READ THIS NOW!

- Print your name in the space provided below.
- There are 6 short-answer questions, priced as marked. The maximum score is 100.
- The grading of each question will take into account whether you obtained a correct solution and how well you presented your analysis and justified your logic. In most cases, as much weight will be given to the presentation and explanation of your analysis as to whether the solution is fully correct. Legibility will be strongly considered in the grading. You may use scratch paper to work out your solution before finalizing it on the exam.
- Externalize! Whether it's a drawing, a table, an equation or something else, externalize! And make the externalization explicit in your answer! Label things for clarity!
- You may use the supplied extra paper for scratch work. Write your name on any scratch work sheets you use and turn those in with your exam.
- All final answers must be written on the test form itself.
- When you have finished, sign the pledge at the bottom of this page and turn in the test.
- This examination is closed book and closed notes, aside from the permitted one-page formula sheet. Your fact sheet may contain definitions and examples, but it may not contain questions and/or answers taken from old tests or homework.
- No calculators or other computing devices may be used. The use of any such device will be interpreted as an indication that you are finished with the test and your test form will be collected immediately.
- Until solutions are posted, you may not discuss this examination with any student who has not taken it.
- Failure to adhere to any of these restrictions is an Honor Code violation.

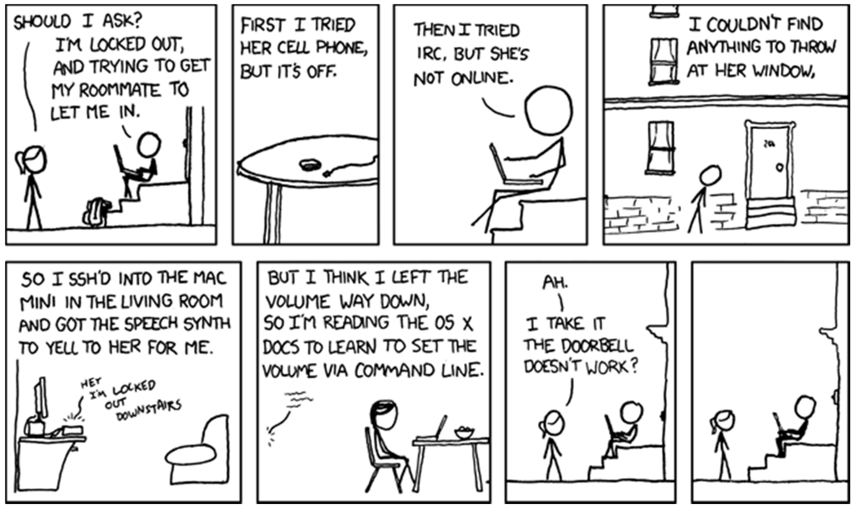
Name (Last, First) Solution

printed

Pledge: On my honor, I have neither given nor received unauthorized aid on this examination.

____

signed


SHOULD I ASK?
I'M LOCKED OUT, AND TRYING TO GET MY ROOMMATE TO LET ME IN.
FIRST I TRIED HER CELL PHONE, BUT IT'S OFF.
THEN I TRIED IRC, BUT SHE'S NOT ONLINE.
I COULDN'T FIND ANYTHING TO THROW AT HER WINDOW,
SO I SSH'D INTO THE MAC MINI IN THE LIVING ROOM AND GOT THE SPEECH SYNTH TO YELL TO HER FOR ME.
HEY I'M LOCKED OUT DOWNSTAIRS
BUT I THINK I LEFT THE VOLUME WAY DOWN, SO I'M READING THE OS X DOCS TO LEARN TO SET THE VOLUME VIA COMMAND LINE.
AH.
I TAKE IT THE DOORBELL DOESN'T WORK?

xkcd.com

1. (The following problem is adapted from Raymond Smullyan.) You must provide a clear explanation of the logic whereby you arrived at your answers to the two questions.

