SIAM J. DISCRETE MATH.
Vol. 19, No. 3, pp. 744–761

THE PMU PLACEMENT PROBLEM*

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Abstract. The PMU placement problem is an optimization problem abstracted from an approach to supervising an electrical power system. The power system is modeled as a graph, and adequate supervision of the system requires that the voltage at each node and the current through each edge be observable. A phasor measurement unit (PMU) is a monitor that can be placed at a node to directly observe the voltage at that node, as well as the current and its phase through all incident edges. The PMU placement problem is to place PMUs at a minimum number of nodes so that the entire electric power system is observed. A new simpler definition of graph observability and several complexity results for the PMU placement problem are presented. The PMU placement problem is shown to be NP-complete even for planar bipartite graphs. Several fundamental properties of PMU placements are proven, including the property that a minimum PMU placement requires no more than 1/3 of the nodes in a connected graph of at least 3 nodes.

Key words. phasor measurement unit, power system graph observability, domination, electric power monitoring, NP-completeness

AMS subject classifications. 05C69, 05C70, 05C85, 68R10, 94C15

DOI. 10.1137/S0895480103432556

1. Power systems and PMUs.

An electrical power system includes a set of *buses* and a set of *transmission lines* connecting the buses. A *bus* is a substation where lines are joined. A power system also includes a set of *generators,* which supply power, and a set of *loads,* into which the power is directed. To securely control a power system, its state must be monitored [8, 14, 19]. The state of a power system is expressed in terms of state variables, such as voltage at a load and phase angle at a generator. Typically, measurement devices are placed at selected points in the power system to monitor values of the state variables, which are fed back to the central control. The central control adjusts the power system to compensate for imbalances and to prevent hazardous (e.g., fault) situations [23]. For proper control, it is essential that all state variables be communicated to the central control in real time.

A phasor measurement unit (PMU) is a measurement device placed on a bus to monitor voltage at the bus and current phase along outgoing lines [5, 11, 12, 13, 25]. The ability to measure the current phasors as well as the voltage gives the PMU an advantage over other measurement units, enabling the deployment of fewer PMUs than is required in other types of measurement systems, some of which require one measurement unit per bus. PMUs track transients in the power system at high sampling rates, allowing automated real-time monitoring and control [21]. It is important to place the PMUs on buses so as to minimize their number while maintaining system observability, as PMUs are expensive [1, 18].

Stability problems of real-time control using PMUs have been studied before, including neural network approaches to control [16, 17]. Synchronization of the control unit and the PMUs may be done by satellite, using the global positioning system [3, 4, 20, 24], and communications of measurements can be implemented via the

*Received by the editors July 31, 2003; accepted for publication (in revised form) March 1, 2005; published electronically December 7, 2005.
http://www.siam.org/journals/sidma/19-3/43255.html
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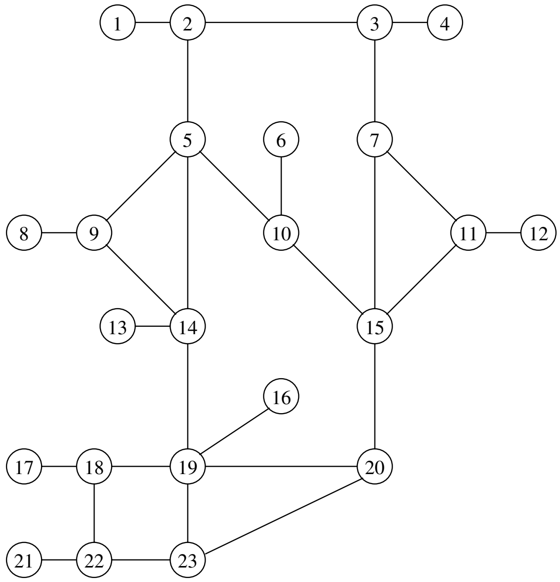

FIG. 1. *Sample power system graph.*

Internet [22]. The problem of optimal placement of PMUs has been studied before. El-Shal and Thorp [6] give an algorithm to optimally place two PMUs to minimize their notion of measurement error. Palmer and Ledwich [20] propose an optimization algorithm based on measurement sensitivity. Baldwin, Mili, Boisen, and Adapa [1] first formulate the PMU placement problem as a problem of minimizing cost and investigate heuristics for the problem.

Brueni [2] recasts the PMU placement problem in a more formal graph-theoretic setting. Haynes, Hedetniemi, Hedetniemi, and Henning [9] also study the problem in a graph-theoretic setting, using the notion of a power dominating set in a graph. Specifically, a power system is modeled as an undirected graph $G = (V, E)$, where V is the set of buses, generators, and loads, and where an edge $(u, v) \in E$ exists if there is a transmission line connecting u and v. For convenience of discussion, such a graph G is called a *power system graph (PSG)*. A *PMU placement* Π is a subset of V on which PMUs are placed. System observability is defined as a function of a PSG G and a PMU placement Π that returns the subgraph of G that is observed by Π (see section 2 for the precise definition of observability). A *PMU cover* Π of G is a placement that observes all of G. A *minimum PMU cover* is a PMU cover Π whose size $|\Pi|$ is minimum. Given a PSG G, the PMU placement problem is to find a minimum PMU cover for G. A more formal definition of the problem, together with an example, is given in section 3. Without loss of generality, we assume henceforth that a PSG is a connected graph with at least two nodes.

We make a few observations about a typical PSG, which are illustrated by the sample PSG in Figure 1. A PSG is planar or nearly so; it is uncommon for power lines to intersect, except, of course, at a bus. A PSG has large induced subgraphs that are trees, due to the fact that power distribution is most economical using only a tree; cycles in power systems provide redundancy. A PSG has many degree one nodes—generators and loads. The maximum degree of a PSG is low, because it is impractical to connect a bus to a large number of lines.

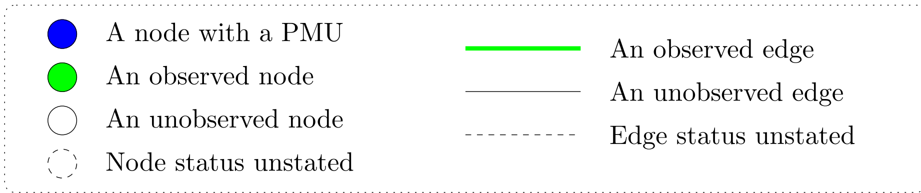

FIG. 2. *Graphical notation for PMU observability.*

In this paper, we address observability and PMU placement as graph-theoretic and algorithmic problems. In section 2, we first take the definition of observability from the power system literature [1] and give an equivalent, much simplified, graph-theoretic definition. Employing the simplified definition of observability, we show how to compute observability in linear time. In section 3, we formally define the minimum PMU placement problem and explore its graph-theoretic properties. In particular, we show that a PSG of at least 3 nodes requires a PMU cover that occupies no more than 1/3 of its nodes. Finally, in section 4, we prove that the PMU placement problem is NP-complete even for planar bipartite graphs.

2. Observability.

2.1. Definitions of observability. In this section, we provide two definitions of PSG observability and prove the two definitions equivalent. We require some notation and terminology. Fix a PSG $G = (V, E)$. Let $V' \subseteq V$. The *node induced subgraph* $<V'>$ of an undirected graph $G = (V, E)$ is

$$<V'>= (V', \{(u, v) \mid (u, v) \in E \text{ and } u, v \in V'\}),$$

where $V' \subseteq V$. For any node v, its *(open) neighborhood* is $\Gamma_G(v) = \Gamma(v) = \{u \in V \mid (u, v) \in E\}$. Its *closed neighborhood* is $\Gamma_G[v] = \Gamma[v] = \Gamma(v) \cup \{v\}$. A *placement* $\Pi \subseteq V$ is a set of the buses on which PMUs are placed. A bus or a line is *observed* if its state variables are monitored. A PMU *cover* Π is a placement where the entire graph is observed. Figure 2 summarizes the graphical notation used for observability in the remainder of the paper.

