Discrete Finite Probability

In these notes, I will consider only the finite discrete case.

That is, in every situation the possible outcomes are all distinct cases, which can be modeled by integers, and there are a finite number of such outcomes.

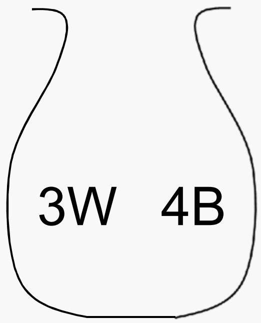
The discussion that follows was drawn from a number of sources, including:

Probability by G E Bates,
Addison-Wesley, 1965 0-201-00405-4

Puzzle-based Learning by Michalewicz & Michalewicz,
Hybrid, 2008 978-1-876462-5

Urns and Balls

Suppose we have an urn that contains 3 white balls and 4 black balls:



Now suppose we shake the urn to randomly (i.e., unpredictably) mix the balls together and then we draw out one ball.

How likely is it that we will draw out a white ball?

Suppose, instead, we draw out two balls.

How likely is it that we will draw out one white ball and one black ball?

Basic Concepts

A *probabilistic experiment* is any action (e.g., tossing a coin, rolling a die, drawing a ball from an urn) where the outcome of the action is not known in advance.

Such an experiment can lead to any of a number of results, called *outcomes*.

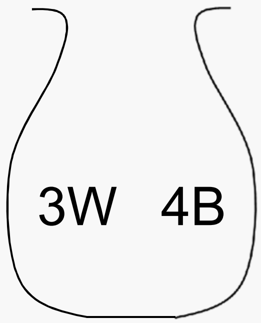
The set of all possible outcomes is called the *sample space*.

A subset of the sample space is called an *event*.

A subset of the sample space that contains a single outcome is called an *elementary event* or simply an *outcome*.

Urns and Balls

Suppose we have an urn that contains 3 white balls and 4 black balls:



Our first experiment is to draw a single ball from the urn.

There are 7 possible outcomes, since we could draw any of 3 white balls or any of 4 black balls, so the sample space is:

$$\{w_1, w_2, w_3, b_1, b_2, b_3, b_4\}$$

The event of interest is that we draw a white ball:

$$\{w_1, w_2, w_3\}$$

Naïve Probability

The likelihood that an outcome actually occurs is its *probability*.

An outcome that is impossible (e.g., drawing a green ball in our experiment) is assigned the probability 0 (or 0%).

An outcome that is inevitable (e.g., getting "heads" when flipping a two-headed coin) is assigned the probability 1 (or 100%).

Of course, most interesting outcomes are neither impossible nor inevitable; those outcomes have probabilities that are larger than 0 and less than 1.

But, how do we determine the probability of an outcome?

First, consider the simplest case: every possible outcome is just as likely to occur as any other possible outcome.

In this case, we say the outcomes are *equally likely* (or *equiprobable*).

Equally-likely Outcomes

If the balls in our urn are truly mixed in an unpredictable (or random) manner, it seems reasonable to declare that each ball is just as likely to be drawn as any other ball.

Since some outcome must occur, the sum of the probabilities of all the outcomes must equal 1.

So, the probability that any particular ball will be drawn from the urn is 1/7.

Probability of an Event

The probability of an event is the sum of the probabilities of the outcomes that belong to that event.

So, if the event of drawing a white ball is:

$$DrawWhite = \{w_1, w_2, w_3\}$$

then the probability of drawing a white ball is:

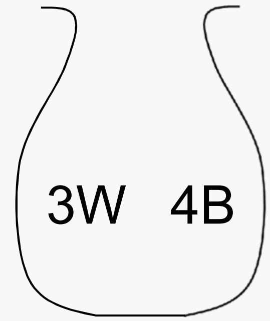
$$P(DrawWhite) = P(w_1)+P(w_2) +P(w_3) = 1/7 + 1/7 + 1/7 = 3/7$$

Notice that if the outcomes are equally-likely then the probability of the event is just the number of outcomes in the event divided by the number of outcomes in the sample space.

The Second Experiment

Suppose we draw out two balls.

How likely is it that we will draw out one white ball and one black ball?



We need a different model:

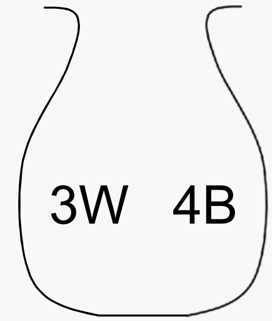
- The possible outcomes are pairs of balls.
- There are a lot of such pairs:

$\{w_1, w_2\}$ $\{w_1, w_3\}$ $\{w_2, w_3\}$
$\{w_1, b_1\}$ $\{w_1, b_2\}$ $\{w_1, b_3\}$ $\{w_1, b_4\}$ $\{w_2, b_1\}$
$\{w_2, b_2\}$ $\{w_2, b_3\}$ $\{w_2, b_4\}$
$\{w_3, b_1\}$ $\{w_3, b_2\}$ $\{w_3, b_3\}$ $\{w_3, b_4\}$
$\{b_1, b_2\}$ $\{b_1, b_3\}$ $\{b_1, b_4\}$
$\{b_2, b_3\}$ $\{b_2, b_4\}$
$\{b_3, b_4\}$

- The outcomes are equally-likely.

