Discrete Finite Probability

In these notes, I will consider only the finite discrete case.

That is, in every situation the possible outcomes are all distinct cases, which can be modeled by integers, and there are a finite number of such outcomes.

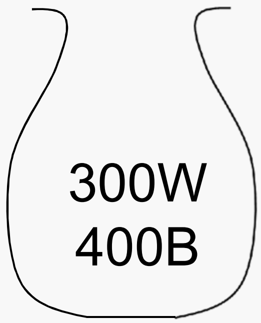
The discussion that follows was drawn from a number of sources, including:

Probability by G E Bates,
Addison-Wesley, 1965 0-201-00405-4

Puzzle-based Learning by Michalewicz & Michalewicz,
Hybrid, 2008 978-1-876462-5

A Challenge

Suppose we have an urn that contains 300 white balls and 400 black balls:



Suppose we draw 2 balls. The probability that we get 1 black ball and 1 white ball is:

$$\frac{\binom{400}{1}\binom{300}{1}}{\binom{700}{2}} = \frac{400 \cdot 300}{(700 \cdot 699)/2} \approx 0.49$$

Analysis

Here are the details of that last example:

Note the use of the Multiplication Rule here

$$\frac{\binom{\text{\# ways to choose}}{\text{1 black ball}}\binom{\text{\# ways to choose}}{\text{1 white ball}}}{\binom{\text{\# ways to choose 2 balls}}{\text{w/o restrictions}}} = \frac{\binom{400}{1}\binom{300}{1}}{\binom{700}{2}}$$

$$\binom{400}{1} = \frac{400!}{1!(400-1)!} = \frac{400!}{399!} = 400$$

$$\binom{700}{2} = \frac{700!}{2!(700-2)!} = \frac{700!}{2 \cdot 698!} = \frac{700 \cdot 699}{2} = 244650$$

Examples

With an urn that contains 300 white balls and 400 black balls:

Suppose we draw 5 balls. The probability that we get 2 black balls and 3 white balls is:

$$\frac{\binom{400}{2}\binom{300}{3}}{\binom{700}{5}} = \frac{\frac{400!}{2!398!}\frac{300!}{3!297!}}{\frac{700!}{5!695!}} = \frac{\frac{400\cdot 399}{2}\frac{300\cdot 299\cdot 298}{6}}{\frac{700\cdot 699\cdot 698\cdot 697\cdot 696}{120}}$$

$$= \frac{200\cdot 399\cdot 50\cdot 299\cdot 298}{35\cdot 699\cdot 698\cdot 697\cdot 116} \approx 0.257$$

Examples

With an urn that contains 300 white balls and 400 black balls:

Suppose we draw 5 balls. What's the probability that we get at least 3 black balls?

The key question is: in how many ways can we choose 5 balls and include at least 3 black balls?

So, we have three distinct (nonoverlapping) cases:

- 3B and 2W
- 4B and 1W
- 5B and 0W

Note the use of the Addition Rule here

$$\binom{400}{3}\binom{300}{2}+\binom{400}{4}\binom{300}{1}+\binom{400}{5}\binom{300}{0}$$

Examples

So, the probability that we get at least 3 black balls is:

$$\frac{\binom{400}{3}\binom{300}{2}+\binom{400}{4}\binom{300}{1}+\binom{400}{5}\binom{300}{0}}{\binom{700}{5}}$$

And, the probability that we get fewer than 3 black balls is:

$$1-\frac{\binom{400}{3}\binom{300}{2}+\binom{400}{4}\binom{300}{1}+\binom{400}{5}\binom{300}{0}}{\binom{700}{5}}$$