Baldwin, Mili, Boisen, and Adapa [1] develop the rules in the following definition of the nodes and edges observed. The rules follow from elementary laws of electrical networks.

DEFINITION 1 (Observability). *Let Π be a placement of PMUs on the nodes of $G = (V, E)$. These rules determine the set of observed nodes Π^R and the set of observed edges Π^-.*

R1. *By definition: A bus with a PMU and any line extending from the bus is observed. Formally, if $v \in \Pi$ and $u \in \Gamma(v)$, then $v \in \Pi^R$ and $(v, u) \in \Pi^-$.*

R2. *Ohm's law, $P = IR$: Any bus that is incident to an observed line connected to an observed bus is observed (the known current in the line, the known voltage at the observed bus, and the known resistance of the line determines the voltage at the bus). Formally, if $(u, v) \in \Pi^-$ and $u \in \Pi^R$, then $v \in \Pi^R$.*

R3. *Ohm's law, $I = P/R$: Any line joining two observed buses is observed (the known voltage at both observed buses and the known resistance of the line determines the current on the line). Formally, if $u, v \in \Pi^R$ and $(u, v) \in E$, then $(u, v) \in \Pi^-$.*

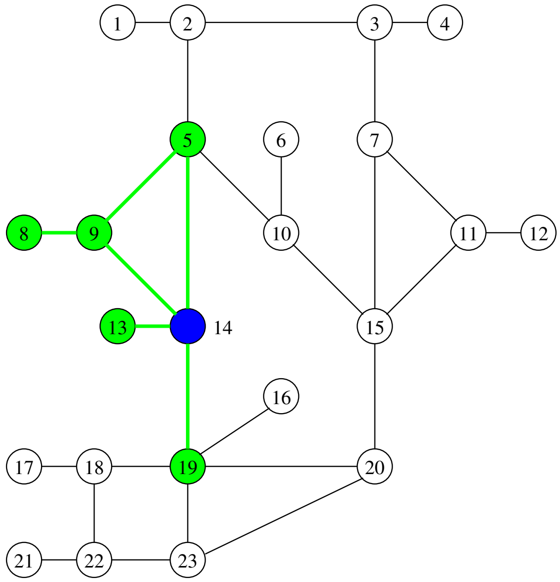

FIG. 3. *Example of definition of observability.*

R4. *Kirchoff current: If all the lines incident to an observed bus are observed, save one, then all of the lines incident to that bus are observed (the net current flowing through a bus is zero). Formally, if* $v \in \Pi^R$ *and* $|\Gamma(v) \cap (V - \Pi^R)| \leq 1$, *then* $\Gamma[v] \subseteq \Pi^R$.

R5. *Derived: Any bus incident only to observed lines is observed. Formally, if, for all* $u \in \Gamma(v)$, $(v, u) \in \Pi^-$, *then* $v \in \Pi^R$.

*Proof. An observed line must be connected to at least one observed bus (*R1 *and* R3*). If all lines incident to a bus are observed, the bus must either be observed itself or each bus adjacent to it is observed. Hence, by* R2, *the bus is observed.* $\square$

This definition does not take into account any inductance or capacitance in the system, which will have effects on the dynamic behavior of the system.

To illustrate the definition, consider the graph of Figure 1 and the placement $\Pi = \{14\}$. Since $14 \in \Pi$, by rule R1, we have

$$14 \in \Pi^R$$
$$(5, 14), (9, 14), (13, 14), (19, 14) \in \Pi^-.$$

By R3, we have $(5, 9) \in \Pi^-$, as $5, 9 \in \Pi^R$. By R4, we have $(8, 9) \in \Pi^-$, as 2 of the 3 lines incident to bus 9 are known to be observed. Finally, we have $8 \in \Pi^R$ by R2; see Figure 3 for the annotated result.

We now provide a simplified definition of observability (originally in Brueni [2]) that requires only 2 rules. Our definition of observability is restricted to observing nodes (buses).

DEFINITION 2 (Simplified Observability). *Let* Π *be a placement of PMUs on the nodes of* $G = (V, E)$. *The two rules below determine the set of observed nodes* $\Pi^S \subseteq V$.

S1. *If a node* v *has a PMU, then all nodes in* $\Gamma[v]$ *are observed. Formally, if* $v \in \Pi$, *then* $\Gamma[v] \subseteq \Pi^S$.

S2. *If a node v is observed and all nodes in $\Gamma(v)$ are observed, save one, then all nodes in $\Gamma[v]$ are observed. Formally, if $v \in \Pi^S$ and $|\Gamma(v) \cap (V - \Pi^S)| \leq 1$, then $\Gamma[v] \subseteq \Pi^S$.*

We now demonstrate that Definitions 1 and 2 are equivalent.

THEOREM 1. *Let $G = (V, E)$ be a PSG, and let $\Pi \subseteq V$ be a placement. Then $\Pi^R = \Pi^S$.*

Proof. We first show that $\Pi^S \subset \Pi^R$. The set Π^S can be obtained one node at a time by a sequence of applications of S1 and S2. For purposes of induction, choose a sequence of steps—applications of S1 and S2—that yields Π^S. The base case of the induction is zero steps, in which case the set of nodes obtained is $\emptyset \subset \Pi^R$. Now assume that $v \in \Pi^S$ is obtained in step $k > 0$ and that all nodes obtained at earlier steps are in Π^R. If step k is an S1 step, then either $v \in \Pi$ and $v \in \Pi^R$ by R1 or there is a node $u \in \Gamma(v) \cap \Pi$, in which case $(u, v) \in \Pi^-$ by R1 and $v \in \Pi^R$ by R2. If step k is an S2 step, then there exists an observed node u such that $v \in \Gamma(u)$ and every node $w \in (V - \Pi^S)$ is obtained in an earlier step and hence is in Π^R. By R4, $v \in \Pi^R$. By induction, we conclude that $\Pi^S \subset \Pi^R$.

We now show that $\Pi^R \subset \Pi^S$. The set Π^R can be obtained one node at a time by a sequence of applications of R1–R4 (R5 is derived and an application of R5 can be rewritten using applications of R1–R4). For purposes of induction, choose a sequence of steps—applications of R1–R4—that yields Π^R. The base case of the induction is zero steps, in which case the set of nodes obtained is $\emptyset \subset \Pi^S$. Now assume that $v \in \Pi^R$ is obtained in step $k > 0$ and that all nodes obtained at earlier steps are in Π^S. If step k is an R1 step, then $v \in \Pi$ and $v \in \Pi^S$ by S1. If step k is an R2 step, then there exists an observed node u such that $v \in \Gamma(u)$, $(u, v) \in \Pi^-$, and $u \in \Pi^R$. If $u \in \Pi$, then $v \in \Pi^S$ by S1. Otherwise, $(u, v) \in \Pi^-$ because of an R3 step, at which point $u, v \in \Pi^R$ and hence $v \in \Pi^S$. Rule R3 only observes edges, so an R3 step does not place any node in Π^R. If step k is an R4 step, then there exists an observed node u such that $v \in \Gamma(u)$ and every node $w \in (V - \Pi^R)$ is obtained in an earlier step and hence is in Π^S. By S2, $v \in \Pi^S$. By induction, we conclude that $\Pi^R \subset \Pi^S$.

The theorem follows. $\square$

By eliminating the concern for observing edges, this definition simplifies proofs and algorithms. All results in this paper are presented using Definition 2.

