2,d_n],\{d_1\}\cup C_1[c_2,c_m])$. Since c_1 is leftmost and d_2 rightmost in $C_1\cup C_2$, the pair $(C_1'\cup\{c_1\},C_2')$ is also noncrossing. Let $a_z=\min_{LR} C_2[d_2,d_n]$. We have $r<z\le y$ and $\langle C_2[d_2,d_n]\rangle=y-z$. By property 4 of the inductive hypothesis, we have

$$\begin{aligned}\langle C_1'\rangle+\langle C_2'\rangle &\le \langle C_2[d_2,d_n]\rangle+\langle\{d_1\}\cup C_1[c_2,c_m]\rangle\\ &= (y-z)+(x-r).\end{aligned}$$

We proceed to show property 4 for $(C_1'\cup\{c_1\},C_2')$. First,

$$\begin{aligned}\langle C_1'\cup\{c_1\}\rangle+\langle C_2'\rangle &= \langle C_1'\rangle+\langle C_2'\rangle+(z-p)\\ &\le \langle C_2[d_2,d_n]\rangle+\langle\{d_1\}\cup C_1[c_2,c_m]\rangle+(z-p)\\ &= (y-z)+(x-r)+(z-p)\\ &= (x-p)+(y-r)\\ &= \langle C_1\rangle+\langle C_2\rangle.\end{aligned}$$

If this inequality is strict, then we are done. Otherwise, $\langle C_1'\cup\{c_1\}\rangle+\langle C_2'\rangle=(x-p)+(y-r)$ and $\langle C_2'\rangle=x-r$. We have

$$\begin{aligned}\min\{\langle C_1'\cup\{c_1\}\rangle,\langle C_2'\rangle\} &= \min\{y-p,x-r\}\\ &= x-r\\ &< \min\{y-r,x-p\}\\ &= \min\{\langle C_1\rangle,\langle C_2\rangle\}.\end{aligned}$$

Hence, Property 4 holds for $(C_1'\cup\{c_1\},C_2')$.

Case 2.2. $c_1=\alpha$ *and* $d_1=\delta$. In this case, C_1 and C_2 always cross. If we succeed in replacing these with two noncrossing chains C_1' and C_2' having the same nodes, then $\max_{LR} C_1'<\min_{LR} C_2'$. Hence, property 4 follows easily for every $(C_1',C_2')=\mathcal{NC}(C_1,C_2)$ constructed for this case.

Again, let v be the first intersection of D_1 and D_2. If $v\in A$, then all of $C_1(v,c_m]$ is to the right of v, and all of $C_2(v,d_n]$ is to the left of v. If $v\notin C_1\cup C_2$, then setting $\mathcal{NC}(C_1,C_2)=(C_1[c_1,v)\cup C_2(v,d_n],C_2[d_1,v)\cup C_1(v,c_m])$ gives the desired pair of chains. If $v\in C_1\cup C_2$, then setting $\mathcal{NC}(C_1,C_2)=(C_1[c_1,v]\cup C_2(v,d_n],C_2[d_1,v)\cup C_1(v,c_m])$ gives the desired pair of chains. In either case, C_1' and C_2' do not cross.

If $v\notin A$, then the argument is a bit more involved. Otherwise, if $c_2<_{LR}\gamma$, then let $(C_1',C_2')=\mathcal{NC}(C_1-\{c_2\},C_2)$. Setting $\mathcal{NC}(C_1,C_2)=(C_1'\cup\{c_2\},C_2')$ gives the desired pair of chains. If $\beta<_{LR} d_2$, then let $(C_1',C_2')=\mathcal{NC}(C_1,C_2-\{d_2\})$. Setting $\mathcal{NC}(C_1,C_2)=(C_1',C_2'\cup\{d_2\})$ gives the desired pair of chains. Hence, suppose $\gamma<_{LR} c_2$ and $d_2<_{LR}\beta$. Since the rainbow arcs of c_2 and d_2 are barriers, we have $v\le_P c_2$, $d_2\le_P v$, and $d_2<_{LR} c_2$. By Observation 1, there are four possibilities.