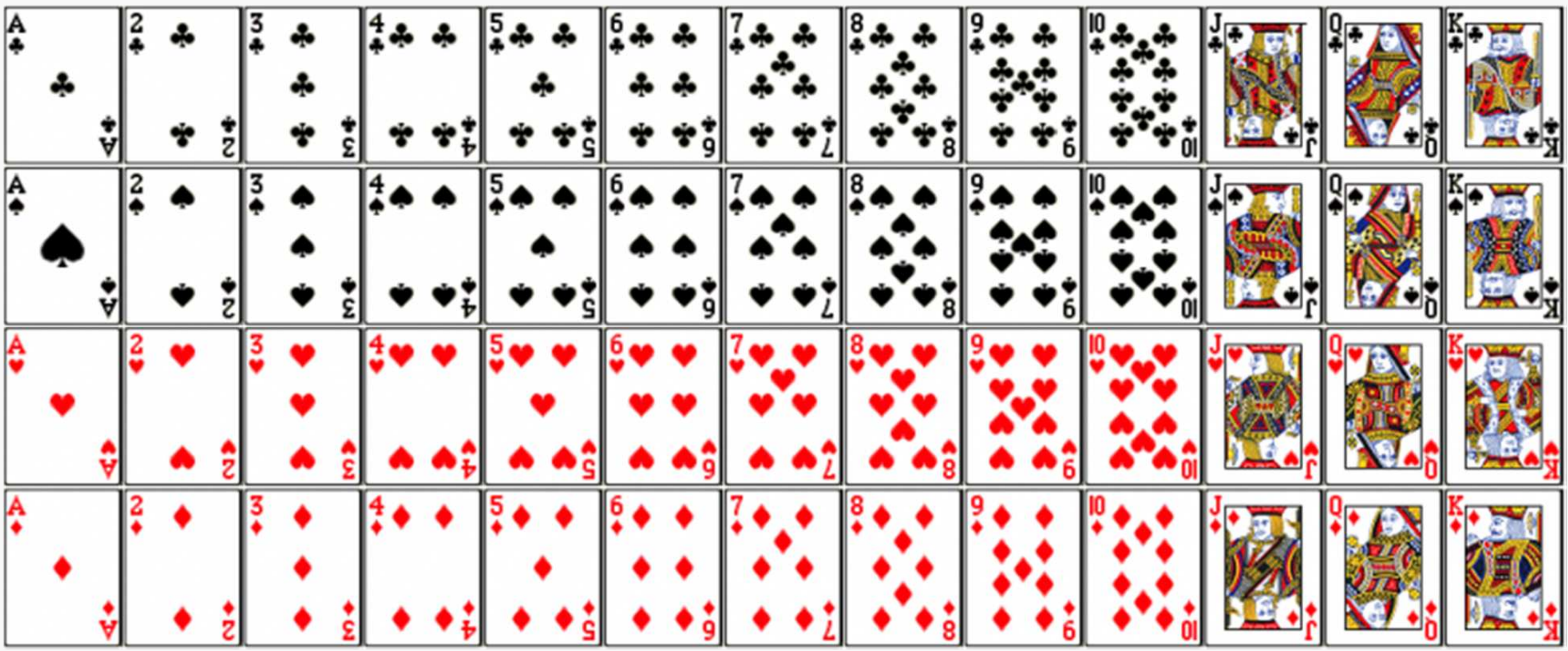
Poker Deck

A standard poker deck:

contains 52 cards,

divided into 4 suites (spades, clubs, hearts, diamonds),

with 13 values in each suite (Ace, 2, 3, …, 9, 10, Jack, Queen, King)



Counting Poker Hands

A *full house* is a poker hand that consists of 3 cards of one value and 2 cards of another value.

For example:

3ofHearts, 3ofClubs,3ofDiamonds,JackofSpades,JackofClubs

So, how many different full houses are there? (Not all at once.)

Number of Full Houses

We can construct a full house by carrying out the following sequence of tasks:

- choose the value for the three-of-a-kind
- choose 3 cards of that value
- choose a different value for the pair
- choose 2 cards of that value

Hence, using the Multiplication Rule:

$$\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}$$

Number of Two Pairs

Two pairs consists of two cards of one value, two cards of another value, and a fifth card of a third value.

We can construct such a hand by:

- choose the values for the two pairs
- choose 2 cards of the higher value
- choose 2 cards of the lower value
- choose a different value for the fifth card
- choose 1 card of that value

Hence, using the Multiplication Rule:

$$\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{11}{1}\binom{4}{1}$$

Number of Two Pairs

This one's tricky. If you pick the values for the two pairs separately, you will double-count:

- choose a value for one pair
- choose 2 cards of that value
- choose a different value for the other pair
- choose 2 cards of that value
- choose a different value for the fifth card
- choose 1 card of that value

$$\binom{13}{1}\binom{4}{2}\binom{12}{1}\binom{4}{2}\binom{11}{1}\binom{4}{1}$$

That's incorrect… but the error is subtle.

Number of Two Pairs

What if you try to pick the value for the higher pair first, then pick the value for the lower pair?

- choose a value for the higher pair
- choose 2 cards of that value
- choose a lower value for the other pair
- choose 2 cards of that value
- choose a different value for the fifth card
- choose 1 card of that value

$$\binom{12}{1}\binom{4}{2}?$$

The problem is that we don't know how many possible values we have left to choose from unless we know what the higher value was…

A Puzzle

There are three cards in a bag.

One card has the symbol **X** written on both sides; one card has the symbol **O** written on both sides; one card has the symbol **X** on one side and the symbol **O** on the other side.

You draw one card at random and examine one side of the card.

You see the symbol **X** on that side of the card.

What is the probability that there is also an **X** on the other side?

Solution

There are three cards in the bag:

XX OO XO

Now, we know we did not draw the card **OO**.

So, the probability is 1/2, right?

Nope. (Although we're right to eliminate **OO**.)

We could be seeing any of three different **X**s:

XX XO

If we're seeing either of the first two, then there's an **X** on the other side.

If we're seeing the third **X**, there's an **O** on the other side.

So, the probability is 2/3, not 1/2.

Expectation

Suppose we have a probabilistic experiment that has a sample space

$$\{e_1, e_2, \ldots, e_N\}$$

and that for each outcome in the sample space we have a probability $p(e_K)$.

And, suppose that corresponding to each outcome in the sample space we have an associated value v_K.

Then the *expectation* or *expected value* of the experiment is the weighted average of the values, using the outcome probabilities as weights:

$$\sum_{K=1}^{N} p(e_K) v_K$$

Example

A game is played by drawing balls out of an urn; the players gain $2 when a green ball is drawn and lose $1 when a red ball is drawn.

Suppose the urn contains 30 green balls and 70 red balls.

What is the expected payoff for a round of this game?

Solution

The probability of drawing a green ball is 0.3 and the probability of drawing a red ball is 0.7.

The value of drawing a green ball is +2 and the value of drawing a red ball is -1.

So, the expectation for a single draw would be

$$0.3 * 2 + 0.7 * -1 = -0.1$$

Hence, you'd expect, on average, to lose 10 cents each time you play the game.