2.2. Observability computation in linear time. The computation of Π^S for a PSG $G = (V, E)$ can be accomplished in time linear in $|V| + |E|$; see Algorithm OBSERVE in Figure 4. (The algorithm of Haynes, Hedetniemi, Hedetniemi, and Henning [9] that implements Definition 1 is not obviously linear time.) For each node $v \in V$, the variable *observedneighbors*$[v]$ maintains the number of nodes in $\Gamma(v)$ that are currently known to be observed. The *degree* of v is DEGREE$(v) = |\Gamma(v)|$.

THEOREM 2. *For $G = (V, E)$ and $\Pi \subseteq V$, Algorithm* OBSERVE *computes Π^S in $O(|V| + |E|)$ time.*

Proof. An examination of Algorithm OBSERVE shows that it implements rules S1 and S2 of Definition 2. The **for** loop for rule S1 marks all the nodes in Π and all their neighbors observed. To implement rule S2, every node u that is observed and whose observed neighbor count reaches the S2 threshhold of DEGREE$(u) - 1$ is placed in the queue Q. In the rule S1 **for** loop, neighbors of nodes in Π that reach the S2 threshhold are enqueued. (There is no need to enqueue a node whose observed neighbor count equals its degree.) The **while** loop implements the propagation of observation of rule S2. Each dequeued node v was enqueued at a time when it was already marked observed and had observed neighbor count DEGREE$(v) - 1$. At the time it is dequeued,

```
Observe(G, Π)
Q ← ∅
for each v ∈ V
    do observed[v] ← false
for each v ∈ V
    do observedneighbors[v] ← 0
for each v ∈ Π
    do ▷ Rule S1—observe all elements of Π and their neighbors
        for each u ∈ Γ[v]
            do if not observed[u]
                    then  observed[u] ← true
                          for each w ∈ Γ(u)
                              do observedneighbors[w] ← observedneighbors[w] + 1
        for each u ∈ Γ(v)
                do ▷ Enqueue neighbors of Π that reach the S2 threshold
                    if observedneighbors[u] = degree(u) − 1
                        then  Enqueue(Q, u)
while Q ≠ ∅
    do ▷ v is observed and has at most one unobserved neighbor
        v ← Dequeue(Q, w)
        if observedneighbors[v] = degree(v) − 1
            then  u ← unobserved neighbor of v
                  observed[u] ← true
                  for each w ∈ Γ(u)
                      do observedneighbors[w] ← observedneighbors[w] + 1
                          if observed(w)
                              if observedneighbors[w] = degree(w) − 1
                                  then  Enqueue(Q, w)
                  if observedneighbors[u] = degree(u) − 1
                      then  Enqueue(Q, u)
return {v ∈ V | observed[v]}
```

Fig. 4. *Algorithm Observe to compute the observability function.*

the observed neighbor count of v may have increased to DEGREE(v). Otherwise, v has a unique neighbor u that is not marked observed. It is u that becomes observed as a consequence of rule S2. In both places in OBSERVE where a node u is marked observed, the count of observed neighbors of u is incremented, so that the *observedneighbors* values are correctly maintained. Moreover, in the **while** loop, whenever an observed node reaches the S2 threshhold, it is enqueued. We conclude that Algorithm OBSERVE correctly computes Π^S.

Every node is marked observed at most once and is enqueued at most once. The tests for the S2 threshhold are executed at most $|E|$ times and require at most $(|E|)$ work. The remaining work is done at most once for each node and is hence $O(|V|)$. We conclude that the time complexity of OBSERVE is $O(|V| + |E|)$. $\square$

3. Properties of PMU placement. In this section, we explore graph-theoretic properties of PMU placement.

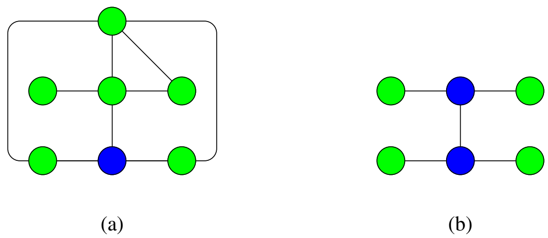

FIG. 5. *Minimum covers for* (a) *a graph* G *and* (b) *an induced subgraph of* G.

3.1. The PMU placement problem. The PMU placement problem (PMUP) of finding a minimum cover is stated formally here.

Problem: PMU Placement (Optimization Version)

INSTANCE: Graph $G = (V, E)$.

QUESTION: Find a cover $\Pi \subseteq V$ such that for any cover $\Pi' \subseteq V$, $|\Pi| \leq |\Pi'|$.

Such a placement Π is called a *minimum PMU cover*. The reader may verify, with some effort, that $\Pi = \{3, 10, 14, 19, 22\}$ is a minimum PMU cover for the PSG of Figure 1.

Haynes, Hedetniemi, Hedetniemi, and Henning [9] call the same problem the power domination problem (PDS) and explore the analogy between PDS and the traditional domination set problem. Though both problems involve some kind of observation of part of a graph, there is the significant difference that observation in dominating sets has bounded locality, while observation in PMUP can propagate more globally. For example, a single PMU suffices to observe a path or cycle PSG. Given an undirected graph G, a dominating set for G is also a PMU cover for G, although it is a poor one in many cases. The converse is, of course, seldom true.

3.2. Induced subgraphs. One might expect that an induced subgraph of a PSG G would always have a minimum PMU cover no larger than the size of a minimum PMU cover of G. However, this expectation is incorrect, as illustrated by the graph G in Figure 5. The single PMU in Figure 5(a) directly observes three nodes, two of which are of degree two. These degree two nodes then allow the node at the top to be observed, after which the observability of the remaining two nodes follows. While the graph in Figure 5(b) is induced by all but one of the nodes of G, it is clearly impossible to observe all of this subgraph of G without two PMUs.

3.3. Placement substitution. The following theorem shows that certain placement sets may replace others.

THEOREM 3 (Substitution). *Given a PSG* $G = (V, E)$ *and two placements* $\Pi_1, \Pi_2 \subseteq V$, *if* ${\Pi_1}^S \subseteq {\Pi_2}^S$, *then for any placement* Π, $(\Pi \cup \Pi_1)^S \subseteq (\Pi \cup \Pi_2)^S$.

Proof. For purposes of induction, choose a sequence of steps—applications of S1 and S2—that yields $(\Pi \cup \Pi_1)^S$. The base case of the induction is zero steps, in which case the set of nodes obtained is $\emptyset \subset (\Pi \cup \Pi_2)^S$. Now assume that $v \in (\Pi \cup \Pi_1)^S$ is obtained in step $k > 0$ and that all nodes obtained at earlier steps are in $(\Pi \cup \Pi_2)^S$. If step k is an S1 step, then there exists $u \in \Pi \cup \Pi_1$ such that $v \in \Gamma[u]$. If $u \in \Pi$, then $v \in (\Pi \cup \Pi_2)^S$ by S1. If $u \in \Pi_1$, then $v \in (\Pi \cup \Pi_2)^S$ since ${\Pi_1}^S \subseteq {\Pi_2}^S$. If step k is an S2 step, then there exists an observed node u such that $v \in \Gamma(u)$ and every node $w \in (V - (\Pi \cup \Pi_1)^S)$ is obtained in an earlier step and hence is in $(\Pi \cup \Pi_2)^S$. By S2, $v \in (\Pi \cup \Pi_2)^S$. By induction, we conclude that $(\Pi \cup \Pi_2)^S \subset (\Pi \cup \Pi_2)^S$. $\square$

If $|\Pi_2| < |\Pi_1|$ with ${\Pi_1}^S \subseteq {\Pi_2}^S$, then substituting Π_2 for Π_1 in a PMU cover results in a smaller cover, without loss of system observability.