Case 2.2.1. *$C_1(c_2, c_m]$ is to the left of c_2 and $C_2(d_2, d_n]$ is to the left of d_2.* If $C_1(c_2, c_m]$ remains to the right of d_2, then set $\mathcal{NC}(C_1, C_2) = (\{c_1\} \cup C_2[d_2, d_n], \{d_1\} \cup C_1[c_2, c_m])$. Otherwise, (c_2, c_m, d_1, d_2) is a crossing, and, by Observation 2, $c_2 <_P d_j$, for all $j \geq 2$. Hence $C_2 \cup \{c_2\}$ is a chain, and there are no nodes of $C_1 \cup C_2$ between c_2 and d_1. Let $(C_1', C_2') = \mathcal{NC}(C_1 - \{c_2\}, C_2)$. Setting $\mathcal{NC}(C_1, C_2) = (C_1', C_2' \cup \{c_2\})$ gives the desired pair of chains.

Case 2.2.2. *$C_1(c_2, c_m]$ is to the left of c_2 and $C_2(d_2, d_n]$ is to the right of d_2.* Let $(C_1', C_2') = \mathcal{NC}(C_1[c_2, c_m], C_2[d_2, d_n])$. Setting $\mathcal{NC}(C_1, C_2) = (\{c_1\} \cup C_2', \{d_1\} \cup C_1')$ gives the desired pair of chains.

Case 2.2.3. *$C_1(c_2, c_m]$ is to the right of c_2 and $C_2(d_2, d_n]$ is to the left of d_2.* Here $C_1[c_2, c_m]$ and $C_2[d_2, d_n]$ do not cross. Setting $\mathcal{NC}(C_1, C_2) = (\{c_1\} \cup C_2[d_2, d_n], \{d_1\} \cup C_1[c_2, c_m])$ gives the desired pair of chains.

Case 2.2.4. *$C_1(c_2, c_m]$ is to the right of c_2 and $C_2(d_2, d_n]$ is to the right of d_2.* This is the left-to-right mirror image of 2.2.1. The same argument applies, *mutatis mutandis.*

Case 2.3. *$c_1 = \beta$ and $d_1 = \gamma$.* This case cannot occur because the rainbow arcs of c_1 and d_1 are barriers to the paths D_1 and D_2. It would require both D_1 to go below d_1 and D_2 to go below c_1, which is impossible.

Case 2.4. *$c_1 = \beta$ and $d_1 = \delta$.* This case is the left-to-right mirror image of Case 2.1. The same argument applies, *mutatis mutandis.*

Case 3. *$\alpha <_{LR} \gamma <_{LR} \delta <_{LR} \beta$.* In this case, C_1 and C_2 may cross. There are again four subcases.

Case 3.1. *$c_1 = \alpha$ and $d_1 = \gamma$.* First suppose $c_2 <_{LR} d_1$. Let $(C_1', C_2') = \mathcal{NC}(C_1 - \{c_1\}, C_2)$. The desired pair of chains is $\mathcal{NC}(C_1, C_2) = (C_1' \cup \{c_1\}, C_2')$. Suppose that $d_1 <_{LR} c_2 <_{LR} \delta$. Then D_1 and D_2 necessarily have an intersection before c_2 and before δ. This is handled as in Case 2.1. Suppose that $\delta <_{LR} c_2$ and $c_2 \neq \beta$. Then $C_1(c_2, c_m]$ is to the right of c_2, C_2 is between c_1 and c_2, and C_1 and C_2 do not cross. Finally, suppose $\delta <_{LR} c_2$ and $c_2 = \beta$. Let $(C_1', C_2') = \mathcal{NC}(C_1 - \{c_1\}, C_2)$. Since all of $C_1(c_2, c_m] \cup C_2$ is between c_1 and c_2 with respect to $\leq_{LR}$, and since $c_2 \in C_1'$, it follows that $C_1' \cup \{c_1\}$ and C_2' do not cross. The desired pair of chains is $\mathcal{NC}(C_1, C_2) = (C_1' \cup \{c_1\}, C_2')$. It is necessary to justify property 4. Let $\tau = \min_{2 \leq i \leq m} c_i$, where min is taken with respect to $\leq_{LR}$. There are three subcases.

