Addition Rule

Suppose you are supposed to select and carry out one of a collection of N tasks, and there are T_K different ways to carry out task K.

Then the number of different ways to select and carry out one of the tasks is just the sum of the numbers of ways to carry out the individual tasks.

That is, the number of ways to complete the sequence of tasks is:

$$\sum_{K=1}^{N} T_K = T_1 + T_2 + \cdots + T_N$$

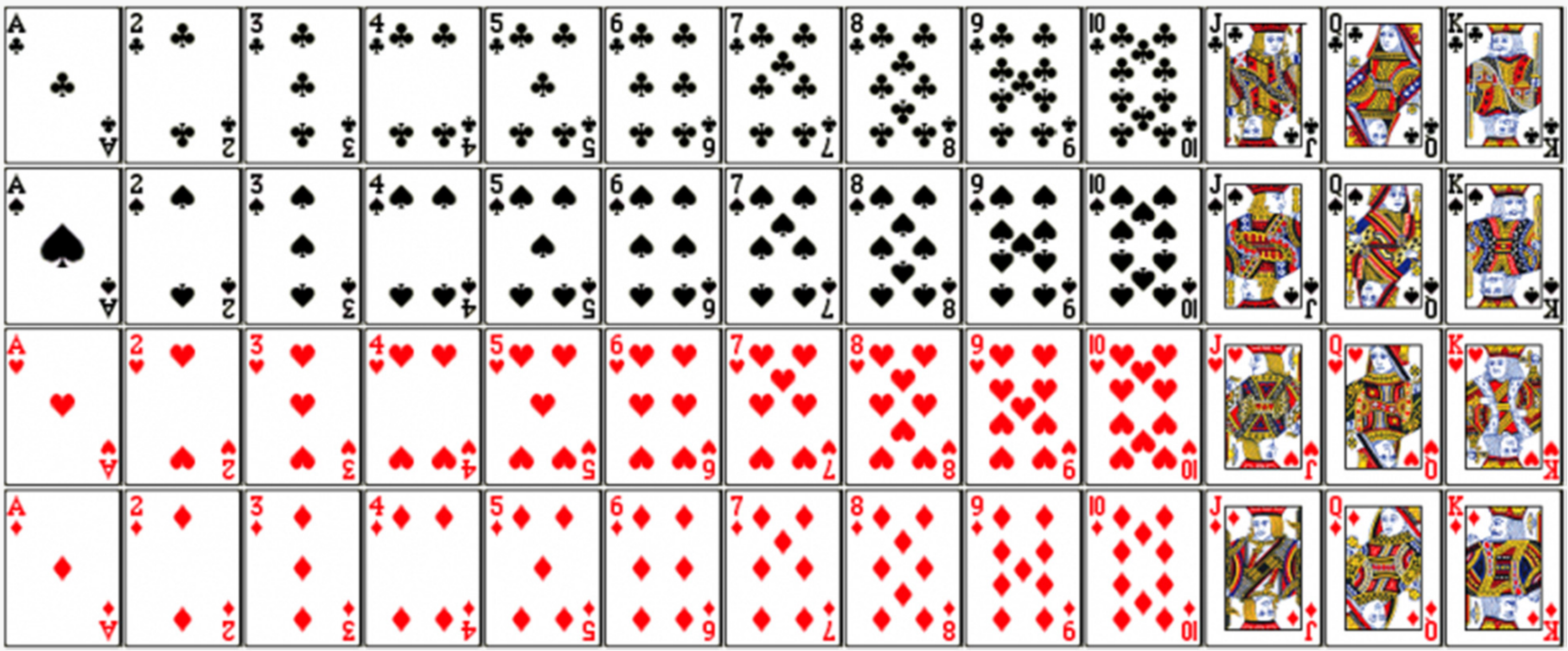
Poker Deck

A standard poker deck:

contains 52 cards,

divided into 4 suites (spades, clubs, hearts, diamonds),

with 13 values in each suite (Ace, 2, 3, …, 9, 10, Jack, Queen, King)



Example

In how many different ways could you lay out a Jack or a Heart?

You have:

4 choices for a Jack

13 choices for a Heart

So, the number of ways to lay out a Jack or a Heart would be:

$$4 + 13 = 17$$

Multiplication Rule

Suppose you are supposed to carry out a sequence of *N* tasks, and there are T_K different ways to carry out task *K* no matter how the other tasks are carried out.

Then the number of different ways to complete the sequence of tasks is just the product of the numbers of ways to carry out the individual tasks.

That is, the number of ways to complete the sequence of tasks is:

$$\prod_{K=1}^{N} T_K = T_1 \cdot T_2 \cdot \cdots \cdot T_N$$

Example

Suppose you have a standard poker deck.

In how many different ways could you lay out a sequence of 5 cards:

You have:

52 choices for card 1
51 choices for card 2 (no matter what card 1 was)
50 choices for card 3
49 choices for card 4
48 choices for card 5

So, the number of ways to lay out the sequence would be:

$$52 \times 51 \times 50 \times 49 \times 48 = 311{,}875{,}200$$

Counting Permutations

Suppose you are given a collection of *N* different objects (you can tell them apart).

Then an arrangement of all *N* of the objects in a row is called a *permutation* of the set of objects.

The number of different permutations of *N* things equals:

$$\prod_{K=1}^{N} K = 1 \cdot 2 \cdot \cdots \cdot N = N!$$

The last expression is read "*N* factorial", and is very convenient shorthand for the expressions that precede it.

Counting Permutations

Suppose you are given a collection of *N* different objects (you can tell them apart).

Then an arrangement of a subset of *R* of the objects in a row is called a *permutation of the N objects, taken R at a time*.

