# Inference in Bayesian Networks

CS 5804 Introduction to Artificial Intelligence
Virginia Tech

# Plan

- Briefly review probability
- Brute-force inference
- Smarter inference: variable elimination

# Probability Identities

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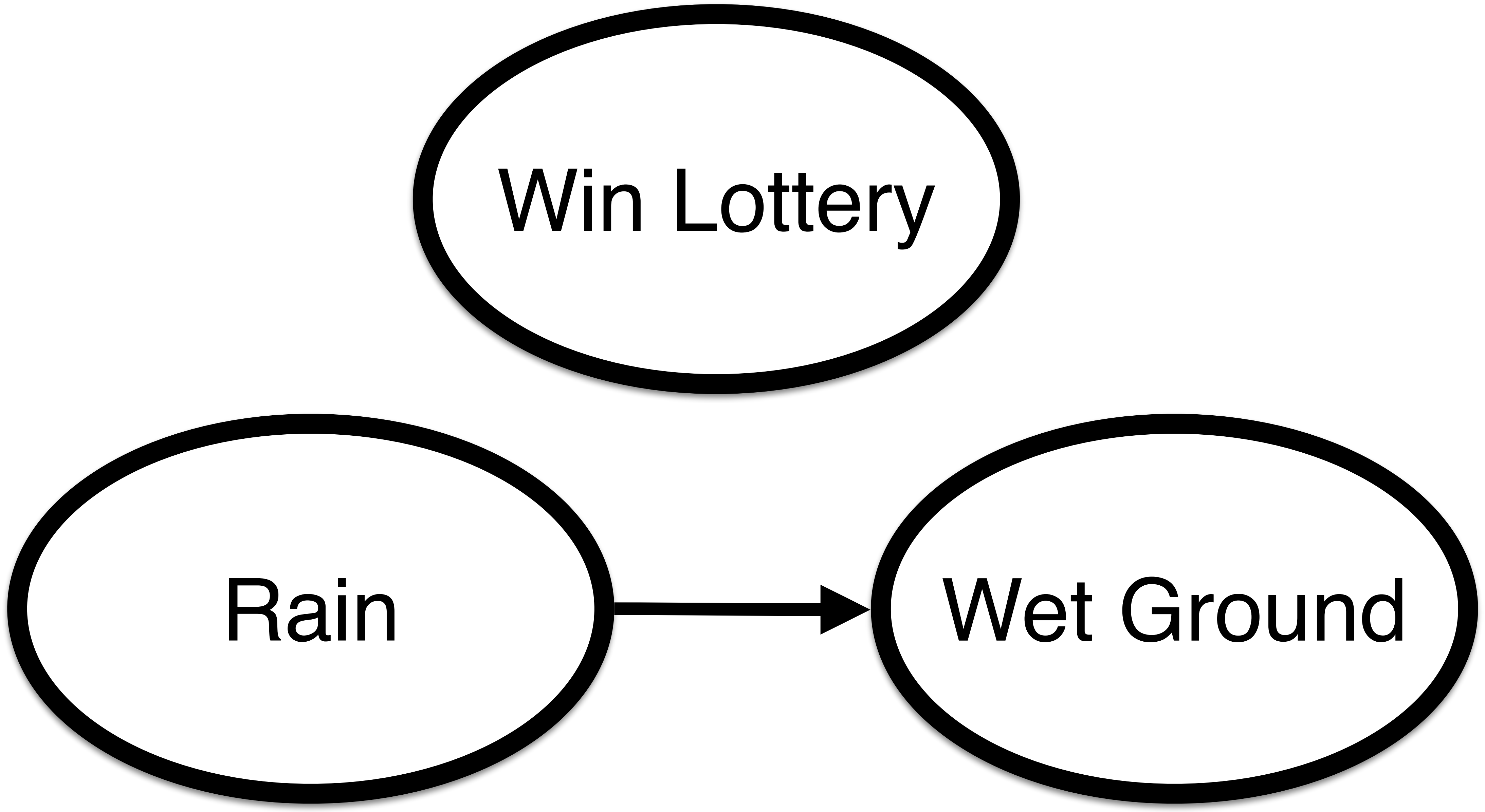
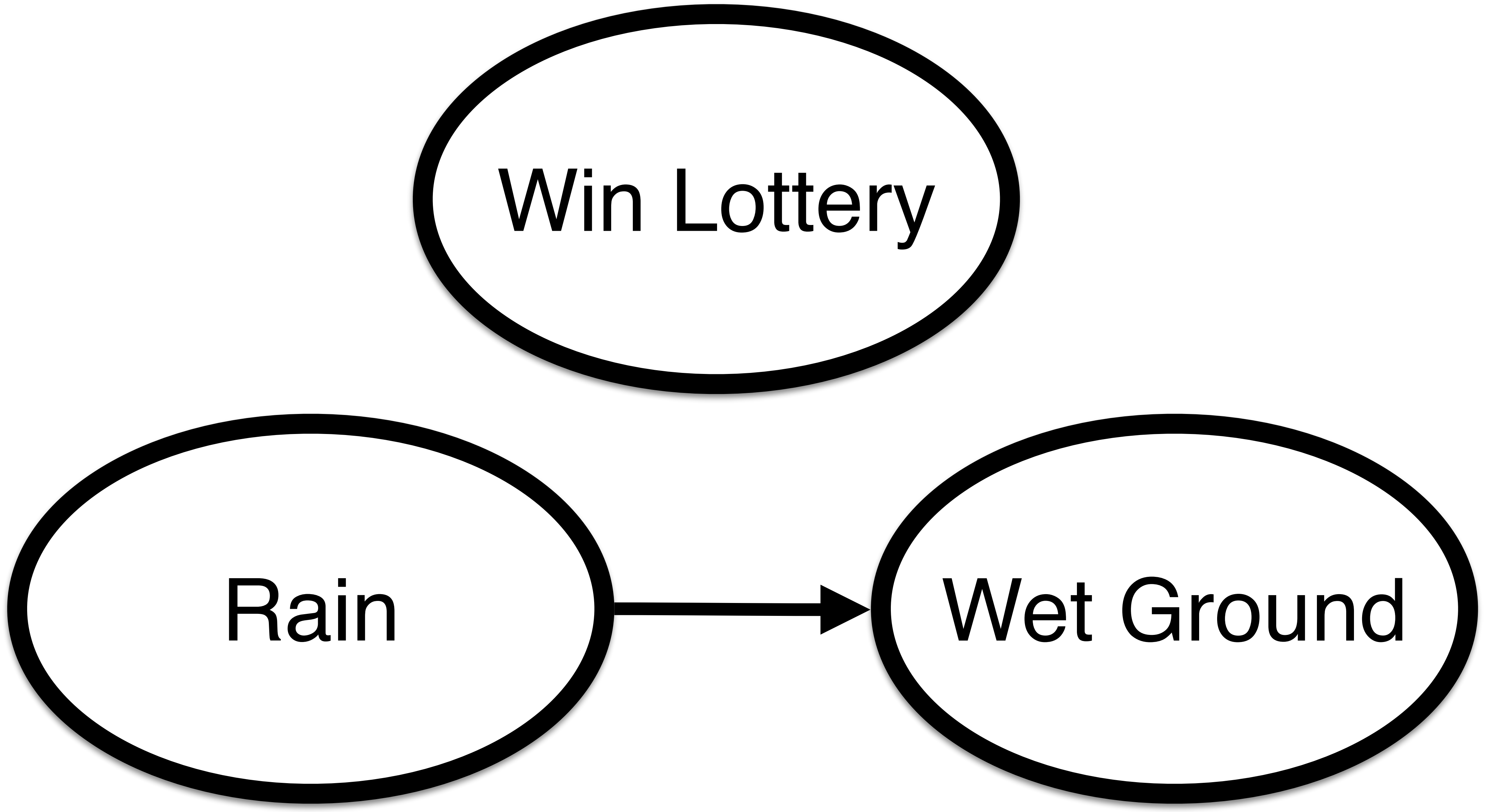
# Bayesian Networks