You are visiting an island that is inhabited by two clans: the knights and the knaves. Knights always tell the truth and knaves always lie. It is impossible to tell which clan a native belongs to, based upon his or her appearance. Moreover, some of the inhabitants of the island are werewolves, and may turn into wolves at night and eat you. Werewolves can be either knights or knaves.

One afternoon, you find yourself alone with three natives whom you do not know. Since we cannot trust what they say about their own names, we will call them A, B and C. However, you do have reliable information that exactly one of them is a werewolf. Each of the natives has invited you to travel to his or her home that night for dinner. During your conversation, the following statements are made:

A: C is a werewolf.
B: I am not a werewolf.
C: At least two of us are knaves.

a) [8 points] Is the werewolf a knight or a knave? Justify your conclusion in the space provided below.

The werewolf must be a knave.

b) [8 points] You must travel with one of these natives for dinner at his or her home tonight. Naturally, you wish to avoid travelling with a werewolf, even if the alternative is to travel with a knave. Which of the natives should you go with? Justify your conclusion in the space provided below.

You should travel with A.

Space for justifications:

If C is a knight, then there must be two knaves, A and B, and so they are both lying. Hence C is NOT a werewolf and B IS a werewolf. And so A is NOT a werewolf since there is only one.

So, in this case, the werewolf is a knave.

If C is a knave, then there are fewer than two knaves, which means there is only one knave, C. And, since A and B must both be knights, and therefore are telling the truth, C must be a werewolf and B must not be a werewolf. And again, A is NOT a werewolf.

So, in this case, the werewolf is also a knave.

The only person guaranteed to NOT be a werewolf is A. That makes A the safe choice to travel with, even if A might be a knave.

2. [20 points] Five cousins, Grant, Ryan, Charlie, Jack and Ollie, went to the candy store. Jack bought 6 of the 28 items they purchased collectively. Grant's purchases included 1 box of Nards, two Chocky Bars, 4 packs of Snittles, and one Snackers Bar. Charlie's purchases included Nards, Spanish Fish, a Chocky Bar, and a pack of Snittles. Jack purchased a number of items, including 3 boxes of Nards, 2 Chocky Bars, and 1 Snackers Bar. The cousins bought a total of 7 boxes of Nards. Ollie got two items: 1 bag of Spanish Fish and 1 pack of Snittles. And, Ryan bought a total of 7 items, including 2 bags of Spanish Fish, one pack of Snittles, and 1 Snackers Bar. Charlie purchased a total of 4 items.

Note: a good, well-labeled externalization for this problem will serve as an explanation of your logic.

	Grant	Ryan	Charlie	Jack	Ollie	totals
Nards	1	2	1	3	0	7
Chocky Bar	2	1	1	2	0	6
Snittles	4	1	1	0	1	7
Snackers	1	1	0	1	0	3
Spanish Fish	1	2	1	0	1	5
totals	9	7	4	6	2	28

The directly given information is entered above in RED.
Inferred information is entered above in GREEN.

The inferences are usually just a matter of looking for rows and columns where we already know all but one value and then inferring what the final value must be:

- Grant bought a total of 9 items, in order to make the last column work out.
- Therefore, Grant bought 1 pack of Spanish fish, in order to make his row work out.
- Charlie bought 0 Snacker Bars, since we've already accounted for all of his purchases.
- So, a total of 4 Snacker Bars were purchased.
- All of Jack's purchases are accounted for, so he bought no Spanish Fish and no Snittles.
- So, we know that a total of 5 packs of Spanish Fish were purchased.
- And, all of Ollie's purchases are accounted for, so he bought no Nards, no Chock Bars, and no Snittles.
- And now, we know a total of 6 packs of Snittles were purchased…
- … and that Ryan bought 2 boxes of Nards (to make the Nards column work out)…
- … and that 6 Chocky Bars were purchased (to make the bottom row work out)…
- … and Ryan bought 1 Chocky Bar (to make the Chocky Bar column work out).