The number of different permutations of *N* things taken *R* at a time equals:

$$P(N,R) = N \cdot (N-1) \cdots (N-R+1) = \frac{N!}{(N-R)!}$$

Counting Combinations

Suppose you are given a collection of *N* different objects (you can tell them apart).

Then a selection of a subset of *R* of the *N* objects is called a *combination of the N objects taken R at a time*.

The number of different combinations of *N* things taken *R* at a time equals:

$$C(N,R) = \frac{P(N,R)}{R!} = \frac{N!}{R!(N-R)!}$$

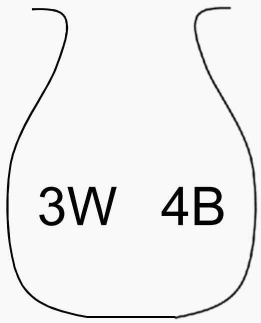
This makes intuitive sense because there are *R!* permutations of *R* things, and each of those corresponds to the same combination.

The more common notation is:

$$C(N,R) = \binom{N}{R}$$

Urns and Balls

Suppose we have an urn that contains 3 white balls and 4 black balls:



Our first experiment is to draw a single ball from the urn.

There are 7 possible outcomes, since we could draw any of 3 white balls or any of 4 black balls, so the sample space is:

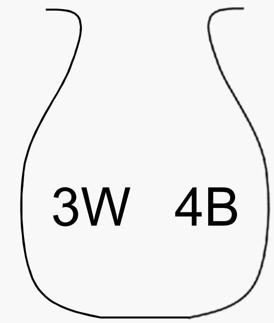
$$\{w_1, w_2, w_3, b_1, b_2, b_3, b_4\}$$

The event of interest is that we draw a white ball:

$$\{w_1, w_2, w_3\}$$

Example

Suppose we draw out two balls.



How many ways are there to draw out one white ball and one black ball?

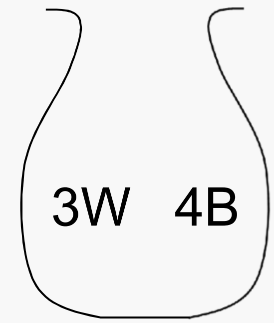
The possible outcomes are pairs of balls.

There are a lot of such pairs:

$\{w_1, b_1\}$	$\{w_1, b_2\}$	$\{w_1, b_3\}$	$\{w_1, b_4\}$
$\{w_2, b_1\}$	$\{w_2, b_2\}$	$\{w_2, b_3\}$	$\{w_2, b_4\}$
$\{w_3, b_1\}$	$\{w_3, b_2\}$	$\{w_3, b_3\}$	$\{w_3, b_4\}$

Applying Counting Formulas

Suppose we draw out two balls.



How many ways are there to draw out one white ball and one black ball?

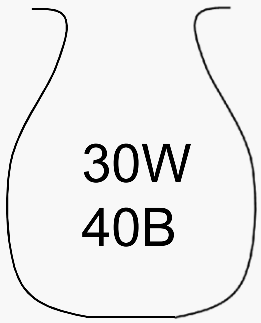
We need a different approach:

- choose one white ball
- choose one black ball

$$\binom{3}{1}\binom{4}{1}$$

More Choices

Suppose we have an urn that contains 30 white balls and 40 black balls:



We reach into the urn and draw out 10 balls.

In how many different ways could we get 6 white balls and 4 black balls?

$$\binom{30}{6}\binom{40}{4}$$

Card Example

A *full house* is a poker hand that consists of 3 cards of one value and 2 cards of another value.

For example:

3ofHearts, 3ofClubs,3ofDiamonds,JackofSpades,JackofClubs

So, how many different full houses are there? (Not all at once.)

Number of Full Houses

We can construct a full house by carrying out the following sequence of tasks:

- choose the value for the three-of-a-kind
- choose 3 cards of that value
- choose a different value for the pair
- choose 2 cards of that value

Hence, using the Multiplication Rule:

$$\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}$$

Number of Two Pairs

This one's tricky. If you pick the values for the two pairs separately, you will double-count:

- choose a value for one pair
- choose 2 cards of that value
- choose a different value for the other pair
- choose 2 cards of that value
- choose a different value for the fifth card
- choose 1 card of that value

$$\binom{13}{1}\binom{4}{2}\binom{12}{1}\binom{4}{2}\binom{11}{1}\binom{4}{1}$$

That's incorrect… but the error is subtle.

Number of Two Pairs

This one's tricky. If you pick the values for the two pairs separately, you will double-count:

- choose a value for one pair
- choose 2 cards of that value
- choose a different value for the other pair
- choose 2 cards of that value
- choose a different value for the fifth card
- choose 1 card of that value

This logic would treat this sequence of choices

3ofHearts, 3ofClubs,5ofDiamonds,5ofSpades,JackofClubs

as being different from this sequence

5ofDiamonds,5ofSpades, 3ofHearts, 3ofClubs,JackofClubs

Number of Two Pairs

What if you try to pick the value for the higher pair first, then pick the value for the lower pair?

- choose a value for the higher pair
- choose 2 cards of that value
- choose a lower value for the other pair
- choose 2 cards of that value
- choose a different value for the fifth card
- choose 1 card of that value

$$\binom{12}{1}\binom{4}{2}?$$

The problem is that we don't know how many possible values we have left to choose from unless we know what the higher value was…

Number of Two Pairs

Two pairs consists of two cards of one value, two cards of another value, and a fifth card of a third value.

We can construct such a hand by:

- choose the values for the two pairs
- choose 2 cards of the higher value
- choose 2 cards of the lower value
- choose a different value for the fifth card
- choose 1 card of that value

Hence, using the Multiplication Rule:

$$\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{11}{1}\binom{4}{1}$$