# Bayesian Networks

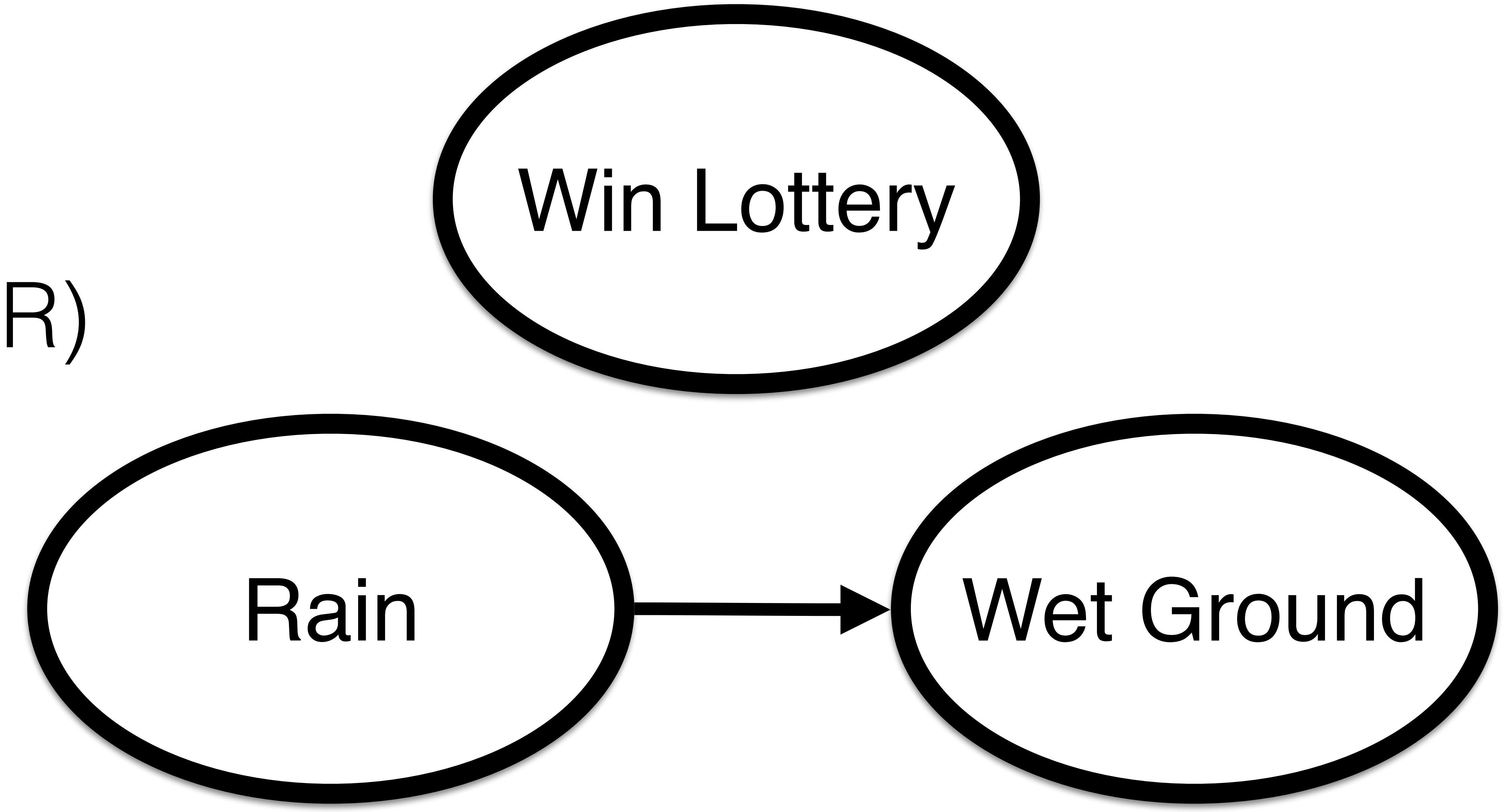
P(L, R, W)



# Bayesian Networks

$P(L, R, W)$

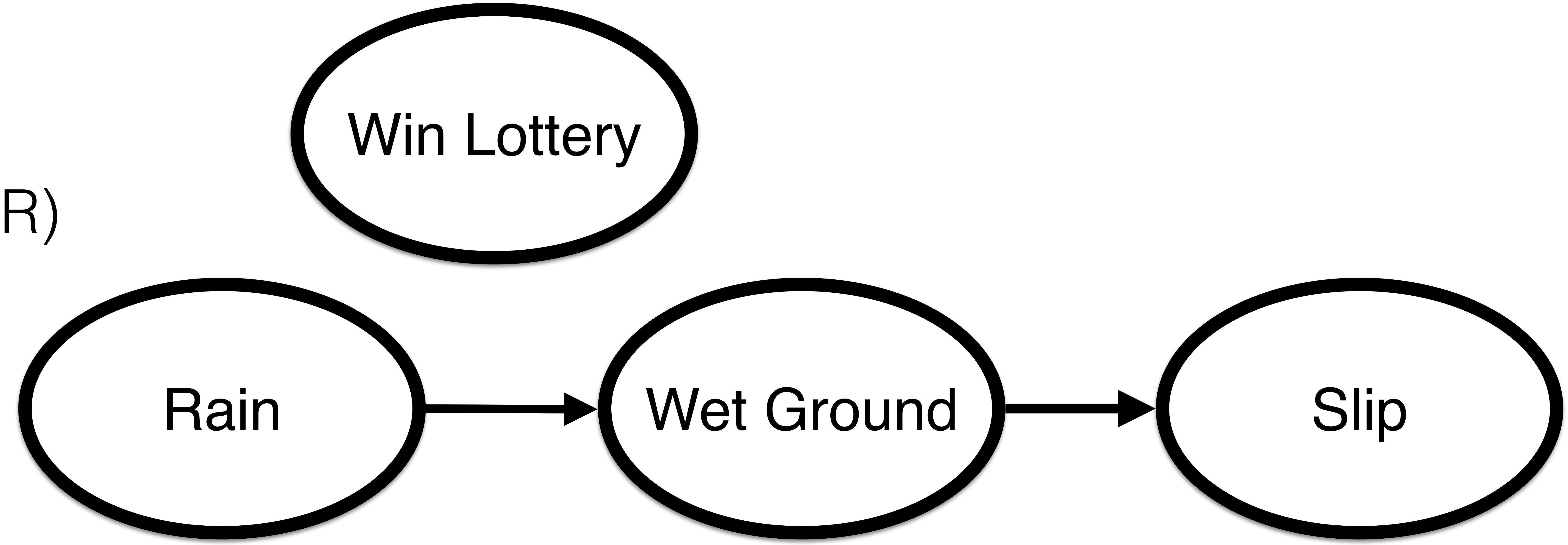
$= P(L)\, P(R)\, P(W \mid R)$



# Bayesian Networks

P(L, R, W)

= P(L) P(R) P(W | R)



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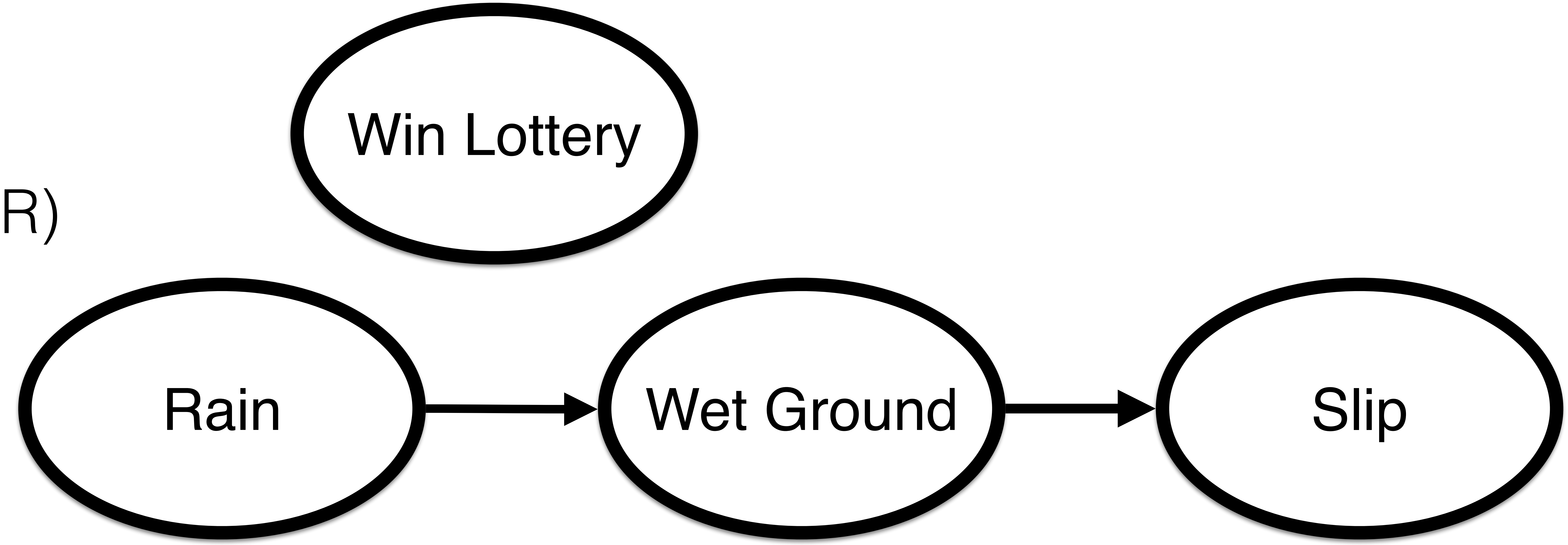


P(L, R, W, S) = P(L) P(R) P(W | R) P(S | W)