The Second Experiment

Suppose we draw out two balls.



How likely is it that we will draw out one white ball and one black ball?

The event is:

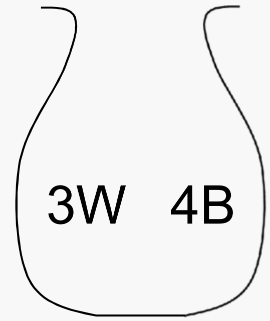
$$\{ \{w_1, b_1\}, \{w_1, b_2\}, \{w_1, b_3\}, \{w_1, b_4\}, \{w_2, b_1\}, \{w_2, b_2\}, \{w_2, b_3\}, \{w_2, b_4\}, \{w_3, b_1\}, \{w_3, b_2\}, \{w_3, b_3\}, \{w_3, b_4\} \}$$

So, the probability of drawing one white ball and one black ball is:

$$P(1W1B) = 12/21 = 4/7$$

QTP: what is the probability of drawing two black balls?

The Third Experiment



Suppose we draw out two balls, <u>one at a time</u>.

Does that make any difference?

Sure. Now the sample space is the set of all ordered pairs of balls:

(w_1, w_2) (w_2, w_1) (w_2, w_3) (w_3, w_2)

. . .

So, we now have two different possible outcomes for each one we had when the balls were drawn together.

Interestingly, this doesn't change the probabilities of events that make sense in both experiments:

$$P(1W1B) = 24/42 = 4/7$$

QTP: what new events would this experiment entail?

Recap So Far

Given an experiment with possible outcomes $e_1, e_2, \ldots, e_n$:

- The set of all possible outcomes is the sample space.
- There is a probability function p() that assigns a real number to each of the possible outcomes in the sample space.
- For each outcome, e_k, $0 <= p(e_k) <= 1$.
- $p(e_1) + p(e_2) + p(e_3) + . . . + p(e_n) = 1$
- An event is a subset of the sample space.
- If E is an event, then the probability of E is the sum of the probabilities of the individual outcomes in E.

A Pair of Bears

You see two bears, one black and one white, and wonder:

- What is the probability that both bears are males?
- What is the probability that both bears are males, IF you are correctly told that one of them is male?
- What is the probability that both bears are males, IF you are correctly told that the white bear is male?

Now, we cannot proceed unless we know something about bears: if we randomly select a bear from among all bears, what is the probability that bear will be male?

In the absence of any experts on the family Ursidae, we must make an assumption… we'll assume the probability our random bear is male is1/2.

A Pair of Bears – Q1

You see two bears, one black and one white, and wonder:

- What is the probability that both bears are males?

Our experiment is to determine the genders of the two bears.

The possible outcomes (listing the black bear first) are:

FF FM MF MM

Now, under our assumption, each of these outcomes is equally-likely, and therefore the answer to our first question is 1/4.

QTP: what is the probability that the bears are not both males?

A Pair of Bears

You see two bears, one black and one white, and wonder:

- What is the probability that both bears are males, IF you are correctly told that one of them is male?

- What is the probability that both bears are males, IF you are correctly told that the white bear is male?

Do you see, intuitively, why these two questions lead to different answers?

A Pair of Bears – Q2

You see two bears, one black and one white, and wonder:

- What is the probability that both bears are males, IF you are correctly told that (at least) one of them is male?

Given the fact that (at least) one of the bears is male, we find that the sample space has changed:

FM MF MM

Again, under our assumption these are equally-likely, and so the probability that both bears are males is 1/3.

QTP: what if we were told that at most one of the bears was male?

A Pair of Bears – Q3

You see two bears, one black and one white, and wonder:

- What is the probability that both bears are males, IF you are correctly told that the white bear is male?

Now, we see that the sample space is:

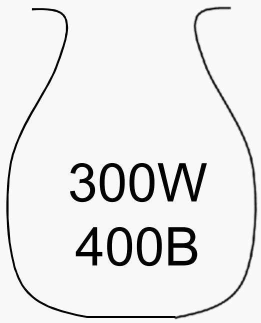
FM MM

(Remember we are listing the black bear first.)

So, now the probability both bears are males is 1/2.