The following corollary to Theorem 3 shows that it is counterproductive to place a PMU on a degree one node (unless, of course, $|V| = 2$).

COROLLARY 1. *Given a PSG $G = (V, E)$ with a cover $\Pi \subseteq V$ such that there is a degree one node $v \in \Pi$, there exists a cover Π' such that $v \notin \Pi'$ and $|\Pi'| \leq |\Pi|$.*

Proof. Let $\{u\} = \Gamma(v)$ and $\Pi' = (\Pi - \{v\}) \cup \{u\}$. Clearly, $\{v\}^S \subseteq \{u\}^S$. By Theorem 3, Π' is a PMU cover for G such that $v \notin \Pi'$ and $|\Pi'| \leq |\Pi|$. ∎

A second corollary shows that it is counterproductive to place a PMU on a degree two node (unless, of course, G is a path or a cycle).

COROLLARY 2. *Given a PSG $G = (V, E)$ with a cover $\Pi \subseteq V$ such that there is a degree two node $v \in \Pi$, there exists a cover Π' such that $v \notin \Pi'$ and $|\Pi'| \leq |\Pi|$.*

Proof. Let $\{u, w\} = \Gamma(v)$ and $\Pi' = (\Pi - \{v\}) \cup \{u\}$. Note that $w \in \{u\}^S$ by application of S1 and S2. Since $\Gamma[v] \subseteq \{u\}^S$, we have $\{v\}^S \subseteq \{u\}^S$. By Theorem 3, Π' is a PMU cover for G such that $v \notin \Pi'$ and $|\Pi'| \leq |\Pi|$. ∎

Corollaries 1 and 2 are implicit in Observation 4 of Haynes, Hedetniemi, Hedetniemi, and Henning [9].

3.4. Placing a PMU on a separation node. A *separation node* in a connected graph is one whose removal leaves a subgraph with two or more components. Baldwin, Mili, Boisen, and Adapa [1] claim that if a PMU placed at a separation node v observes all of the nodes in any one of the subgraphs resulting from the deletion of v, then v is an element of some minimum cover. This claim may fail if the observed subgraph is a path, due to the propagation of observability using S2, even when v has no PMU. The following restatement is correct.

THEOREM 4. *Let $G = (V, E)$ have separation node x. Let $u, w \in \Gamma(x)$ be distinct nodes. Let U and W be the components of $<V - x>$ containing u and w, respectively. If $U \cup W \subseteq \{x\}^S$, then there exists a minimum cover for G containing x.*

Proof. Note that U and W do not have to be distinct. Let Π_1 be any minimum PMU cover of G. If $x \in \Pi_1$, then we are done. Otherwise, by S2, there must be a node $y \in (U \cup W) \cap \Pi_1$. Let $\Pi_2 = \{x\} \cup (\Pi_1 - \{y\})$. Then Π_2 is a minimum cover for G containing x. ∎

3.5. Upper bound on the size of a minimum PMU cover. In this section, we show that, in a PSG having $n \geq 3$ nodes, at most $\lfloor n/3 \rfloor$ PMUs suffice to cover the PSG and that this upper bound is tight. Haynes, Hedetniemi, Hedetniemi, and Henning [9] show the same upper bound just for trees (their Theorem 14).

In a PSG, a node u is *symmetric* to a node v, written $u \equiv v$, if $\Gamma(u) - \{v\} = \Gamma(v) - \{u\}$.

THEOREM 5. *Node symmetry is an equivalence relation.*

Proof. Let $G = (V, E)$ be a connected graph. *Reflexivity.* For any $x \in V$, $\Gamma(x) - \{x\} = \Gamma(x) - \{x\}$ and hence $x \equiv x$. *Symmetry.* For any $x, y \in V$, $x \equiv y$ implies $\Gamma(x) - \{y\} = \Gamma(y) - \{x\}$, which implies $y \equiv x$. *Transitivity.* For any $x, y, z \in V$, $x \equiv y$ and $y \equiv z$ implies $\Gamma(x) - \{y\} = \Gamma(y) - \{x\}$ and $\Gamma(y) - \{z\} = \Gamma(z) - \{y\}$. These imply that $(x, z) \in E$ if and only if $(y, z) \in E$ and $(x, y) \in E$ if and only if $(x, z) \in E$. Consequently, $(x, y) \in E$ if and only if $(y, z) \in E$. Let $N = \Gamma(x) \cup \Gamma(y) \cup \Gamma(z) - \{x, y, z\}$. Thus $\Gamma(x) - \{z\} = (\Gamma(x) \cap \{y\}) \cup N = (\Gamma(z) \cap \{y\}) \cup N = \Gamma(z) - \{x\}$. Hence, $x \equiv z$.

Thus, node symmetry is an equivalence relation. ∎

For a PSG $G = (V, E)$, let S be the set of equivalence classes of V under $\equiv$. For every $P \in S$, $<P>$ is either a clique or an independent set. For two distinct equivalence classes $P, Q \in S$, P is *adjacent* to Q if for every $u \in P$, we have $Q \subseteq \Gamma(u)$. Note that P adjacent to Q implies Q adjacent to P. Define $A(S) = \{(P, Q) \mid P, Q \in S \text{ and } P \text{ adjacent to } Q\}$. The graph $H(S) = (S, A(S))$ is the *adjacency graph* of S. For any $R \subseteq S$, define $\pi(R) = \cup_{U \in R} U$.

LEMMA 1. *Let $G = (V, E)$ be a PSG, and let S be the set of equivalence classes of V under $\equiv$. Let $U_1, U_2 \in V$ be distinct equivalence classes, and let $u_1 \in U_1$ and $u_2 \in U_2$. Then $(u_1, u_2) \in E$ if and only if $(U_1, U_2) \in A(S)$. Consequently, $H(S)$ is connected.*

An equivalence class of $\equiv$ containing more than one node represents a kind of node redundancy. The following lemma identifies a small placement that dominates all but one node of each equivalence class.

LEMMA 2. *Let $G = (V, E)$ have 3 or more nodes. There exists a placement Π such that*

1. *for every distinct $u, v \in V$ such that $u \equiv v$, either $u \in \Gamma[\Pi]$ or $v \in \Gamma[\Pi]$; and for every $U \in S$, $|\Gamma[\Pi] \cap U| \geq |U| - 1$; and*
2. *$|\Gamma[\Pi]| \geq 3|\Pi|$.*

Proof. First suppose that G is a clique. Then $S = \{V\}$ and $H(S) = (S, \emptyset)$. Let $\Pi = \{v\}$, where $v \in V$. Clearly, Π satisfies (1) and (2).

Now suppose that G is not a clique. Then $|S| \geq 2$. We proceed by induction on $|S|$ to show that there exists a Π that satisfies (1) and (2), as long as $|\pi(S)| \geq 3$. The base case is $|S| = 2$. Let $S = \{U_1, U_2\}$, where $|U_1| \geq |U_2| \geq 1$. If $|U_2| = 1$ or $|U_2| = 2$, then let $\Pi = \{u\}$ for any $u \in U_2$. If $|U_2| \geq 3$, then let $\Pi = \{u_1, u_2\}$ for any $u_1 \in U_1$ and any $u_2 \in U_2$. In both cases, $U_1 \cup U_2 \subseteq \Gamma[\Pi]$, and $|U_1 \cup U_2| \geq 3|\Pi|$. Hence, (1) and (2) hold for Π.

