Each question is worth 10 points unless otherwise specified.

In this HW you will explore the (ODE -based) model of the pandemic discussed in class, see the corresponding Mathematica notebook and other mathematica examples. The main equation here is

$$\frac{dx}{dt} = R_0(1 - \frac{x}{P})x - R_1 x, \quad x(0) = x_0. \tag{1}$$

where x is the number of people infected, R_0 is the infectivity rate, that is how many people a single person infects. $R_0 = 1.1$ for seasonal flu, and $R_0 = 5.0$ for COVID-19. $R_1 = 1.0$ is the recovery rate: one person can only recover himslef. x_0 is the initial condition at $t = 0$, that is the number of infected people at the start, for example $x(0) = x_0 = 1$. P is the population size, *e.g.* $P = 300000000$. You will always use $R_1 = 1$, but other prameters may vary. In questions 3 through 6, assume parameter values relevant to Covid-19. While doing all of the calculations within the same Mathematica code may be convenient, one may occasionally run into stubborn syntax issues. It is totally acceptable to do some of the calculations externally, e.g. output values of a function from Mathematica script into a spreadsheet, to find the maximum. Reading the answer off of a good graph is also acceptable, as along as the logic behind it is correct.

Question 1

What are the stationary points of Eq. 1, that is points x^s such that, if $x(0) = x^s$ then $x(t) = x^s = const$ – never changes? Derive analytical expressions for x^s, then show corresponding numerical plots using NDSolve with Mathematica's default settings. Hint: what does dx/dt equal to at a stationary point?

Question 2

"Herd immunity" or "HI" can be thought of as a value of x (number of infected people) such that, once reached, it does not grow anymore. Relate to the above. What % of the population is needed to reach HI for seasonal flu ($R_0 = 1.1$)? COVID-19 ($R_0 = 5.0$)?

Question 3

Suppose you are solving Eq. 1 using 1st order Euler's method. Which step size h can you recommend as reasonable guess? You may find it helpful to explore plots of $x(t)$ and dx/dt. Remember that for finding the h one is normally interested in *relative* accuracy of the solution, so you want to normalize the output by x^m – the max. value of $x(t)$. An approximate x_m will suffice. Plot $\frac{x(t)}{x^m}$, and $\frac{dx/dt}{x^m}$, and then use the analysis in the lecture notes to estimate a decent h. Now plot your solutions (without the normalization) using Euler's with the step found above, and, on the same graph, a solution using Methematica's NDsolve with its default parameters (do not specify the Method, StepSize, etc). Are these solution close to each other? To force NDSolve to use 1st order Euler's, see the example in `ODEUpdated.math`, on the class site.

Question 4

Now do a more thorough exploration of your Euler's method. Call the step found above h_0. First, halve it, and solve the equation using Euler's. Compare the solutions: you can characterize the convergence of the method by the maximum relative deviation between the two solutions $x_{h1}(t)$ and $x_{h2}(t)$: $max\{|\frac{x_{h1}(t)-x_{h2}(t)}{0.5*(x_{h1}(t)+x_{h2}(t))}|\}$ Explore the time interval from $t = 0$ to $t = 5t^*$, where t^* is the point at which dx/dt reaches its peak value. By how much does the error change as you halve h_0? What h do you think you need so that the maximum relative deviation is of an order of 10^{-6}?

Question 5

Now explore how Euler's method becomes unstable (see lecture notes): consider several $h(n) = h_0 * 2^n$, for several $n > 1$, make plots of $x(t)$. At which h do you see a clear onset of the method's instability? Show your plots.

Question 6

Suppose you are interested in the initial stages of the pandemic, that is well before time t^* at which dx/dt reaches its peak value (say, up to $0.2t^*$). Explore stability of the ODE from $t = 0$ to t^* by tweaking the initial condition – number of people infected at $t = 0$ – and watching how the solution evolves with time. (A stable ODE is the one where a small perturbation of the initial conditions results in equally small perturbation of the solution. Conversely, it is called unstable.) Combine several such curves on the same plot, show them. What are your conclusions from pure mathematical stand point (solutions converge, diverge)? Which implications these results might have from the epidemiological standpoint?