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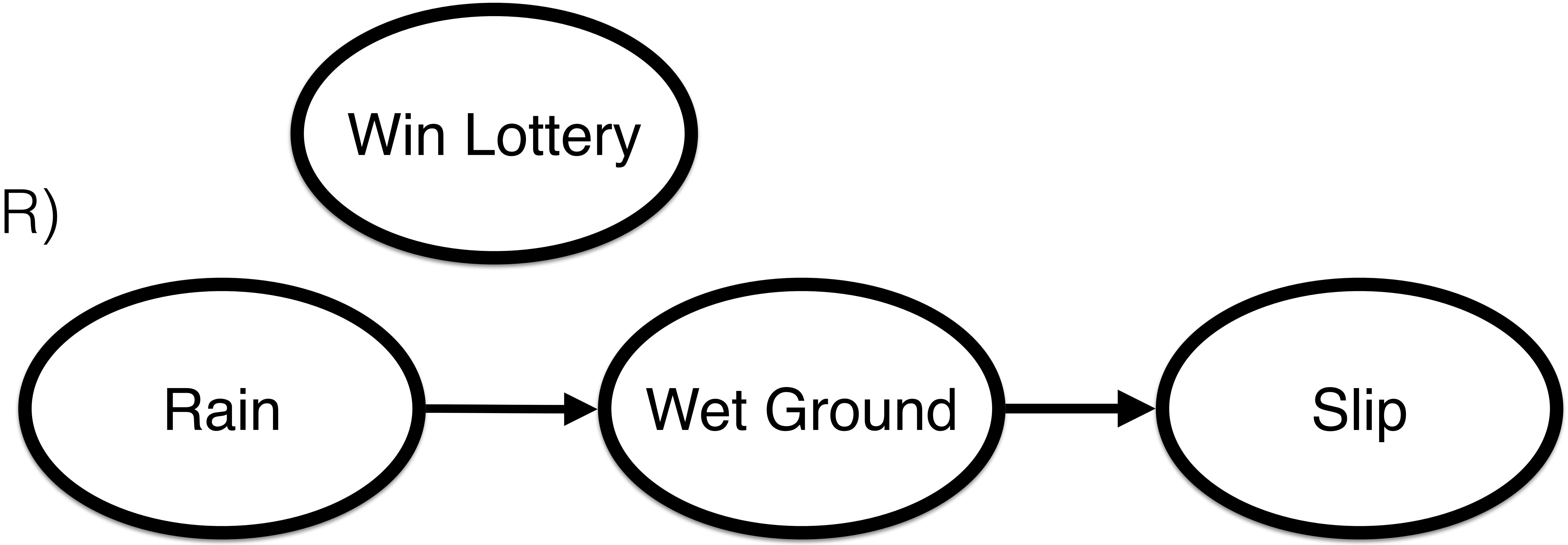


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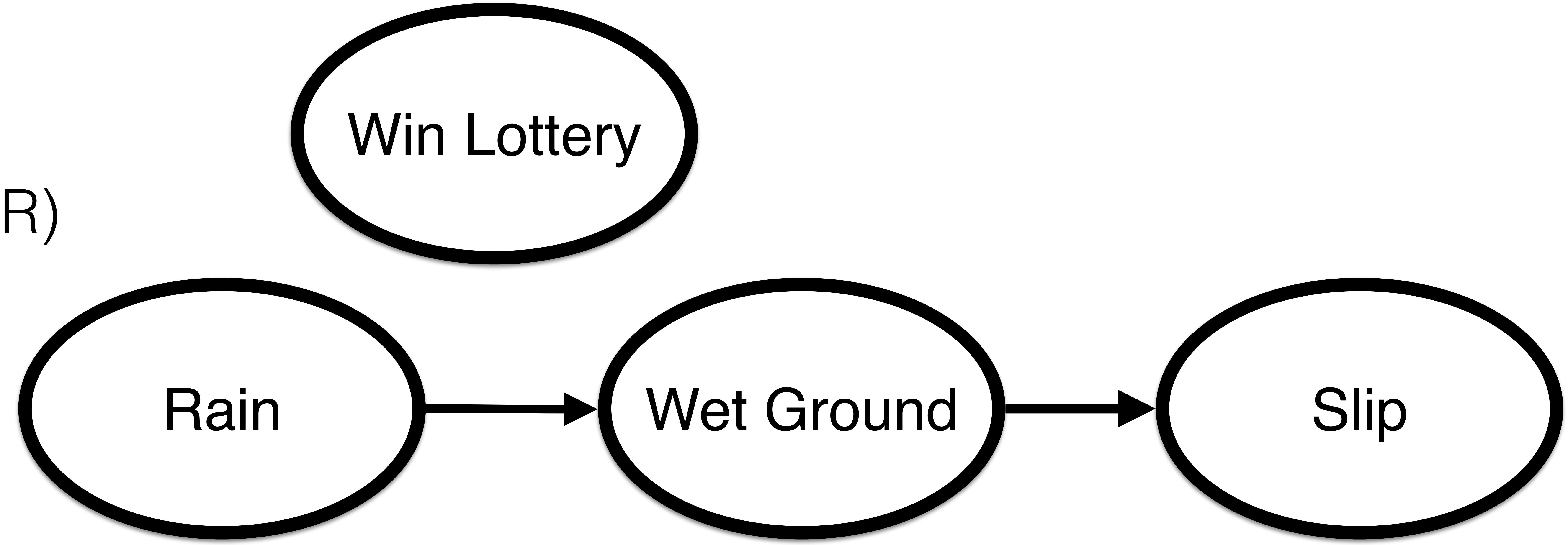


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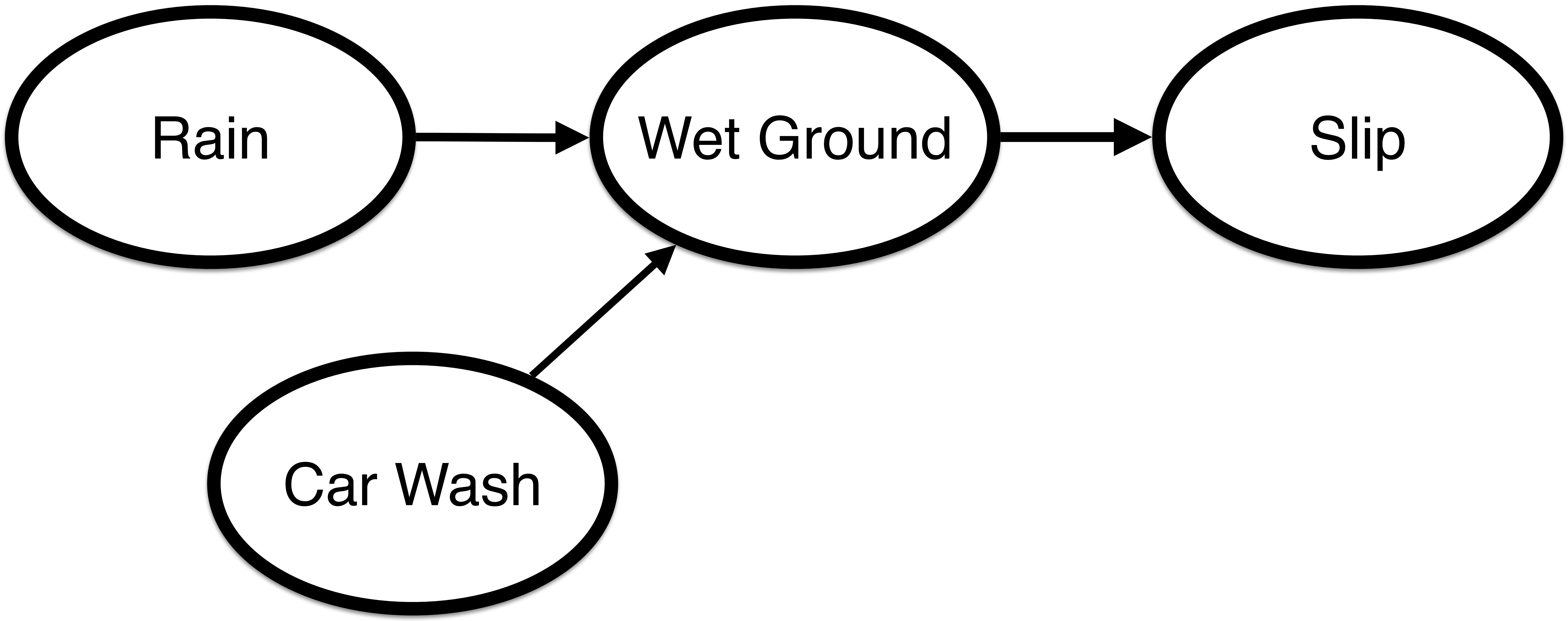
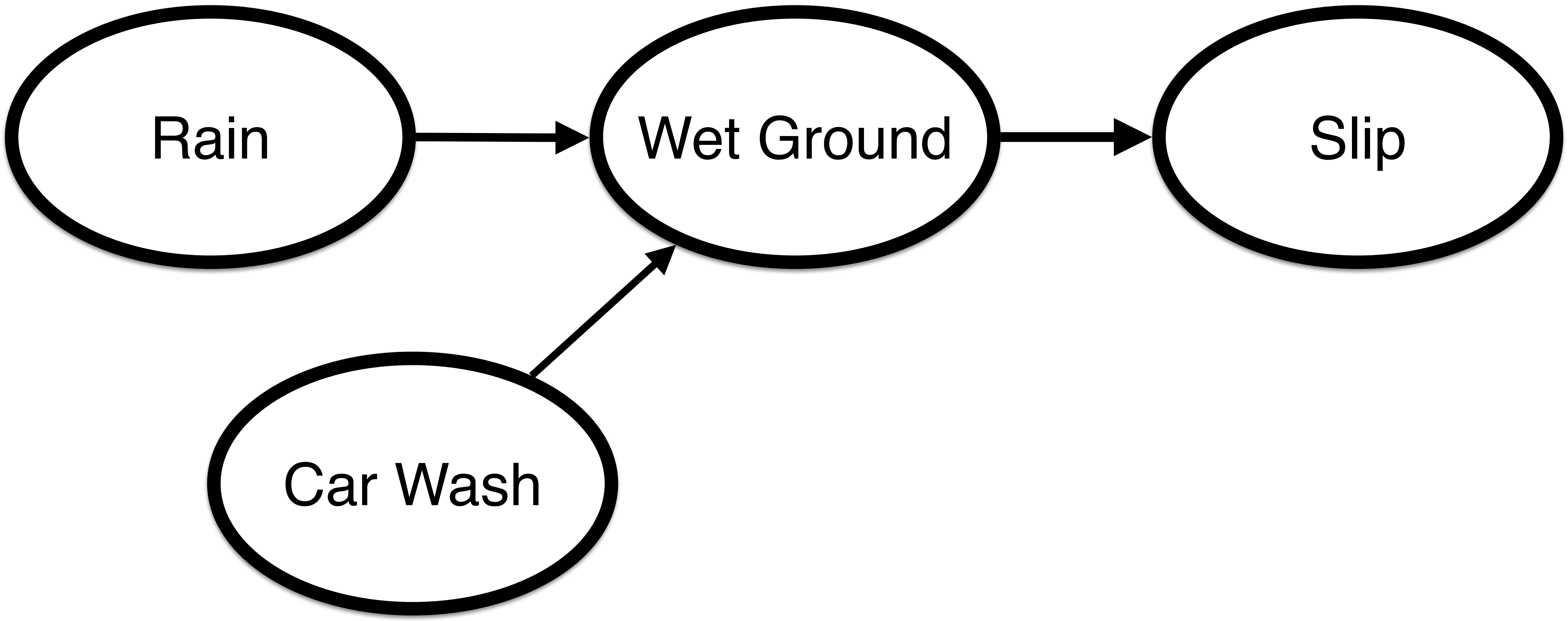