Now assume that $|S| = m \geq 3$ and that the inductive hypothesis holds for any adjacency graph $H(S')$ of size less than m, as long as $|\pi(S')| \geq 3$. Let $T = (S, F)$ be a spanning tree of $H(S)$. Choose $U \in S$ that is not a leaf but is adjacent to at least one leaf in T. Root T at U. Let $T_1, T_2, \ldots, T_r$ be the subtrees under U. Note that $r \geq 2$, since $|S| \geq 3$. For $1 \leq j \leq r$, let R_j be the root of T_j. Without loss of generality, assume that R_j is a leaf of T for $j \leq s$ and a nonleaf for $j > s$, where $1 \leq s \leq r$, and that the T_j, for $1 \leq j \leq s$, are arranged in nondecreasing order by cardinality of $|R_j|$.

First suppose that $|R_1| = 1$. Then R_1 places no constraints on Π with respect to (1) or (2). Let $S' = S - \{R_1\}$. If $|S'| \geq 3$, then, by induction, a placement Π can be found for $H(S')$ that satisfies (1) and (2) for $<V - R_1>$ and hence also for G. If $|S'| = 2$, then $s = r = 2$. Select $u \in U$ and $w \in R_2$. If $|R_2| \leq 2$, then set $\Pi = \{w\}$. If $|R_2| \geq 3$ and $|U| \leq 2$, then set $\Pi = \{u\}$. If $|R_2| \geq 3$ and $|U| \geq 3$, then set $\Pi = \{u, w\}$ (in this case, $|\pi(S)| \geq 6$). In all cases, Π satisfies (1) and (2) for G.

Now suppose that $|R_1| \geq 2$. Select $u \in U$. Consider the case $|U| \leq 2$. If $r = s$, then set $\Pi = \{u\}$. Otherwise, consider each T_j, where $s + 1 \leq j \leq r$, in turn. If $|\pi(T_j)| \geq 3$, then apply the inductive hypothesis to T_j to identify Π_j that satisfies (1) and (2) for T_j. If $|\pi(T_j)| \leq 2$, then T_j is a path of one-node equivalence classes; set $\Pi_j = \emptyset$. Set $\Pi = \{u\} \cup \bigcup_{j=s+1}^{r} \Pi_j$. Then Π satisfies (1) and (2). Now consider the case $|U| \geq 3$. In this case, $|U - \{u\}| \geq 2$ and $| <V - R_1 - \{u\}> | \geq 3$. Let $G' = <V - R_1 - \{u\}>$. By induction, there exists a Π' satisfying (1) and (2) for G'. Set $\Pi = \Pi' \cup \{u\}$. Then Π satisfies (1) and (2).

By induction, we obtain $\Pi \subseteq V$ satisfying (1) and (2). □

THEOREM 6. *Let $G = (V, E)$ be a PSG, and let $n = |V|$. Then there exists a cover Π satisfying $|\Pi| \leq \lfloor n/3 \rfloor$, if $n \geq 3$, and $|\Pi| = 1$, if $1 \leq n \leq 2$.*

Proof. The result for $1 \leq n \leq 2$ is immediate. For $n \geq 3$, the proof is an inductive

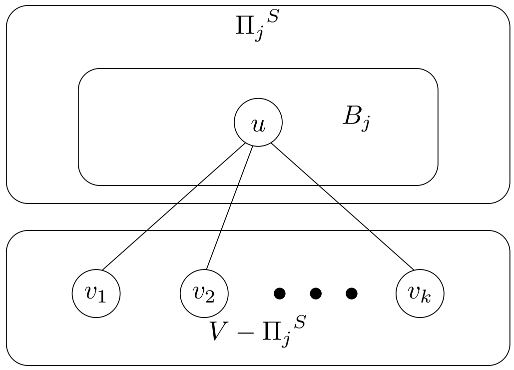

FIG. 6. *Boundary node $u \in B$.*

construction of a sequence of placements $\Pi_0{}^S, \Pi_1{}^S, \ldots, \Pi_\ell{}^S$ such that, for $0 \leq j < \ell$, we have that Π_j is a proper subset of Π_{j+1}; Π_ℓ is a cover of G; and, for $0 \leq j \leq \ell$, we have $\Pi_j \neq \emptyset$ and $|\Pi_j{}^S \geq 3|\Pi_j{}^S|$.

The base case is $j = 0$. Let Π' be the initial placement guaranteed by Lemma 2. If $\Pi' \neq \emptyset$, then set $\Pi_0 = \Pi'$. Otherwise, set $\Pi_0 = \{u\}$, where u is any degree 2 node of G. Clearly, $\Pi_0 \neq \emptyset$ and $|\Pi_j{}^S \geq 3|\Pi_j{}^S|$, as required.

Now suppose that $j \geq 0$ and that, for every $0 \leq i < j$, Π_i is a proper subset of Π_{i+1}, and, for $0 \leq i \leq j$, $\Pi_j \neq \emptyset$ and $|\Pi_i{}^S \geq 3|\Pi_i{}^S|$. Let $B_j = \{u \in \Pi_j{}^S \mid \Gamma(u) \cap (V - \Pi_j{}^S)\}$ be the set of *boundary nodes*—observed nodes adjacent to unobserved nodes. If $B_j = \emptyset$, then Π_j is a cover of G and the theorem is proved for G. Otherwise, $V - \Pi_j{}^S \neq \emptyset$. In that case, we construct Π_{j+1} as follows.

Clearly $B_j \cap \Pi_j = \emptyset$, since $\Gamma[\Pi_j] \subset \Pi^S$. Let $u \in B_j$, and let $\Gamma(u) \cap (V - \Pi_j{}^S) = \{v_1, v_2, \ldots, v_k\}$, as illustrated in Figure 6. Without loss of generality, we may assume that u is selected so that k is as large as possible. Observe that $k \geq 2$, because if v_1 were the only unobserved neighbor of u, then v_1 would be observed by rule S2.

First consider the case $k \geq 3$. Set $\Pi_{j+1} = \Pi_j \cup \{u\}$, a proper superset of Π_j. Then $|\Pi_{j+1}{}^S| \geq |\Pi_j{}^S| + 3$, as desired.

Now consider the case $k = 2$, which means that every node in B_j is adjacent to exactly two nodes of $V - \Pi_j{}^S$. Without loss of generality, assume that $\text{DEGREE}(v_1) \geq \text{DEGREE}(v_2) \geq 1$. Since $v_1 \not\equiv v_2$, we cannot have $\text{DEGREE}(v_1) = \text{DEGREE}(v_2) = 1$. Thus, $\text{DEGREE}(v_1) \geq 2$.

Let $C_1 = (V_1, E_1)$ (respectively, $C_2 = (V_2, E_2)$) be the component of $<V - \Pi_j{}^S>$ containing v_1 (respectively, v_2). First consider the cases where $|V_1| \geq 3$ or where $|V_1| = 2$ and $C_1 \neq C_2$. Select a $v_3 \in V_1 \cap \Gamma(v_1)$ that is not v_2. Set $\Pi_{j+1} = \Pi_j \cup \{v_1\}$, a proper superset of Π_j. We obtain $v_2, v_3 \in \Pi_{j+1}{}^S$; in particular, $v_2 \in \Pi_{j+1}{}^S$ because v_2 is the last unobserved neighbor of u and hence is observed by rule S2. Therefore, $|\Pi_{j+1}{}^S| \geq |\Pi_j{}^S| + 3$, as desired. Now consider the cases where $|V_1| = 1$ or where $|V_1| = 2$ and $C_1 = C_2$. These cases imply that v_1 and v_2 are adjacent only to nodes in B_j and perhaps each other. Since $\text{DEGREE}(v_1) \geq 2$ and $v_1 \not\equiv v_2$, there must be a node $w \in B_j - \{u\}$ adjacent to v_1 and not adjacent to v_2. Let $\Gamma(w) \cap (V - \Pi_j{}^S) = \{v_1, z\}$; see Figure 7. We have $z \neq v_2$, since w is not adjacent to v_2. Set $\Pi_{j+1} = \Pi_j \cup \{v_1\}$, a proper superset of Π_j. We obtain $v_2, z \in \Pi_{j+1}{}^S$ by application of rule S2 to u and w. Therefore, $|\Pi_{j+1}{}^S| \geq |\Pi_j{}^S| + 3$, as desired.