3. You have a box containing 20 red cubes, 20 green cubes and 20 blue cubes. The box has been shaken so that the cubes are mixed together randomly.

a) [8 points] If you randomly draw 6 cubes from the box, what is the probability that they will include exactly 3 green cubes?

You need to pick 3 of 20 green cubes and 3 cubes from the remaining 40 that are not green:

$$\frac{\binom{20}{3}\binom{40}{3}}{\binom{60}{6}}$$

This could also be solved by considering individual cases: 3G3R, 3G2R1B, etc.

b) [10 points] If you randomly draw 7 cubes from the box, what is the probability that they will include 3 cubes of one color, 3 cubes of another color, and 1 cube of a third color?

We must be careful about not letting the order of selection cause us to overcount the possibilities:

- **choose 2 colors for the sets of 3 cubes**
- **choose 3 cubes of each of those colors**
- **choose 1 cube of the remaining color (only one choice for that color)**

$$\frac{\binom{3}{2}\binom{20}{3}\binom{20}{3}\binom{20}{1}}{\binom{60}{7}}$$

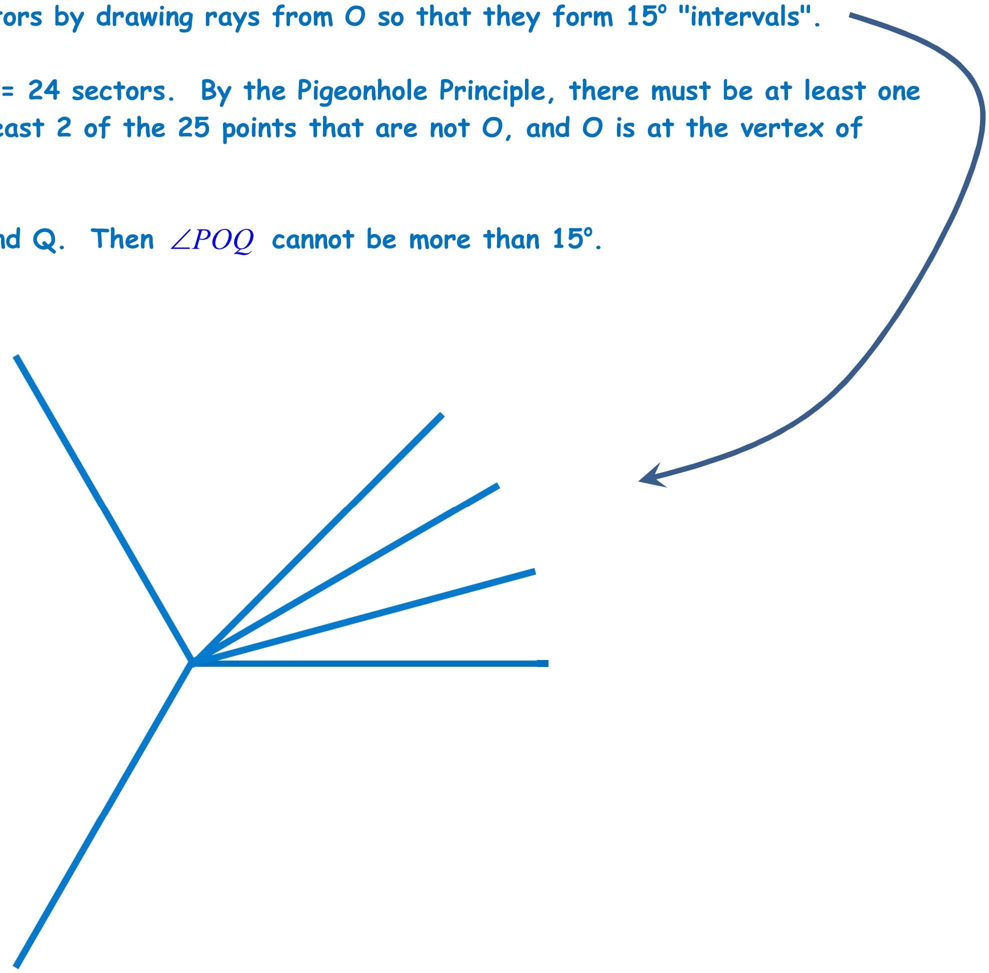
This could also be solved by considering individual cases: 3R3G1B, 3R3B1G, etc.