# Bayesian Networks

$P(R, W, S, C) = P(R)\, P(C)\, P(W \mid C, R)\, P(S \mid W)$



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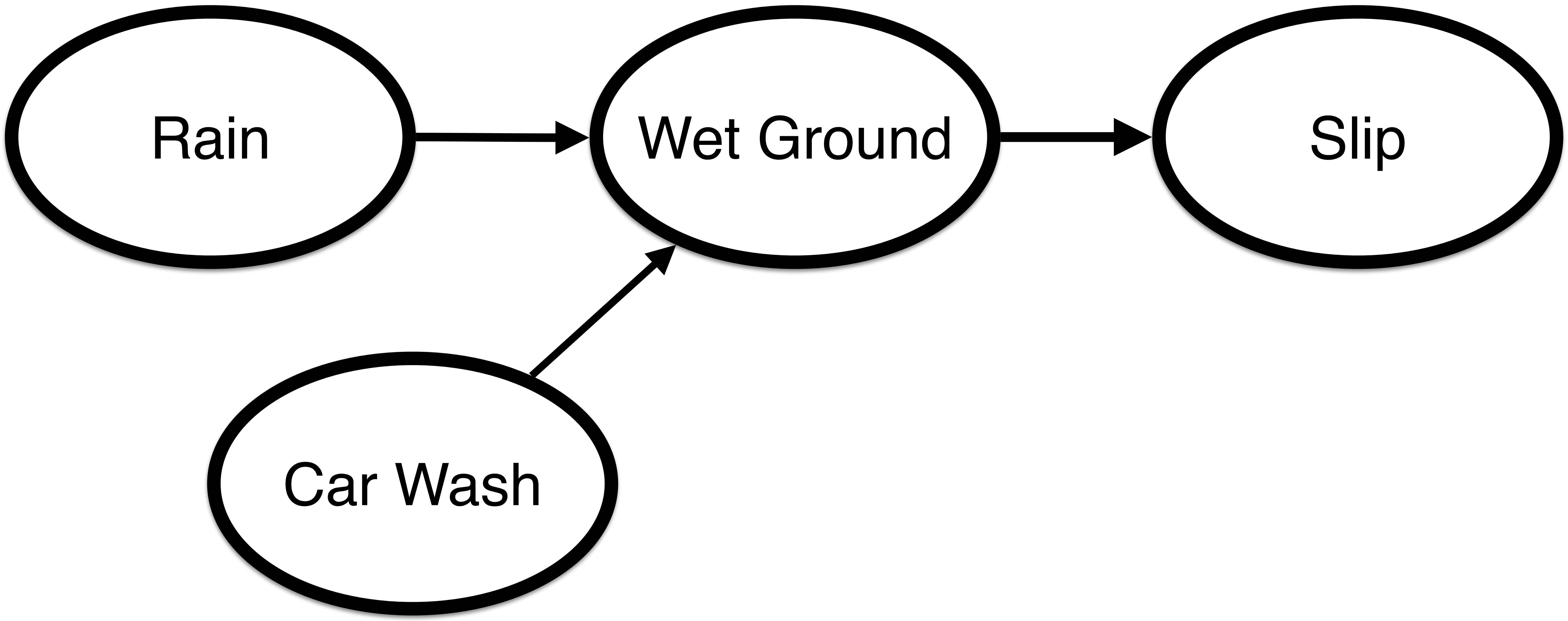

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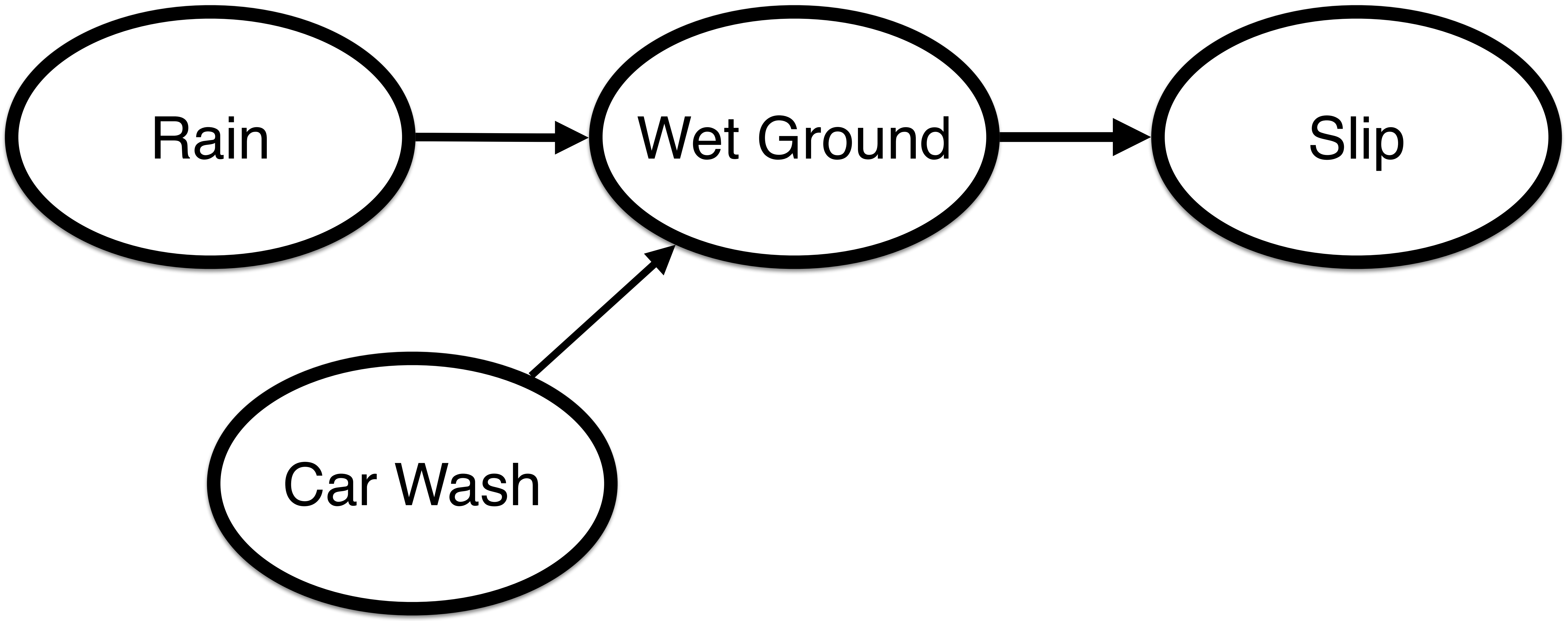