A Challenge

Suppose we have an urn that contains 300 white balls and 400 black balls:



Suppose we draw 2 balls. The probability that we get 1 black ball and 1 white ball is:

$$\frac{\binom{400}{1}\binom{300}{1}}{\binom{700}{2}} = \frac{400 \cdot 300}{(700 \cdot 699)/2} \approx 0.49$$

Analysis

Here are the details of that last example:

Note the use of the Multiplication Rule here

$$\frac{\begin{pmatrix}\text{\# ways to choose} \\ \text{1 black ball}\end{pmatrix}\begin{pmatrix}\text{\# ways to choose} \\ \text{1 white ball}\end{pmatrix}}{\begin{pmatrix}\text{\# ways to choose 2 balls} \\ \text{w/o restrictions}\end{pmatrix}} = \frac{\binom{400}{1}\binom{300}{1}}{\binom{700}{2}}$$

$$\binom{400}{1} = \frac{400!}{1!(400-1)!} = \frac{400!}{399!} = 400$$

$$\binom{700}{2} = \frac{700!}{2!(700-2)!} = \frac{700!}{2 \cdot 698!} = \frac{700 \cdot 699}{2} = 244650$$

Examples

With an urn that contains 300 white balls and 400 black balls:

Suppose we draw 5 balls. The probability that we get 2 black balls and 3 white balls is:

$$\frac{\binom{400}{2}\binom{300}{3}}{\binom{700}{5}} = \frac{\frac{400!}{2!398!}\,\frac{300!}{3!297!}}{\frac{700!}{5!695!}} = \frac{\frac{400\cdot 399}{2}\,\frac{300\cdot 299\cdot 298}{6}}{\frac{700\cdot 699\cdot 698\cdot 697\cdot 696}{120}}$$

$$= \frac{200\cdot 399\cdot 50\cdot 299\cdot 298}{35\cdot 699\cdot 698\cdot 697\cdot 116} \approx 0.257$$

Examples

With an urn that contains 300 white balls and 400 black balls:

Suppose we draw 5 balls. What's the probability that we get at least 3 black balls?

The key question is: in how many ways can we choose 5 balls and include at least 3 black balls?

So, we have three distinct (nonoverlapping) cases:

- 3B and 2W
- 4B and 1W
- 5B and 0W

Note the use of the Addition Rule here

$$\binom{400}{3}\binom{300}{2}+\binom{400}{4}\binom{300}{1}+\binom{400}{5}\binom{300}{0}$$

Examples

So, the probability that we get at least 3 black balls is:

$$\frac{\binom{400}{3}\binom{300}{2}+\binom{400}{4}\binom{300}{1}+\binom{400}{5}\binom{300}{0}}{\binom{700}{5}}$$

And, the probability that we get fewer than 3 black balls is:

$$1-\frac{\binom{400}{3}\binom{300}{2}+\binom{400}{4}\binom{300}{1}+\binom{400}{5}\binom{300}{0}}{\binom{700}{5}}$$

A Puzzle

There are three cards in a bag.

One card has the symbol **X** written on both sides; one card has the symbol **O** written on both sides; one card has the symbol **X** on one side and the symbol **O** on the other side.

You draw one card at random and examine one side of the card.

You see the symbol **X** on that side of the card.

What is the probability that there is also an **X** on the other side?

Solution

There are three cards in the bag:

XX **OO** **XO**

Now, we know we did not draw the card **OO**.

So, the probability is 1/2, right?

Nope. (Although we're right to eliminate **OO**.)

We could be seeing any of three different **X**s:

XX **XO**

If we're seeing either of the first two, then there's an **X** on the other side.

If we're seeing the third **X**, there's an **O** on the other side.

So, the probability is 2/3, not 1/2.

Solution

A clearer way of saying this is that we are thinking of the sample space (the set of all possible outcomes) incorrectly:

XX **XO**

We are looking at one side of a card, and we know that's an **X**.

A precise description of the sample space is that when we have the following possible cases:

X on front of another **X**

X on back of another **X**

X on front of an **O**

So, in 2 cases out of 3, we're looking at the card with 2 **X**'s.

Expectation

Suppose we have a probabilistic experiment that has a sample space

$$\{e_1, e_2, \ldots, e_N\}$$

and that for each outcome in the sample space we have a probability $p(e_K)$.

And, suppose that corresponding to each outcome in the sample space we have an associated value v_K.

Then the *expectation* or *expected value* of the experiment is the weighted average of the values, using the outcome probabilities as weights:

$$\sum_{K=1}^{N} p(e_K) v_K$$

Example

A game is played by drawing balls out of an urn; the players gain $2 when a green ball is drawn and lose $1 when a red ball is drawn.

Suppose the urn contains 30 green balls and 70 red balls.

What is the expected payoff for a round of this game?

Solution

The probability of drawing a green ball is 0.3 and the probability of drawing a red ball is 0.7.

The value of drawing a green ball is +2 and the value of drawing a red ball is -1.

So, the expectation for a single draw would be

$$0.3 * 2 + 0.7 * -1 = -0.1$$

Hence, you'd expect, on average, to lose 10 cents each time you play the game.