Since the sequence of placements are increasing, we must eventually reach the

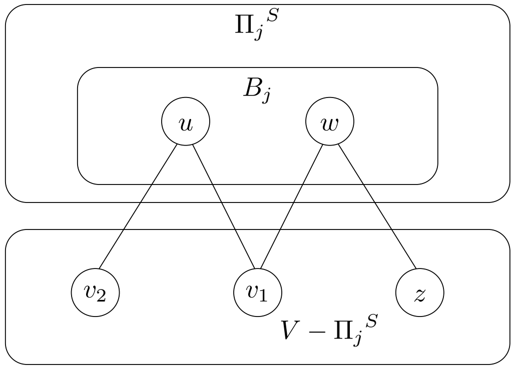

FIG. 7. *Boundary nodes $u, w \in B_j$.*

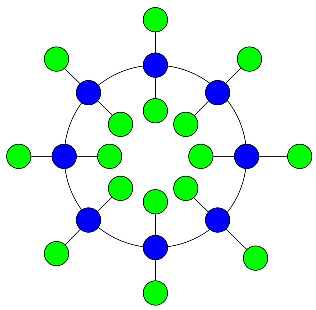

FIG. 8. *Corona $B_{\ell,2}$, which requires $n/3$ PMUs.*

case where $B_j = \emptyset$. The theorem follows. $\square$

We now show that the above bound is existentially tight. To do so, we start with a construction defined in Haynes, Hedetniemi, Hedetniemi, and Henning [9]. If G and H are two graphs, then the *corona* $G \circ H$ of G and H is achieved by making a copy H_v of H for every node v of G and adding an edge from v to every node of H_v. For purposes of notation, let $C_\ell = (U_\ell, E_\ell)$, where

$$\begin{aligned} U_\ell &= \{u_1, u_2, \ldots, u_\ell\} \\ E_\ell &= \bigcup_{i=1}^{\ell-1} \{(u_i, u_{i+1})\} \cup \{(u_\ell, u_1)\}\}, \end{aligned}$$

be a cycle of length ℓ, and let $I_k = (V_k, \emptyset)$, where $V_k = \{v_1, v_2, \ldots, v_k\}$, be a graph of k isolated nodes. For each $u_i \in U_\ell$, define a copy I_{k,u_i} of I_k by $I_{u_i,k} = (V_{k,u_i}, \emptyset)$, where $V_{u_i,k} = \{v_{i,1}, v_{i,2}, \ldots, v_{i,k}\}$. The corona $B_{\ell,k} = C_\ell \circ I_k$ is a graph of $n = k\ell$ nodes that requires exactly ℓ PMUs to be observed when $k \geq 2$. Moreover, the initial placement phase in the proof of Theorem 6 finds exactly the minimum PMU cover of $B_{\ell,k}$. More specifically, $B_{\ell,2}$ requires exactly $n/3$ PMUs to be observed, which we show in Theorem 7. For example, a minimum PMU cover for $B_{\ell,2}$ has exactly ℓ PMUs as shown in Figure 8.

THEOREM 7. *A minimum cover for $B_{\ell,k}$ requires $\lceil \ell/3 \rceil$ PMUs if $k = 1$ and requires ℓ PMUs if $k \geq 2$.*

Proof. First consider the construction of a minimum PMU cover of $B_{\ell,1} = C_\ell \circ I_1$. Starting with an arbitrary point on C_ℓ, place a PMU on every third node of C_ℓ. It is easy to verify that such a placement is a minimum cover, since every degree one node is either adjacent to a PMU or adjacent to a node of C_ℓ that is adjacent to PMU.

Now assume $k \geq 2$ and consider the construction of a minimum PMU cover of $B_{\ell,k}$. Let $V = \cup_{i=1}^{\ell} V_{u_i,k}$. Let Π be a minimum cover of G. Among all minimum covers of G, select $\Pi \cap V$ to be as small as possible. Suppose $v = v_{i,j} \in \Pi \cap V$. Then v is a degree one node adjacent only to u_i, a degree $k+2$ node. The set $\{u_i\} \cup (\Pi - \{v\})$ is a cover of G of the same cardinality as Π, but with one fewer element of V, contradicting the choice of Π. Hence, $\Pi \cap V = \emptyset$. We claim that $\Pi = U_\ell$. Consider any $u_i \in U_\ell$. We know that none of the k neighbors of u_i in V are in Π. If $u_i \notin \Pi$, then the k neighbors are observed via applications of rule S2. But rule S2 can only be applied when at most one neighbor of u_i is unobserved, while $k \geq 2$. We conclude that $u_i \in \Pi$ and, moreover, that $\Pi = U_\ell$. The theorem follows. $\square$

One referee suggested this generalization of Theorem 7.

THEOREM 8. *Let G be a connected graph with ℓ nodes, and let $k \geq 2$. Then a minimum cover for the corona $G \circ I_k$ requires ℓ PMUs.*

The proof is similar to that of Theorem 7.

4. NP-completeness. Haynes, Hedetniemi, Hedetniemi, and Henning [9] show that PMUP is NP-complete for bipartite graphs and for chordal graphs. Here we show that the following decision problem version of PMUP is NP-complete even for planar bipartite graphs.

Problem: PMU Placement (Decision Version)

INSTANCE: Graph $G = (V, E)$, integer $k \geq 1$.

QUESTION: Is there a set $\Pi \subseteq V$ such that $|\Pi| \leq k$ and $\Pi^S = V$?

THEOREM 9. *PMUP is NP-complete even when restricted to the class of planar bipartite graphs.*

Proof. The decision problem is easily in NP. Nondeterministically select k nodes forming a candidate Π and verify observability using the methods described in section 2.

The remainder of the proof is a reduction from planar 3-SAT (P3SAT) [15]. An instance of 3-SAT is a boolean formula ϕ in conjunctive normal form such that each clause contains at most 3 literals [7]. ϕ consists of the variables $\{v_1, v_2, \ldots, v_r\}$ and the set of clauses $\{c_1, c_2, \ldots, c_s\}$. Each c_j is a set containing at most 3 literals, where each literal is either a variable v_i or its complement $\overline{v_i}$. A clause containing exactly k literals is called a *k-clause.* The *graph of* ϕ, $G(\phi) = (V(\phi), E(\phi))$, is a bipartite graph constructed as follows:

$$\begin{aligned} V(\phi) &= \{v_i \mid 1 \leq i \leq r\} \cup \{c_j \mid 1 \leq j \leq s\} \\ E(\phi) &= \{(v_i, c_j) \mid v_i \in c_j \text{ or } \overline{v_i} \in c_j\}. \end{aligned}$$

The edges in $E(\phi)$ represent whether a variable occurs in a clause or not. For example, the graph of the formula

$$\phi = (\overline{v_1} \vee v_2 \vee v_3) \wedge (\overline{v_1} \vee \overline{v_4} \vee v_5) \wedge (\overline{v_2} \vee \overline{v_3} \vee \overline{v_5}) \wedge (v_3 \vee \overline{v_4}) \wedge (\overline{v_3} \vee v_4 \vee \overline{v_5})$$

is shown in Figure 9. ϕ is satisfied if $v_2, \overline{v_4}$, and $\overline{v_5}$ are true; hence ϕ is a satisfiable formula. Lichtenstein shows that 3-SAT is NP-complete even when $G(\phi)$ is planar (the problem P3SAT) [15].