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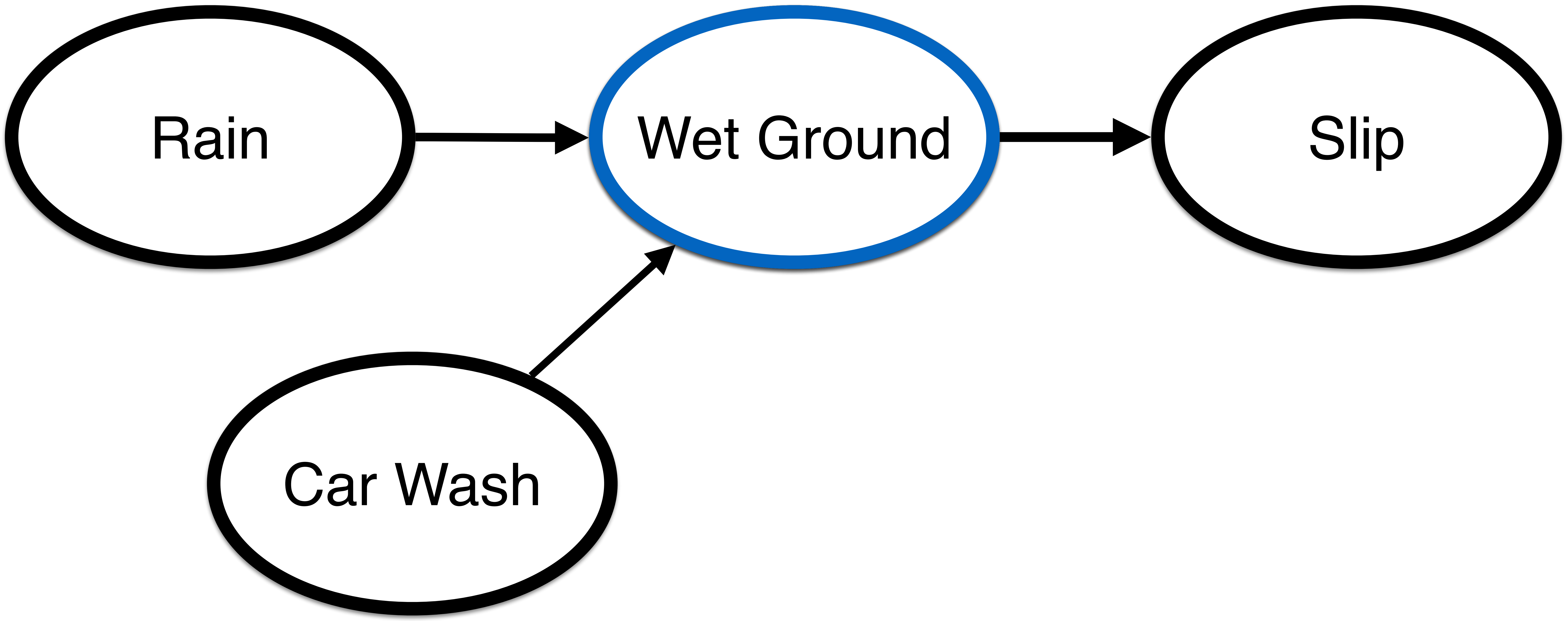

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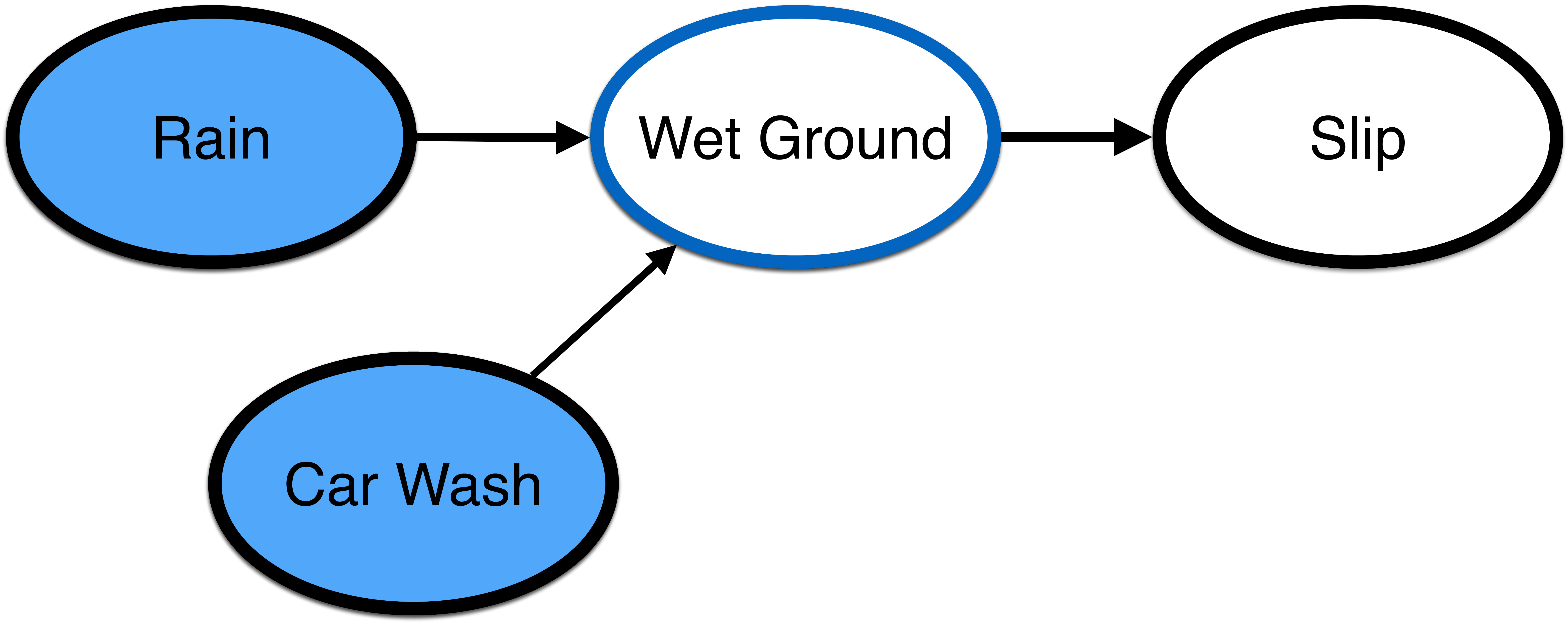

# Inference

- Given a Bayesian Network describing P(X, Y, Z), what is P(Y)
    - First approach: **enumeration**

$$P(R, W, S, C) = P(R)\, P(C)\, P(W \mid C, R)\, P(S \mid W)$$

P(R, W, S, C) = P(R) P(C) P(W | C, R) P(S | W)

$$P(r|s) = \sum_{w} \sum_{c} P(r, w, s, c) / P(s)$$

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$O(2^n)$

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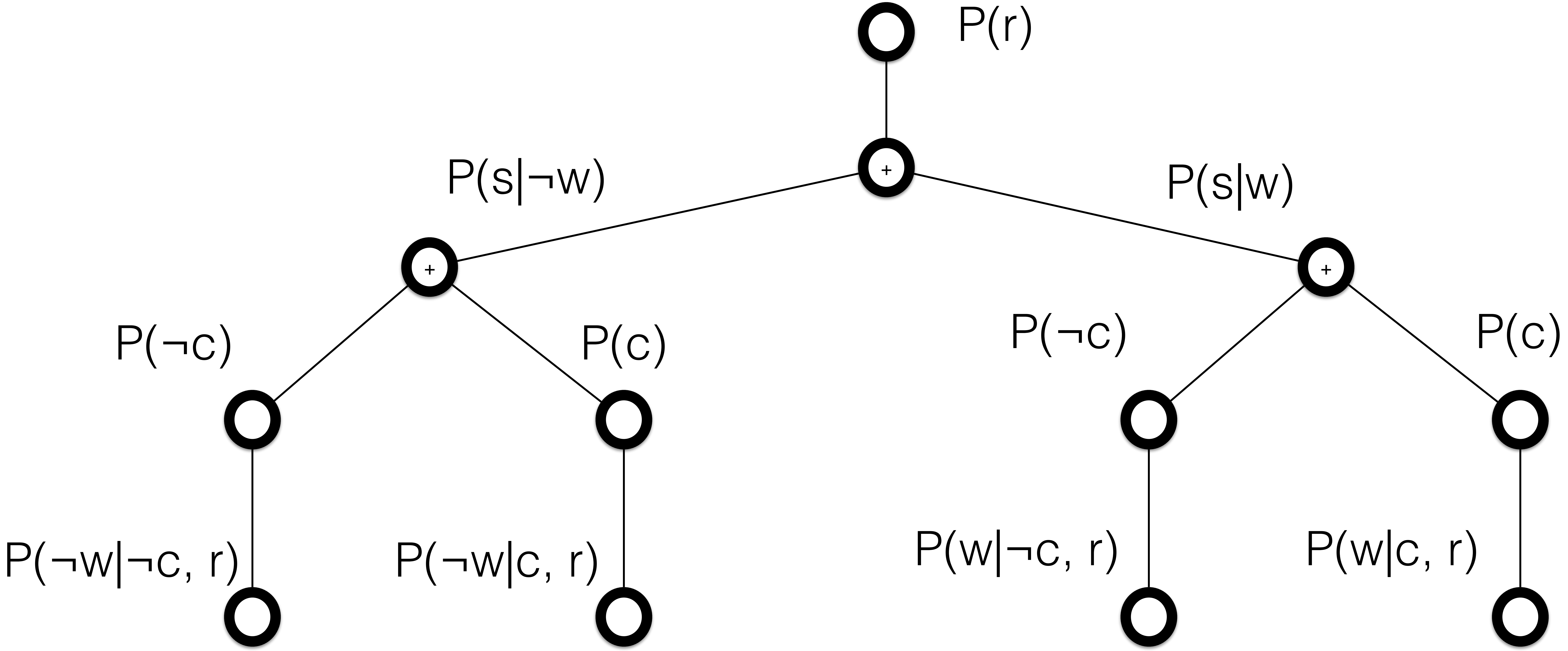