Case 3.1.1. *$\tau <_{LR} \gamma$.* Note that all of $C_2(d_1, d_n]$ is to the right of $d_1 = \gamma$. If τ and γ are unrelated with respect to $\leq_P$ or if $\tau <_P \gamma$, then $\tau \notin C_2'$, since $\gamma = \min_P C_2'$. If $\gamma <_P \tau$, then τ, being to the left of γ, is unrelated to every node in $C_2[d_2, d_n]$; again $\tau \notin C_2'$. Since $\tau \in C_1'$, we have $\langle C_1' \rangle = \langle C_1[c_2, c_m] \rangle$. Applying property 4, we must have $\langle C_2' \rangle < \langle C_2 \rangle$ if $C_1[c_2, c_m]$ and C_2 cross. It follows that $\langle C_1' \cup \{c_1\} \rangle + \langle C_2' \rangle < \langle C_1 \rangle + \langle C_2 \rangle$ if C_1 and C_2 cross.

Case 3.1.2. *$\gamma <_{LR} \tau <_{LR} \delta$.* For $C_2[c_2, c_m]$ and C_1, this case is the same as Case 2.2. For all the possibilities in that case, we get that $\langle C_2' \rangle < \langle C_2 \rangle$. Hence,

$$\begin{aligned}\langle C_1' \cup \{c_1\} \rangle + \langle C_2' \rangle &= (\beta - \alpha) + \langle C_2' \rangle \\ &< (\beta - \alpha) + \langle C_2 \rangle \\ &= \langle C_1 \rangle + \langle C_2 \rangle,\end{aligned}$$

as desired.

Case 3.1.3. *$\delta <_{LR} \tau$.* In this case, $C_1' = C_1[c_2, c_m]$ and $C_2' = C_2$ do no cross. Hence, neither do $C_1' \cup \{c_1\}$ and C_2'.

Case 3.2. *$c_1 = \alpha$ and $d_1 = \delta$.* In this case, C_1 and C_2 do not cross, as the rainbow arc of d_1 is a barrier to D_1 crossing D_2.

Case 3.3. *$c_1 = \beta$ and $d_1 = \gamma$.* This case is the left-to-right mirror image of Case 3.2.

Case 3.4. *$c_1 = \beta$ and $d_1 = \delta$.* This case is the left-to-right mirror image of Case 3.1. ∎

The second and last step in justifying the assumption converts any $\mathcal{C}_A$ into a $\mathcal{C}'_A$ that has no pair of crossing chains.

LEMMA 4.8. *Suppose $\mathcal{C}_A$ is a set of disjoint bottom chains of minimum cardinality that covers A. Then there exists a set $\mathcal{C}'_A$ of disjoint bottom chains that covers A such that $|\mathcal{C}'_A| = |\mathcal{C}_A|$ and no pair of chains in $\mathcal{C}'_A$ cross.*

Proof. If $\mathcal{C}_A$ contains no pair of crossing chains, then $\mathcal{C}'_A = \mathcal{C}_A$ is the set required for the lemma.

Otherwise, let $C_1, C_2 \in \mathcal{C}_A$ be a pair of chains that cross. By Lemma 4.7, there exist chains C'_1 and C'_2 such that by substituting these chains for C_1 and C_2, we get the set $\mathcal{C}''_A = \mathcal{C}_A \cup \{C'_1, C'_2\} - \{C_1, C_2\}$, which is also a set of bottom chains of minimum cardinality that covers A. By property 4, either

(i) the sum of the extents of chains in $\mathcal{C}''_A$ is strictly less than the sum of the extents of chains in $\mathcal{C}_A$ or,

(ii) $\min\{\langle C'_1\rangle, \langle C'_2\rangle\} < \min\{\langle C_1\rangle, \langle C_2\rangle\}$.

Since every chain has extent at least 0, repeated substitution of a pair of crossing chains by a pair of noncrossing chains must eventually reduce the sum of the extents of the chains. Again, since every chain has extent at least 0, the sum of the extents of the chains cannot reduce infinitely, and hence we must eventually arrive at a set $\mathcal{C}'_A$ that contains no pair of noncrossing chains. This set $\mathcal{C}'_A$ is the set required for the lemma. ∎

We are finally prepared to prove our main result.

Proof of Theorem 4.5. By Lemma 4.8, we may assume that $\mathcal{C}_A$ contains no pair of crossing chains. Now let $\mathcal{C}_B$ be a partition of B into at most $W(P)$ chains. Similarly, we may assume that $\mathcal{C}_B$ contains no pair of crossing chains.