It suffices to consider only instances of P3SAT such that each clause contains either 2 or 3 literals. Our planar embedding of $G(\phi)$ positions each node v_i, $1 \leq i \leq r$, along a straight line; this is called the *variable axis.* From Lemma 1 of [15], we may

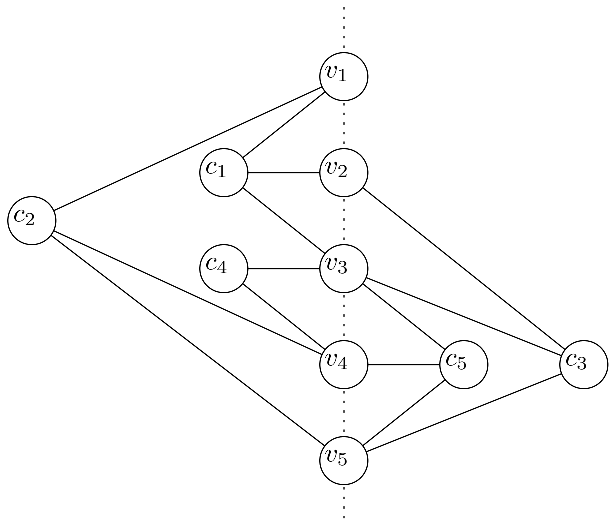

FIG. 9. *Example of planar* 3*-SAT.*

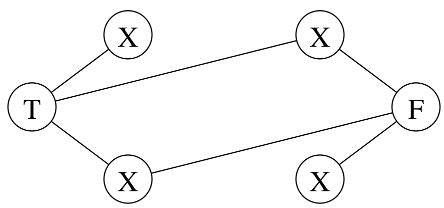

FIG. 10. *A gadget for a variable.*

assume that our planar embedding of $G(\phi)$ satisfies the condition that, for each v_i, all clauses containing the literal v_i are on one side of the variable axis and all clauses containing the literal $\overline{v_i}$ are on the other side. This property of the planar embedding of $G(\phi)$ is called *consistency* [10]. Figure 9 is an example of a consistent planar embedding.

Let $V = \{v_1, \ldots, v_r\}$ and $C = \{c_1, c_2, \ldots, c_s\}$ be an instance of P3SAT such that $G(\phi)$ has a consistent planar embedding. We will construct a corresponding instance of PMUP that also is a planar bipartite graph. The strategy is to replace each node in $G(\phi)$ with a specially constructed graph, or *gadget*. Let $H(\phi)$ denote the resulting graph. Each clause node c_j, $1 \leq j \leq s$, is replaced with a 2-clique $C[j], C'[j]$, effectively making the clause node adjacent to an additional degree one node. The gadgets placed on clauses simply force a clause to be adjacent to at least one node with a PMU. Each variable node v_i is replaced by the gadget shown in Figure 10. Observe that the gadget forces at least one PMU placed on it in order to be covered. This implies that $H(\phi)$ requires a minimum of r PMUs in its cover. We wish to show that a minimum cover for $H(\phi)$ uses exactly r PMUs if and only if ϕ is satisfiable. Thus we are allowed only one PMU per gadget.

The gadget is designed to toggle between two states, representing either a *true* (T) or *false* (F) value for the literal it replaces, depending on which node the PMU is placed on; see Figure 11. For any variable v_i, let $z_i \in \{v_i, \overline{v_i}\}$ denote the variable

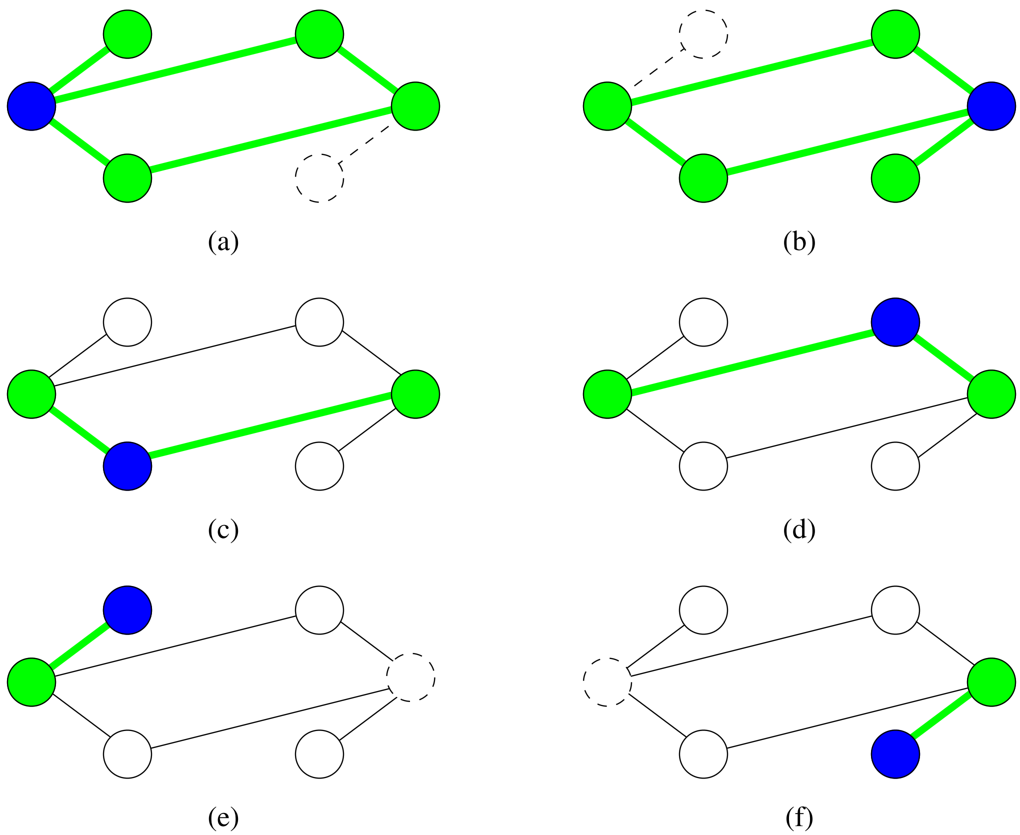

FIG. 11. *Gadget states:* (a) *true;* (b) *false;* (c) *left bridge;* (d) *right bridge;* (e) *left leaf; and* (f) *right leaf.*

appearing in all clauses to the left of the variable axis. The following cases ensue:

1. *true*: In this case, the gadget is indicating that z_i is true. The right leaf of the gadget is observed only if all clauses connected to the rightmost node are observed.
2. *false*: In this case, the gadget is indicating that z_i is false. The left leaf of the gadget is observed only if all clauses connected to the leftmost node are observed.
3. *eliminated (left bridge and right bridge)*: It is impossible to cover the gadget with one PMU on either bridge.
4. *eliminated (left leaf and right leaf)*: It is impossible to cover the gadget with one PMU on a leaf.

For illustration, consider the instance of P3SAT depicted in Figure 9, with gadgets inserted, as shown in Figure 12, and shown with a minimum PMU cover in Figure 13.

We have shown our construction guarantees a graph $H(\phi)$ for which a minimum PMU cover has at least one PMU per gadget. At this time, note that $H(\phi)$ is planar *and* bipartite, as shown in Figure 14. We have also classified the nodes of the gadget semantically as either *true*, *false*, or *illegal*. By the substitution lemma, we do not need to consider illegal nodes when constructing a minimum PMU cover for $H(\phi)$. It remains to show that $H(\phi)$ has a cover of size r if and only if ϕ is satisfiable.