4. [14 points] Let O be the origin of the xy-plane. Suppose you are given 25 more points on the xy-plane, none the same as O. Prove that there must be two points, call them P and Q, among those 25 points, such that the angle $\angle POQ$ is no greater than 15^o.

Divide the plane into sectors by drawing rays from O so that they form 15° "intervals".

That will give us 360/15 = 24 sectors. By the Pigeonhole Principle, there must be at least one sector that contains at least 2 of the 25 points that are not O, and O is at the vertex of every sector.

Call those two points P and Q. Then $\angle POQ$ cannot be more than 15°.



5. [12 points] You have 30 buttons and 10 paperclips. You also have 6 envelopes addressed to 6 different acquaintances. In how many different ways could you distribute all of the buttons and paperclips among the envelopes, if it's OK to leave one or more envelopes empty? It does not matter which buttons or paperclips go into any envelope, just how many.

There are several ways to think about this.

The simplest is to consider the perspective that you begin by dividing the 30 buttons into 6 groups, which will then go into the six envelopes. From that perspective, you can think of it as placing five "dividers" between groups of buttons; so it comes down to putting the buttons and dividers into a row. This will divide the buttons into six groups (some may be empty). Since we have 30 buttons and 5 dividers, the number of ways to choose where to put the dividers would be

$$\binom{30+5}{5} = \binom{35}{5}$$

Once you've decided where to put the dividers, there's only one way to fill the remaining 30 spots with buttons, since the buttons are considered to be the same.

(You could also count the number of ways to choose where to put the buttons first. That yields an equivalent answer.)

A similar approach shows that the number of ways to divide up the paperclips would be

$$\binom{10+5}{5} = \binom{15}{5}$$

Since you must divide up the buttons AND divide up the paperclips, the total number of ways to do that involves the Multiplication Rule:

$$\binom{35}{5} \cdot \binom{15}{5}$$

6. [20 points] At a small company, parking spaces are reserved for the top executives: CEO, president, vice president, secretary, and treasurer—with the spaces lined up in that order. The parking lot guard can tell at a glance if the cars are parked correctly by looking at the color of the cars. The cars are yellow, green, purple, red, and blue, and the executives' names are Alice, Bert, Cheryl, David, and Enid.

i. The car in the first space is red.
ii. A blue car is parked between the red car and the green car.
iii. The car in the last space is purple.
iv. The secretary drives a yellow car.
v. Alice's car is parked next to David's.
vi. Enid drives a green car.
vii. Bert's car is parked between Cheryl's and Enid's.
viii. David's car is parked in the last space.

For each of the executive position, what is the executive's name and what color car does he or she drive?

Note: For this problem, explain all your inferences carefully. Every conclusion you reach should be justified. Be precise and complete. Use externalization.

From i and ii, the cars in the first three spaces are red, blue and green. From iii and iv, the car in the fourth space is yellow and the one in the fifth space is purple. So we have

red blue green yellow purple

Now, from vi, Enid's car is in the fourth space, and then from vii Bert's car must be in the third space and Cheryl's must be in the second space. So, from viii, David's car is in the fifth space and, by elimination, Alice's car must be in the first space:

red	blue	green	yellow	purple
Alice	Cheryl	Bert	Enid	David

Since we are given the spaces are arranged in order of job:

CEO	Pres.	VP	Sec.	Treas.
red	blue	green	yellow	purple
Alice	Cheryl	Bert	Enid	David