Consider an arbitrary bottom chain C and an arbitrary top chain C'. It is possible that a rainbow arc connects a node in C to a node in C'. However, it is not possible for more than one rainbow arc to connect C and C', for then one of the rainbow arcs (the "longest" one) would be a transitive arc in $\vec{H}(P)$. For example, in Fig. 4.5, we cannot have a bottom chain $C = \{a_1, a_2\}$ and a top chain $C' = \{b_1, b_2\}$, for then there is a path from b_2 to b_1 and (a_1, b_1) is a transitive arc.

We now construct a bipartite graph $G = (\mathcal{C}_A, \mathcal{C}_B, E)$, where E contains an edge between $C \in \mathcal{C}_A$ and $C' \in \mathcal{C}_B$ if there is a rainbow arc connecting C to C'. Since every rainbow arc connects exactly one bottom chain to exactly one top chain, there is exactly one edge in G for every rainbow arc; that is, $|E| = k$. Since there is no pair of crossing bottom chains and no pair of crossing top chains, G is planar. As an example, Fig. 4.7 illustrates a graph $G = (\mathcal{C}_A, \mathcal{C}_B, E)$ obtained from the poset of Fig. 4.5. In particular,

$$\mathcal{C}_A = \Big\{\{a_1, a_2\}, \{a_3, a_4\}, \{a_5, a_6\}\Big\}$$

and

$$\mathcal{C}_B = \Big\{\{b_1, b_5\}, \{b_2\}, \{b_3\}, \{b_4\}, \{b_6\}\Big\}.$$

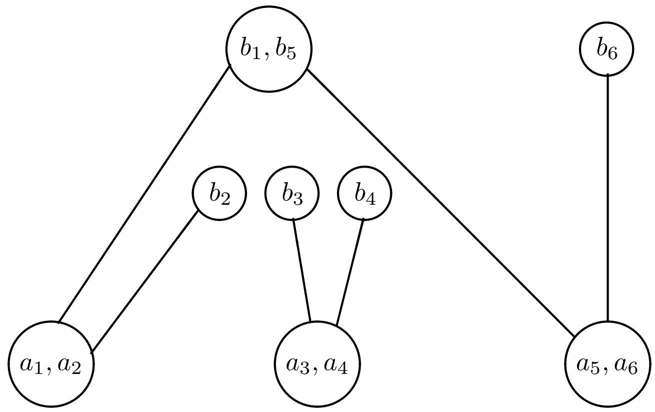

FIG. 4.7. *A bipartite planar graph $G = (\mathcal{C}_A, \mathcal{C}_B, E)$ corresponding to the poset in Fig.* 4.5.

According to Euler's formula for planar graphs, we have

$$|\mathcal{C}_A| + |\mathcal{C}_B| - |E| + f = 1 + t, \tag{4.1}$$

where f is the number of faces in a planar embedding of G and t is its number of connected components. If G consists of a single edge, then $k = 1 \leq W(P)$ and $k < 4W(P)$, as desired. Otherwise, since G is bipartite, we have the following inequality:

$$\begin{aligned} 4f &\leq 2|E|, \\ f &\leq \frac{|E|}{2}. \end{aligned} \tag{4.2}$$

Combining equations 4.1 and 4.2, we obtain

$$\begin{aligned} |\mathcal{C}_A| + |\mathcal{C}_B| - |E| + \frac{|E|}{2} &\geq 1 + t, \\ |\mathcal{C}_A| + |\mathcal{C}_B| &\geq 1 + t + \frac{|E|}{2}, \\ 2 + \frac{|E|}{2} &\leq |\mathcal{C}_A| + |\mathcal{C}_B|. \end{aligned} \tag{4.3}$$

We know that $|E| = k$ and that both $|\mathcal{C}_A|$ and $|\mathcal{C}_B|$ are at most $W(P)$. Substituting into equation 4.3, we obtain

$$k + 4 \leq 4W(P).$$

Hence, the queuenumber of $\vec{H}(P)$ with respect to σ is less than $4W(P)$. ∎

COROLLARY 4.9. *For any planar poset P where $\vec{H}(P)$ contains at least one arc, $QN(P) < 4W(P)$.*

We believe that this result can be improved to show that, for *any* poset P, there exists a $W(P)$-queue layout of $\vec{H}(P)$; see Conjecture 1 in section 7.