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$$P(r|s) \propto \sum_w P(r)P(s|w)f_c(w)$$

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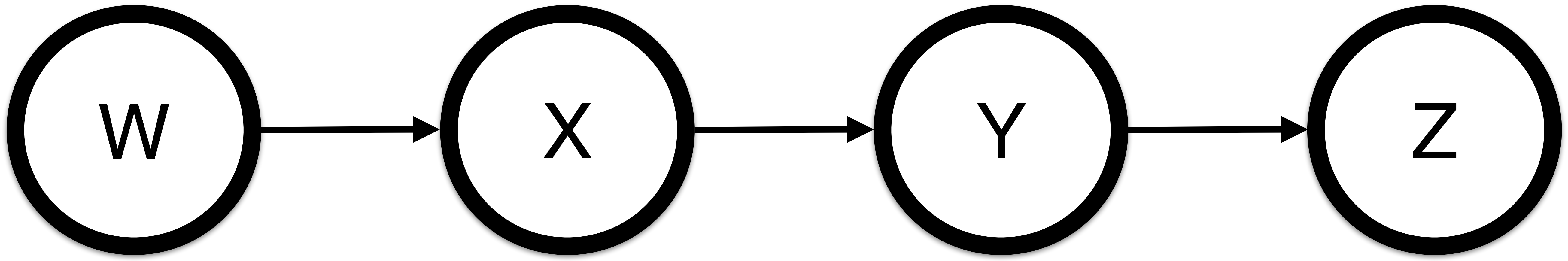


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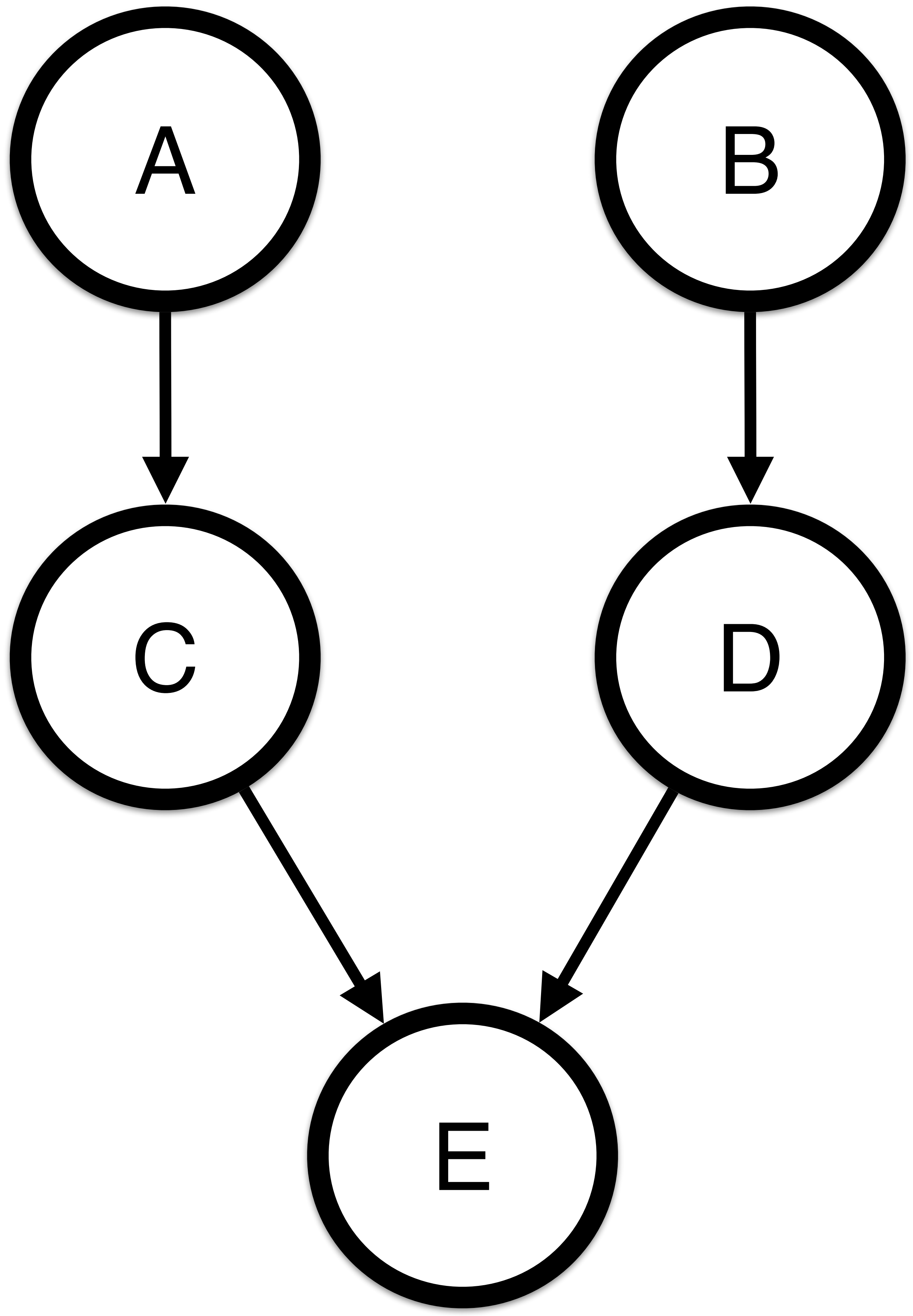
# Variable Elimination

- Every variable that is not an ancestor of a query variable or evidence variable is irrelevant to the query
- Iterate:
  - choose variable to eliminate
  - sum terms relevant to variable, generate new factor
  - until no more variables to eliminate
- Exact inference is #P-Hard
  - in tree-structured BNs, linear time (in number of table entries)

# Independence in Bayes Nets

- Each variable is conditionally independent of its **non-descendents** given its **parents**