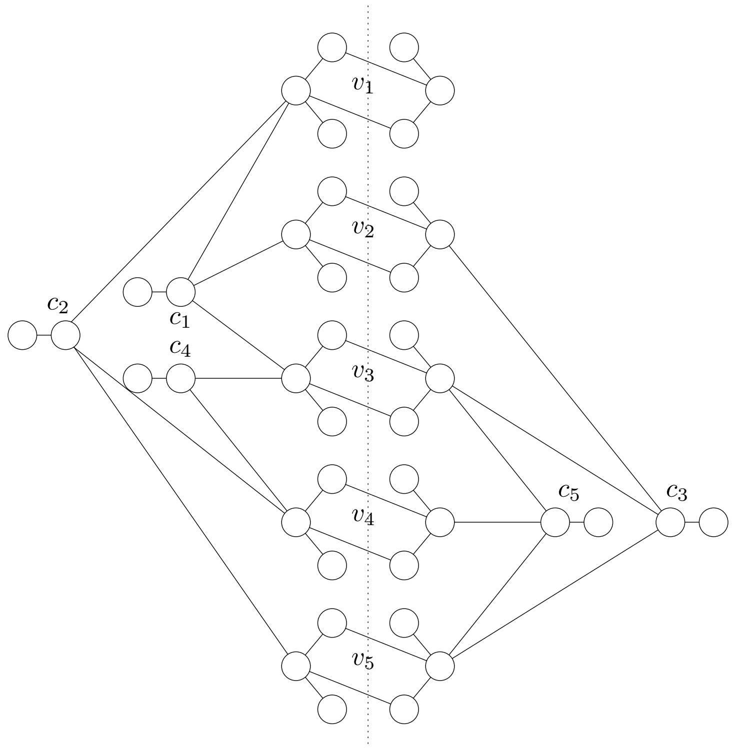

FIG. 12. *Instance of PMUP (planar 3-SAT with gadgets).*

Assume that ϕ is satisfiable. For each variable v_i, $1 \leq i \leq r$, place a PMU on either the leftmost or the rightmost gadget node according to whether v_i is true or false in a given satisfying instance S for ϕ. If ϕ is satisfied, then for each clause c_j, $1 \leq j \leq s$, there exists a literal $v_i \in c_j$ or $\overline{v_i} \in c_j$ which is in S. The PMU placed on the corresponding gadget observes the main node of c_j, as well as the main body of the v_i's gadget. Thus all main clause nodes are observed. Furthermore, all leaf nodes on clauses become observed by S2. Likewise, the remaining leaf nodes on gadgets become observed. Hence, there is a cover of size r for $H(\phi)$.

Now assume that $H(\phi)$ has a cover Π of size r. Each gadget must have at least one PMU. Thus there can be no nongadget PMUs in Π. Since Π is a cover, all clauses are observed. By construction, a clause cannot be observed unless it is adjacent to at least one PMU located on a gadget. Then for each main clause node c_j, $1 \leq j \leq s$, there exists a node $u \in \Gamma(c_j)$ with a PMU. Let v_i be variable containing u. Let $z_i \in \{v_i, \overline{v_i}\}$ be the variable appearing in c_j. The clause c_j is satisfied if z_i is chosen as true. Hence, all clauses in ϕ are satisfied by the truth assignment derived from the minimum cover Π.

In summary, we have transformed instance ϕ of P3SAT into a PSG $H(\phi)$ with the property that ϕ is satisfiable if and only if $H(\phi)$ has a PMU cover of size r. Therefore, P3SAT reduces to PMUP. Since P3SAT is NP-complete [15], we conclude that PMUP is NP-complete even when restricted to planar bipartite graphs. $\square$

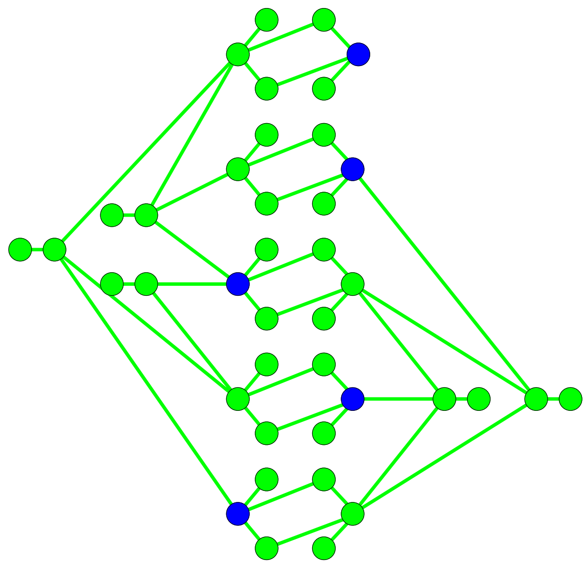

Fig. 13. *Minimum PMU cover (P3SAT with gadgets).*

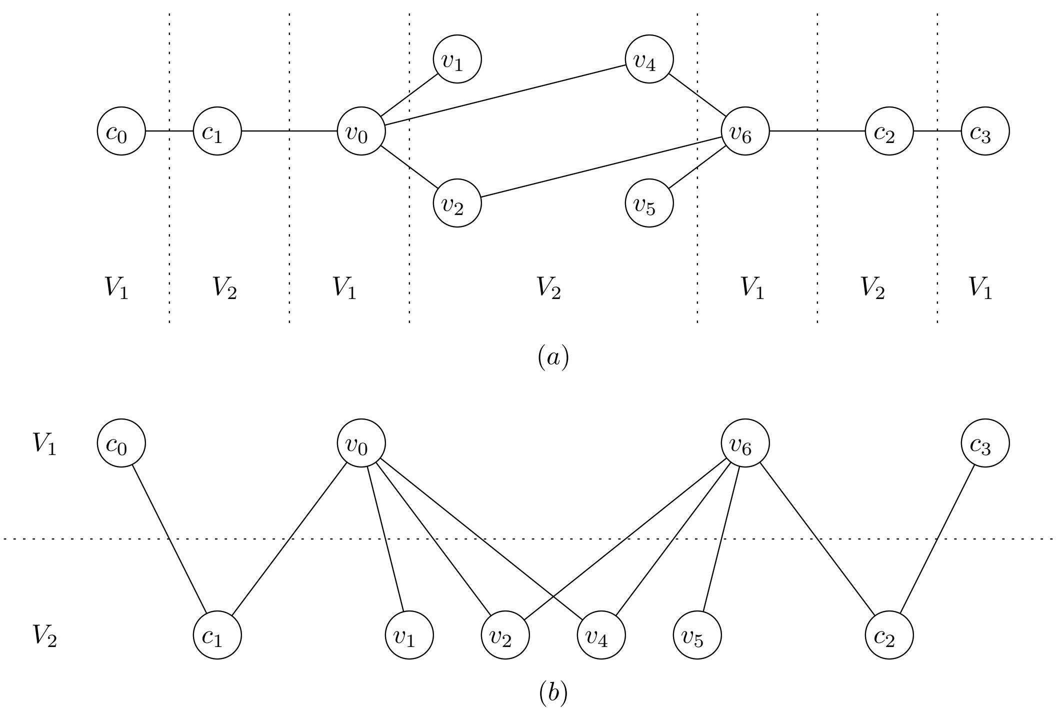

Fig. 14. *Partitions of nodes in $H(\phi)$ showing that $H(\phi)$ is bipartite.*

Acknowledgments. We wish to thank Monte Boisen and Donald Allison for helpful discussions. We also thank Lamine Mili and Thomas Baldwin for advice and contribution of test data. We extend our appreciation to the three referees who helped make this a better paper.

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