5. Stacknumber of posets with planar covering graphs. In this section we construct, for each $n \geq 1$, a $3n$-element poset R_n such that $H(R_n)$ is planar and hence has stacknumber at most 4 (see Yannakakis [21]), but such that the stacknumber of the class $\mathcal{R} = \{R_n \mid n \geq 1\}$ is not bounded.

THEOREM 5.1. *For each $n \geq 1$, there exists a poset R_n such that $|R_n| = 3n$, $H(R_n)$ is planar, and*

$$\left\lceil \frac{n}{2} \right\rceil \;\leq\; SN(R_n) \;\leq\; n.$$

Proof. Suppose $n \geq 1$. Define three disjoint sets U, V, and W as follows:

$$U = \{u_i \mid 1 \leq i \leq n\},$$
$$V = \{v_i \mid 1 \leq i \leq n\},$$
$$W = \{w_i \mid 1 \leq i \leq n\}.$$

The poset $R_n = (U \cup V \cup W, \leq)$ is given by

$$u_i < u_{i+1},$$
$$v_i < v_{i+1},$$
$$w_i < w_{i+1},$$

for $1 \leq i \leq n-1$,

$$u_i \quad < \quad w_i \quad < \quad v_i,$$

for $1 \leq i \leq n$, and

$$u_n < v_1.$$

Figure 5.1 shows $H(R_4)$.

Aside. While the covering graph $H(R_n)$ is clearly planar, the poset R_n is not planar. This can be seen as follows. In any upward embedding of $\vec{H}(R_n)$ in the plane, the nodes

$$u_1, u_2, \ldots, u_n, v_1, v_2$$

have increasing y-coordinates. Thus, any point in the plane whose y-coordinate is between the y-coordinates of u_1 and v_2 lies either on the left or on the right of the path

$$D = u_1, u_2, \ldots, u_{n-1}, u_n, v_1, v_2.$$

Now add the nodes w_1 and w_2 to the embedding. Their y-coordinates are between the y-coordinates of u_1 and v_2 because of $u_1 < w_1 < v_1 < v_2$ and $u_1 < u_2 < w_2 < v_2$. If both w_1 and w_2 are embedded on the same side of D, then the paths u_1, w_1, v_1 and u_2, w_2, v_2 must cross somewhere. If w_1 and w_2 are embedded on different sides of D, then the line segment (w_1, w_2) crosses a line segment in D. *End Aside.*

To prove the lower bound on $SN(R_n)$, let σ be any topological order on $\vec{H}(R_n)$. The order σ contains the elements of $U \cup V$ in the order $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n$, and the elements of W in the order $w_1, w_2, \ldots, w_n$. The elements of W are mingled among the elements of $U \cup V$. Suppose $w_1, w_2, \ldots, w_k$ occur before u_n in σ, while $w_{k+1}, w_{k+2}, \ldots, w_n$ occur after u_n. Then the arcs

$$(w_1, v_1), (w_2, v_2), \ldots, (w_k, v_k)$$

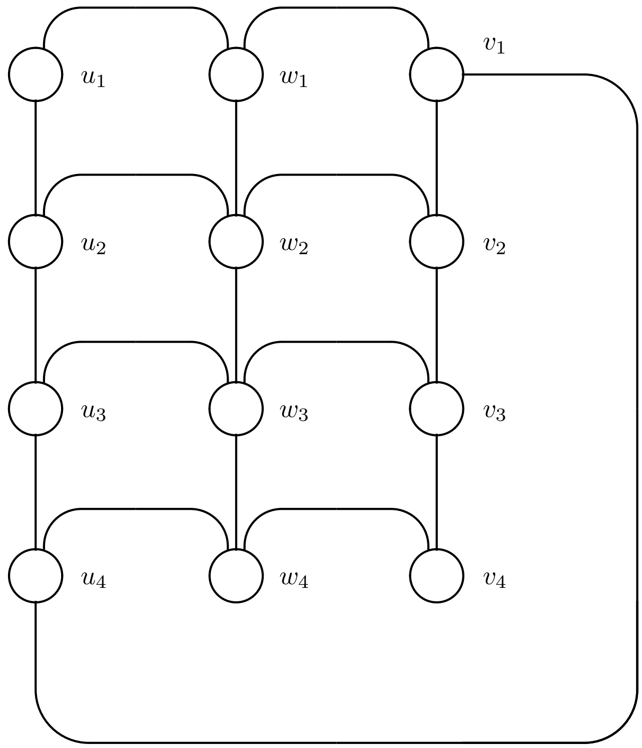

FIG. 5.1. *The covering graph of* R_4.

