Recurrence Relations

The proverbial German phenomenon of the verb-at-the-end about which droll tales of absentminded professors who would begin a sentence, ramble on for an entire lecture, and then finish up by rattling off a string of verbs by which their audience, for whom the stack had long since lost its coherence, would be totally nonplussed, are told, is an excellent example of linguistic **recursion**.

Douglas Hofstadter

recursion – see recursion.

Mythical definition

Recurrence Relations

A *recurrence relation* defines a sequence of values by relating the *n*-th value to *r* previous values.

For example:

$$a_n = 2 \cdot a_{n-1} + 1 \text{ for } n \geq 1$$
$$a_0 = 1$$

first-order, nonhomogeneous,
linear,
constant coefficients

$$b_n = n \cdot b_{n-1} \text{ for } n \geq 1$$
$$b_0 = 1$$

first-order, homogeneous,
linear,
nonconstant coefficients

$$f_n = f_{n-1} + f_{n-2} \text{ for } n \geq 2$$
$$f_0 = 0, f_1 = 1$$

second-order, homogeneous,
linear,
constant coefficients

Why do we care?

The solutions of many important problems lead to recurrence relations.

For example, let H_n be the number of disks that must be moved in order to solve the Towers of Hanoi problem discussed earlier.

There must be an intermediate step must have the *n-1* smallest disks on pole 2, and only the largest disk on pole 1.

From that step, we can move the largest disk to pole 3, and then follow the same logic to move the *n-1* smallest disks from pole 2 to pole 3 (atop the largest disk).

So, the number of disk moves must satisfy the recurrence:

$$H_n = 2 \cdot H_{n-1} + 1 \text{ for } n \geq 2$$
$$H_1 = 1$$

But, we would like to have a nonrecursive formula for H_n.

Solving Recurrence Relations

There are many different kinds of recurrence relations, and a number of different solution techniques that apply to different kinds of recurrence relations.

Given a homogeneous linear recurrence relation with constant coefficients:

$$A_n = C_{n-1}A_{n-1} + C_{n-2}A_{n-2} + \cdots + C_{n-r}A_{n-r} \text{ for } n \geq r \quad (1)$$

Find the roots of the characteristic polynomial:

$$P(t) = t^r - C_{r-1}t^{r-1} - C_{r-2}t^{r-2} - \cdots - C_1 t^1 - C_0$$

If the roots are all distinct, say they are $d_1, d_2, \ldots, d_r$, then (1) is satisfied by:

$$A_n = k_1 d_1^{\ n} + k_2 d_2^{\ n} + k_3 d_3^{\ n} + \cdots + k_r d_r^{\ n}$$

We will consider the case of non-distinct roots a bit later.

Example: Fibonacci

Consider the Fibonacci recurrence: $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$

$$f_0 = 0, f_1 = 1$$

The characteristic polynomial is: $P(t) = t^2 - t - 1$

The roots are easily found by using the Quadratic Formula:

$$d_1 = \frac{1-\sqrt{5}}{2} \text{ and } d_2 = \frac{1+\sqrt{5}}{2}$$

So, the Fibonacci recurrence would be satisfied by:

$$f_n = k_1\left(\frac{1-\sqrt{5}}{2}\right)^n + k_2\left(\frac{1+\sqrt{5}}{2}\right)^n$$

What about the coefficients?

Example: Fibonacci

We can find the coefficients by applying the base cases:

$$f_n = f_{n-1} + f_{n-2} \text{ for } n \geq 2$$

$$f_0 = 1, f_1 = 1$$

$$f_n = k_1 \left(\frac{1-\sqrt{5}}{2} \right)^n + k_2 \left(\frac{1+\sqrt{5}}{2} \right)^n$$

$$f_0 = k_1 \left(\frac{1-\sqrt{5}}{2} \right)^0 + k_2 \left(\frac{1+\sqrt{5}}{2} \right)^0 = k_1 + k_2 = 0$$

$$f_1 = k_1 \left(\frac{1-\sqrt{5}}{2} \right)^1 + k_2 \left(\frac{1+\sqrt{5}}{2} \right)^1 = \left(\frac{1-\sqrt{5}}{2} \right) k_1 + \left(\frac{1+\sqrt{5}}{2} \right) k_2 = 1$$

Now, we have a pair of (ugly) linear equations, which are easily solved:

$$k_1 = -\frac{1}{\sqrt{5}} \text{ and } k_2 = \frac{1}{\sqrt{5}}$$

Example: Fibonacci

A bit of algebraic manipulation yields:

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n \quad \text{for } \; n \geq 0$$

It's interesting that yields an integer value for every nonnegative integer n.

A Theorem

Given a linear homogeneous recurrence relation

$$A_n = C_{n-1}A_{n-1} + C_{n-2}A_{n-2} + \cdots + C_{n-r}A_{n-r} \qquad (1)$$

if the characteristic polynomial

$$P(t) = t^r - C_{r-1}t^{r-1} - C_{r-2}t^{r-2} - \cdots - C_1 t^1 - C_0$$

has a root λ of multiplicity m, then the following are all solutions of (1):

$$\lambda^n,\ n\lambda^n,\ n^2\lambda^n,\ \cdots,\ n^{m-1}\lambda^n$$

and every solution of (1) can be expressed as a sum of constant multiples of all of the solutions described above,

and, if we are given r base cases, we can solve for exact values for those constant multiples and find a unique solution.

Example

Consider the following recurrence: $a_n = 4_{n-1} - 4a_{n-2}$ for $n \geq 1$

$a_0 = 5,\ a_1 = 16$

The characteristic polynomial is: $P(t) = t^2 - 4t + 4 = (t-2)^2$

This has a repeated root, 2.

According to the preceding theorem, the *general solution* of the recurrence is:

$$s_n = c_1 2^n + c_2 n 2^n$$

Then, using the base cases above, we can solve for the coefficients and get the *specific solution*:

$$s_n = 5 \cdot 2^n + 3n2^n$$

A Nonhomogenous Example

Consider the Towers of Hanoi recurrence:

$$H_n = 2 \cdot H_{n-1} + 1 \text{ for } n \geq 2$$
$$H_1 = 1$$

This is not homogeneous, because of the nonzero constant on the right side.

In this case, we can transform the nonhomogeneous recurrence by performing a trick:

$$H_n = 2H_{n-1} + 1$$
$$H_{n-1} = 2H_{n-2} + 1$$

If we subtract, we get second-order linear homogeneous recurrence:

$$H_n - H_{n-1} = 2H_{n-1} - 2H_{n-2}, \text{ or}$$
$$H_n = 3H_{n-1} - 2H_{n-2}$$

The characteristic polynomial has roots 1 and 2.

A Nonhomogeneous Example

Using the roots 1 and 2, the general solution would be:

$$H_n = c_1 1^n + c_2 2^n = c_1 + c_2 2^n$$

We can use the facts that $H_1 = 1$ and $H_2 = 3$ to solve for c_1 and c_2, and get the specific solution:

$$H_n = 1 + 2^n \quad \text{for} \quad n \geq 1$$

(So, if those monks are working with a tower of 64 disks, and moving 1 disk per second, it will take them a total of more than 584,942,417,355 years to finish.)