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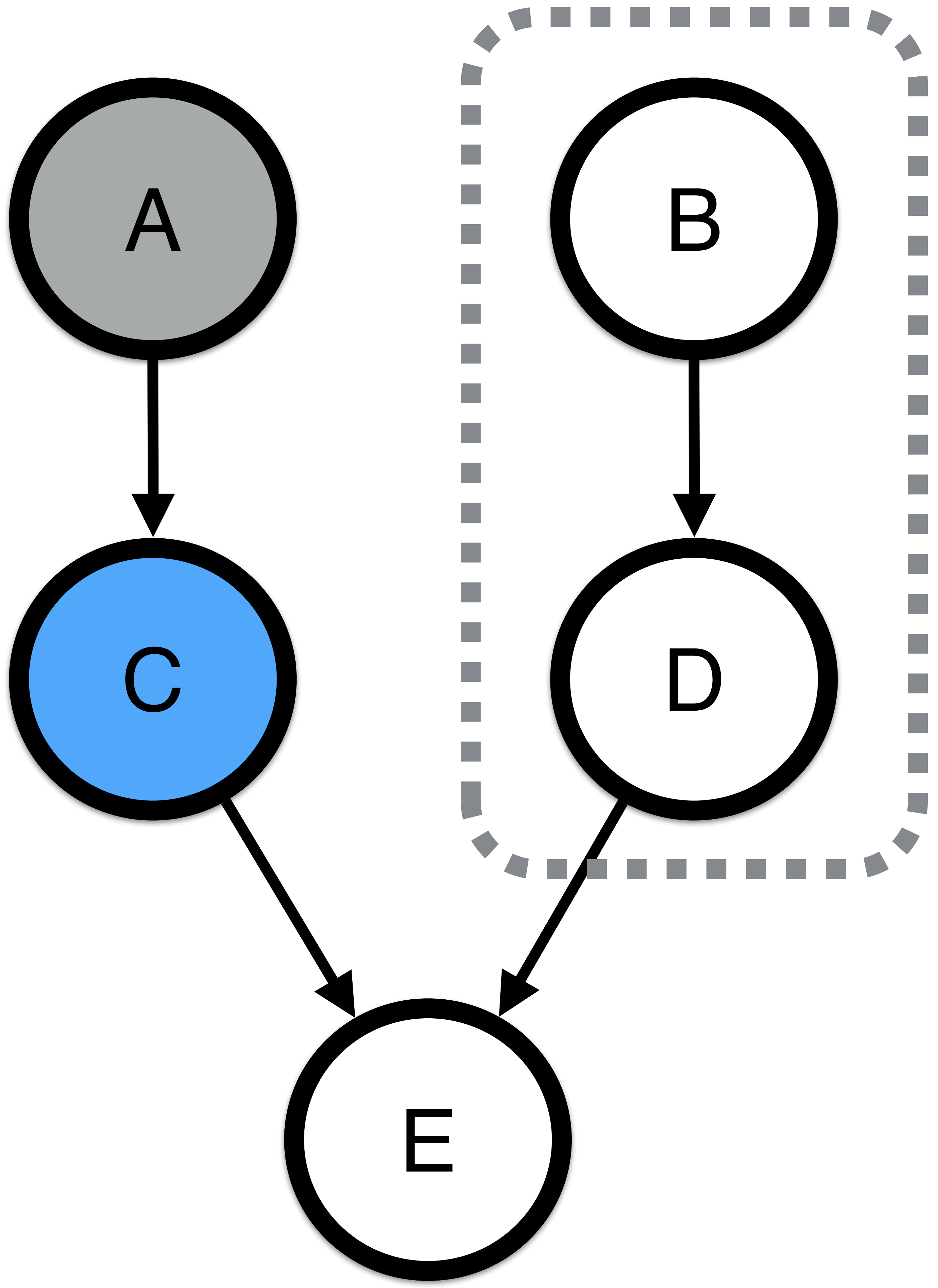

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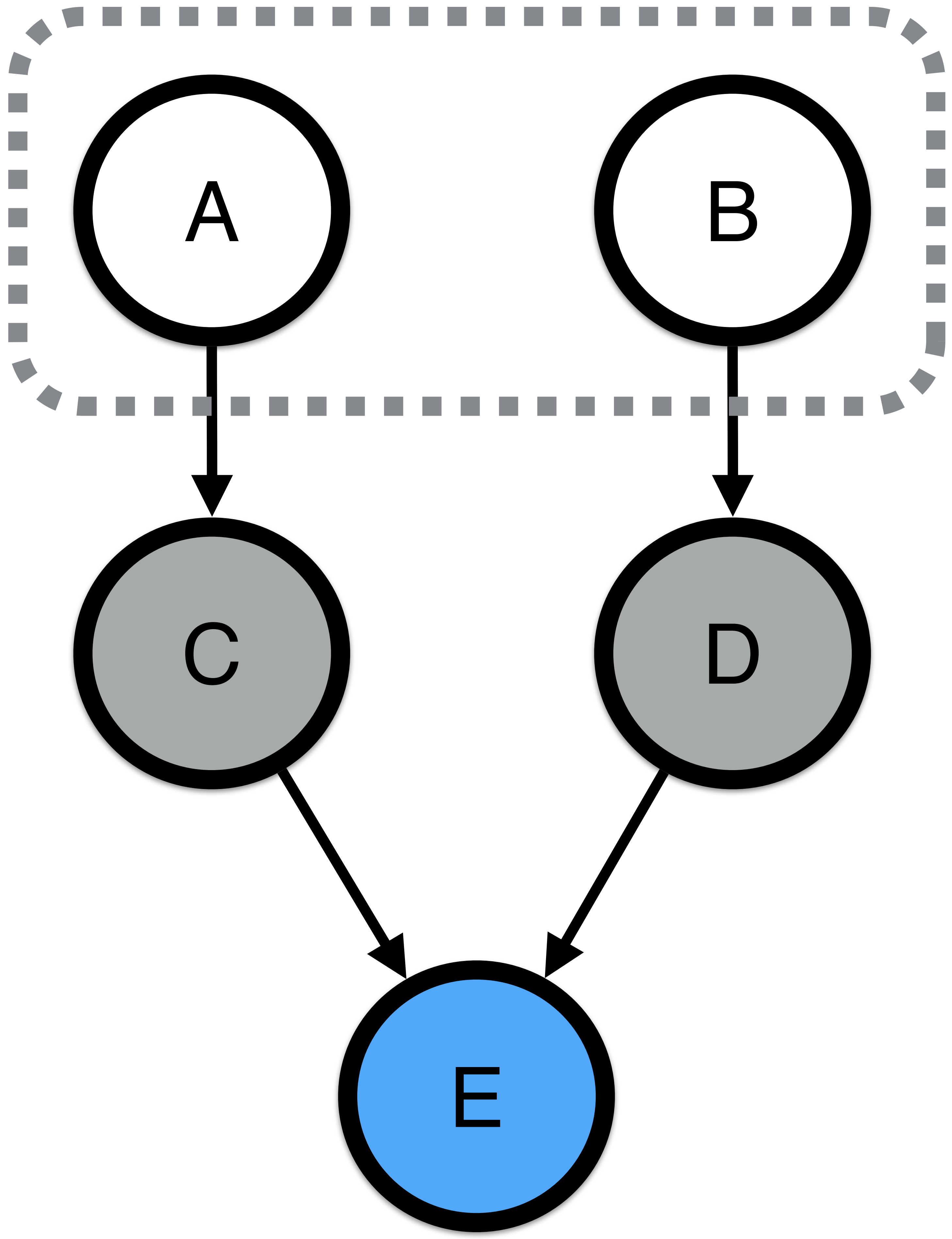

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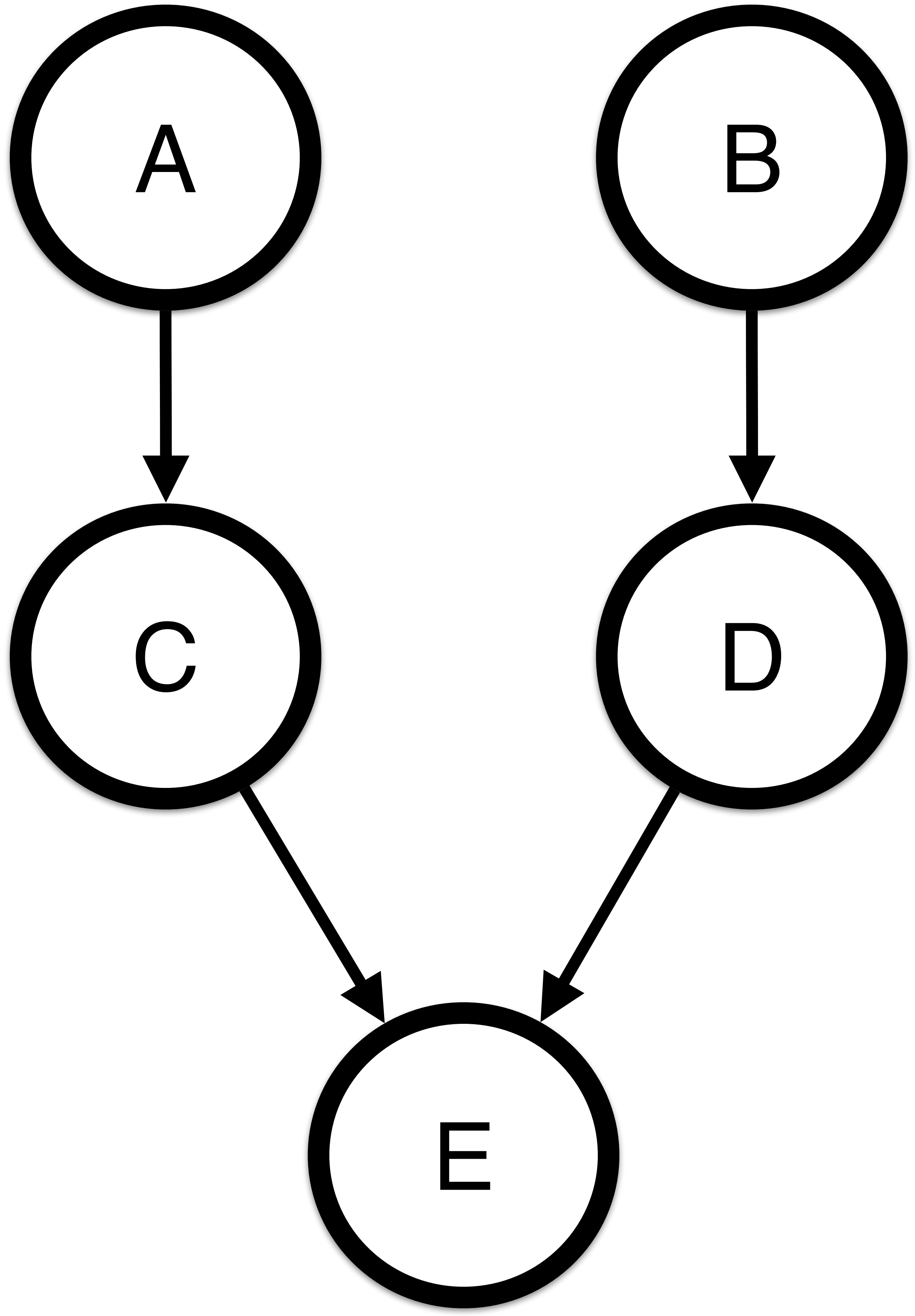
- Each variable is conditionally independent of its **non-descendents** given its **parents**
- Each variable is conditionally independent of any other variable given its **Markov blanket**
  - Parents, children, and children's parents