form a k-twist, while the arcs

$$(u_{k+1}, w_{k+1}), (u_{k+2}, w_{k+2}), \dots, (u_n, w_n)$$

form an $(n-k)$-twist. Hence,

$$SN(R_n) \geq \max(k, n-k) \geq \lceil n/2 \rceil .$$

Therefore, $SN(R_n) \geq \lceil n/2 \rceil$, as desired.

The proof of the upper bound is constructive. An n-stack layout of R_n is obtained by laying out the elements of $U \cup V$ in the only possible order, and then placing each w_i immediately after u_i for all i, $1 \leq i \leq n$. The assignment of arcs to stacks is as follows. Assign each arc in the set $\{(u_i, w_i), (w_i, v_i), (w_i, w_{i+1})\}$ to stack s_i for all i, $1 \leq i \leq n-1$ and assign each arc in the set $\{(u_n, w_n), (w_n, v_n)\}$ to stack s_n. Note that no two arcs assigned to the same stack intersect. The only arcs remaining to be assigned are the arcs in the set

$$\{(u_i, u_{i+1}) \mid 1 \leq i \leq n-1\} \cup \{(v_i, v_{i+1}) \mid 1 \leq i \leq n-1\} \cup \{(u_n, v_1)\}.$$

The arcs (v_i, v_{i+1}) for i, $1 \leq i \leq n-1$, do not intersect any other arc and can be assigned to any stack. Each arc (u_i, u_{i+1}), $1 \leq i \leq n-1$, is assigned to stack s_{i+1} and arc (u_n, v_1) is assigned to stack s_1. An n-stack layout of R_n is obtained. The upper bound follows. ∎

Two observations about the poset R_n constructed in the above proof are in order. The first observation is that $QN(R_n) = 2$. A 2-queue layout of R_4 is shown in Fig. 5.2.

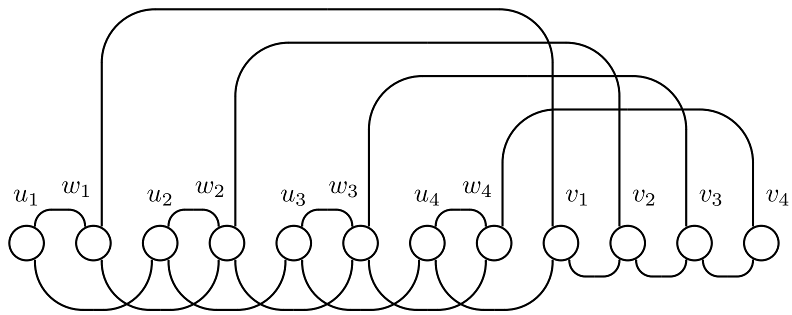

FIG. 5.2. *A 2-queue layout of R_4.*

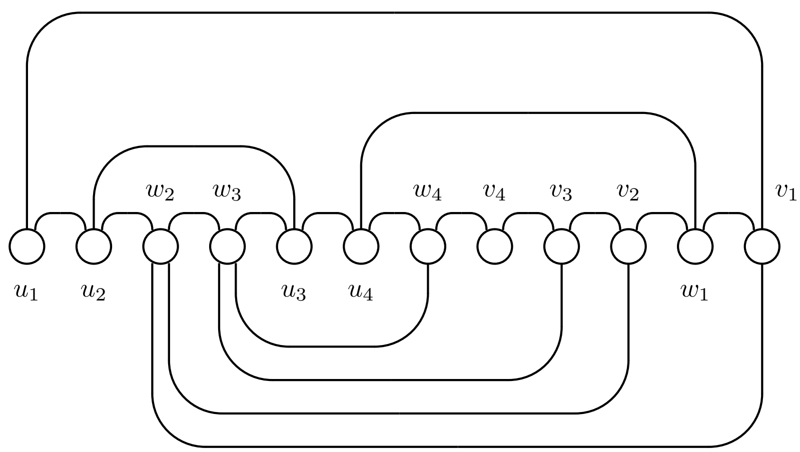

FIG. 5.3. *A 2-stack layout of the covering graph of R_4.*

In general, the total order used in the n-stack layout of R_n described in the above proof yields a 2-queue layout of R_n. The second observation is that the stacknumber and the queuenumber of the covering graph $H(R_n)$ is 2. A 2-stack layout of $H(R_4)$ is shown in Fig. 5.3. In general, a 2-stack layout of $H(R_n)$ can be obtained because $H(R_n)$ is a hamiltonian planar graph [1].

Theorem 5.1 and the above observations lead to the following corollaries.

COROLLARY 5.2. *There exists a class $\mathcal{R} = \{R_n \mid n \geq 1\}$ of posets such that $|R_n| = 3n$, $H(R_n)$ is planar, and*

$$\frac{SN_{\mathcal{R}}(n)}{QN_{\mathcal{R}}(n)} = \Omega(n).$$

COROLLARY 5.3. *There exists a class $\mathcal{R} = \{R_n \mid n \geq 1\}$ of posets R_n such that $|R_n| = 3n$, $H(R_n)$ is planar and*

$$\frac{SN_{\mathcal{R}}(n)}{SN_{H(\mathcal{R})}(n)} = \Omega(n).$$

6. NP-completeness results. Heath and Rosenberg [13] show that the problem of recognizing a 1-queue graph is NP-complete. Since a 1-stack graph is an outerplanar graph, it can be recognized in linear time (Sysło and Iri [18]). But Wigderson

[19] shows that the problem of recognizing a 2-stack graph is NP-complete. Heath and Pemmaraju [9] and Heath, Pemmaraju, and Trenk [11] show that the problem of recognizing a 4-queue poset is NP-complete. They also show that the problem of recognizing a 6-stack dag is NP-complete. We have not been able to extend this NP-completeness result for stack layouts of dags to an analogous result for posets.

Formally, the decision problem for queue layouts of posets is POSETQN.

POSETQN
INSTANCE: A poset P.
QUESTION: Does P have a 4-queue layout?

THEOREM 6.1 ([1, 9]). *The decision problem POSETQN is NP-complete.*

Since the Hasse diagram of a poset is a dag, this result hold for dags in general. This result is in the spirit of the result of Yannakakis [20] that recognizing a 3-dimensional poset is NP-complete.

7. Conclusions and open questions. In this paper, we have initiated the study of queue layouts of posets and have proved a lower bound result for stack layouts of posets with planar covering graph. The upper bounds on the queuenumber of a poset in terms of its jumpnumber, its length, its width, and the queuenumber of its covering graph, proved in section 3, may be useful in proving specific upper bounds on the queuenumber of various classes of posets. We believe that the upper bound of $W(P)^2$ on the queuenumber of an arbitrary poset P, proved in section 3, and the upper bound of $4W(P)$ on the queuenumber of any planar poset P, proved in section 4 are not tight. We have the following conjecture.

CONJECTURE 1. *For any poset P, $QN(P) \leq W(P)$.*

We have established a lower bound of $\Omega(\sqrt{n})$ on the queuenumber of the class of planar posets. We believe that this bound is tight and come to the following conjecture.

CONJECTURE 2. *For any n-element planar poset P, $QN(P) = O(\sqrt{n})$.*

We conjecture that another upper bound on the queuenumber of a planar poset P is given by its length $L(P)$. We believe that it is possible to embed a planar poset in an "almost" leveled-planar fashion with $L(P)$ levels. (See Heath and Rosenberg [13] for a definition of leveled-planar embeddings.) From such an embedding, a queue layout of P in $L(P)$ queues should be obtainable. Therefore we have the following conjecture.

CONJECTURE 3. *For any planar poset P, $QN(P) \leq L(P)$.*

In section 5, we show that the stacknumber of the class of n-element posets having planar covering graphs is $\Theta(n)$. However the stacknumber of the more restrictive class of planar posets is still unresolved.

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