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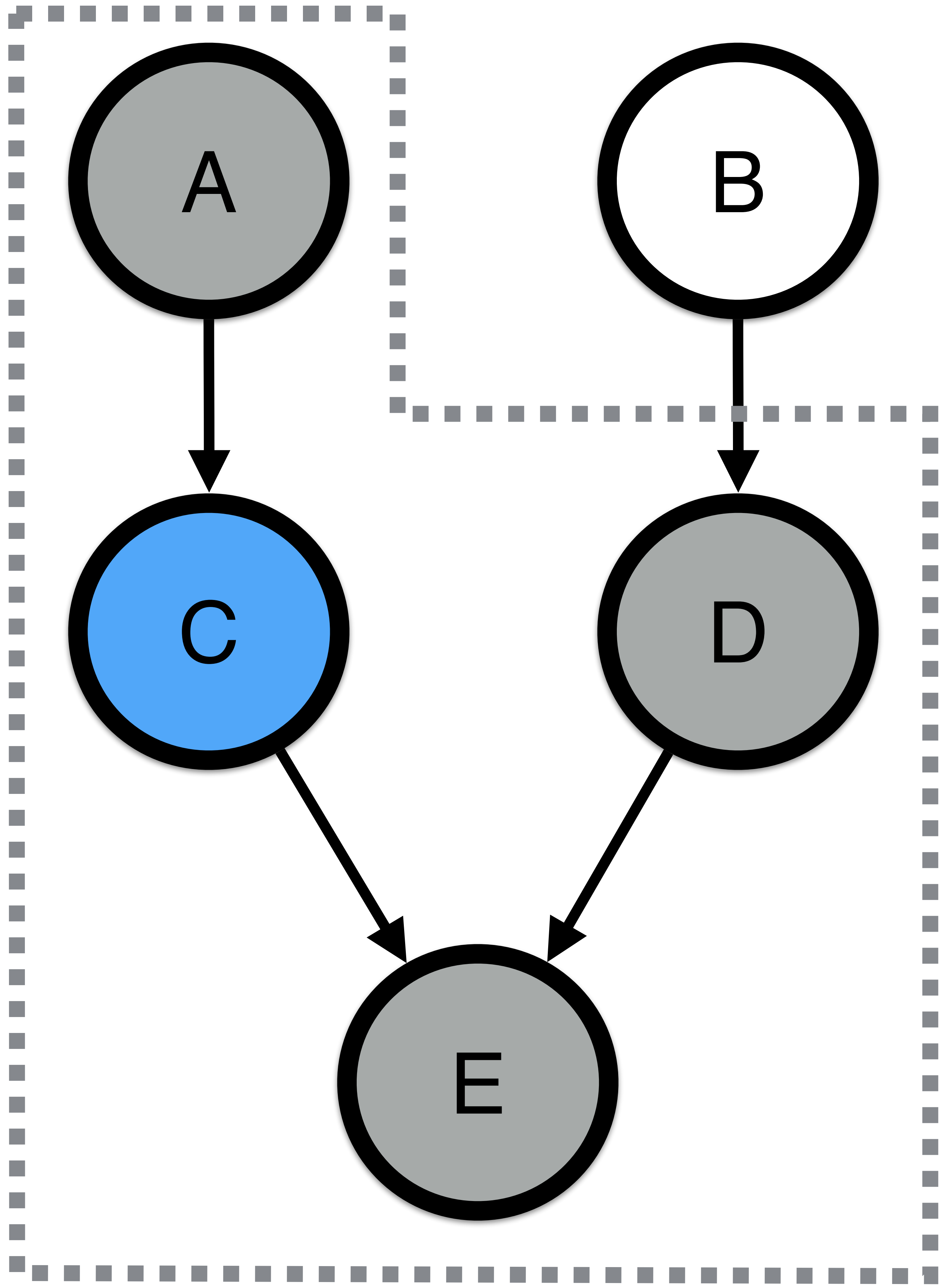
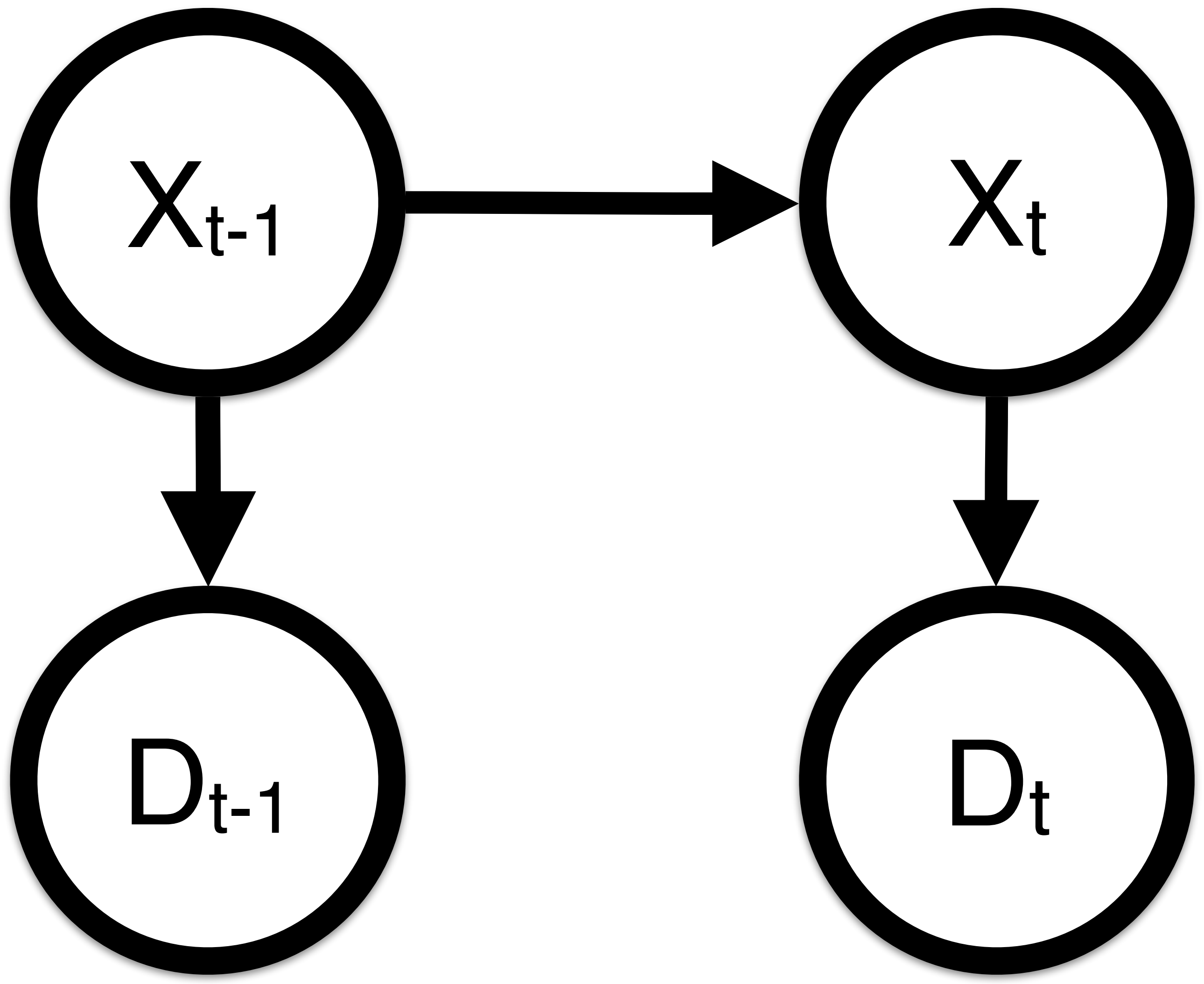


# Pacman Homework Warmup

- Given: P(ghostLocation | prevGhostLocation)
- Given: P(noisyDistance | ghostLocation)
- Goal: P(ghostLocation | noisyDistance)
  - Need: P(ghostLocation | all previous evidence)



# Reading

- Chapter 13
- Chapter 14 - 14.2, 14.4-14.4.3