**<u>You may work in pairs or purely individually for this assignment.</u>** Prepare your answers to the following questions in a plain ASCII text file or MS Word document. Submit your file to the Curator system by the posted deadline for this assignment. No late submissions will be accepted. If you work in pairs, list the names and email PIDs of both members at the beginning of the file, and submit your solution under only one PID. No other formats will be graded.

For this assignment, you may (and are encouraged to) work in pairs; if you do so, you must also write your solutions in such a way that it is clear how each member contributed to deriving the solution.

You will submit your answers to the Curator System (www.cs.vt.edu/curator) under the heading `OOC06`.

For the questions on probability, express your answer first in terms of appropriate notation (factorials, combinations, permutations), and then simplify your answer to a single decimal value, rounded to two digits after the decimal point.

**1.** [15 points each] An urn contains 5 balls of each of 6 colors (red, blue, green, yellow, orange and purple).

a) If two balls are drawn randomly from the urn, what is the probability that they will be of the same color?

The number of ways to choose 2 balls from the urn is obviously C(30,2).

How many ways are there to pick 2 balls of the same color? There are C(6,1) ways to choose the color to be used, and then C(5,2) ways to pick 2 balls from that color.

So, the probability is: $$\frac{\binom{6}{1}\binom{5}{2}}{\binom{30}{2}} = \frac{6 \cdot 10}{435} \approx 0.14$$

b) If six balls are drawn randomly from the urn, what is the probability that they will be of six different colors?

We need to choose one ball of each color. The order doesn't matter, so we can impose an ordering, say pick in alphabetical order (blue, green, etc.):

$$\frac{\binom{5}{1}\binom{5}{1}\binom{5}{1}\binom{5}{1}\binom{5}{1}\binom{5}{1}}{\binom{30}{6}} = \frac{5^6}{593775} \approx 0.026 \approx 0.03$$

c) If six balls are drawn randomly from the urn, what is the probability there will be two balls of one color and four balls of another color?

Here we must choose a color for the pair of balls, choose 2 balls of that color, choose a color for the other balls, and then choose 4 balls of that color:

$$\frac{\binom{6}{1}\binom{5}{2}\binom{5}{1}\binom{5}{4}}{\binom{30}{6}} = \frac{6 \cdot 10 \cdot 5 \cdot 4}{593775} \approx 0.0020 \approx 0.00$$

d) If six balls are drawn randomly from the urn, what is the probability there will be three balls of one color and three balls of another color?

**The difference between this and the previous question is that we must be careful not to count choices twice. Unlike the previous question, we are choosing the same number of balls of each color, and so we must choose both the colors at the same time (or correct in some other manner):**

$$\frac{\binom{6}{2}\binom{5}{3}\binom{5}{3}}{\binom{30}{6}} = \frac{15 \cdot 10 \cdot 10}{593775} \approx 0.0025 \approx 0.00$$

e) If three balls are drawn from the urn, and all of them are yellow, what is the probability that the next ball drawn will also be yellow?

**There would be 3 yellow balls left, and 27 total, so:**

$$\frac{\binom{3}{1}}{\binom{27}{1}} = \frac{3}{27} \approx 0.11$$

f) If three balls are drawn from the urn, and all of them are yellow, what is the probability that the next ball drawn will be blue?

**There would be 5 blue balls left, and 27 total, so:**

$$\frac{\binom{5}{1}}{\binom{27}{1}} = \frac{5}{27} \approx 0.185 \approx 0.19$$

**2.** [10 points] In a country where every couple wants to have a son, each couple continues to have babies until they have a son, at which point they stop having babies. What is the proportion of girls to boys in this country? Assume that the probability of having a boy is 1/2 , and so is the probability of having a girl.

For credit, you must explain carefully the logic you used to derive your answer.

This becomes obvious if you begin by considering a specific example. Let's say there are 128 couples having children. Then 1/2 of the couples (64) will have a boy as their first (and only) child, and 1/2 of the couples (64) will have a girl as their first child. Of the latter 64 couples, 32 will have a boy as their second (and last) child, and 32 will have a girl as their second child.

So, if we continue that logic we see that the following situation will arise:

| # children | # families | # boys in family | total # boys | # girls in family | total # girls |
|---|---|---|---|---|---|
| 1 | 64 | 1 | 64 | 0 | 0 |
| 2 | 32 | 1 | 32 | 1 | 32 |
| 3 | 16 | 1 | 16 | 2 | 32 |
| 4 | 8 | 1 | 8 | 3 | 24 |
| 5 | 4 | 1 | 4 | 4 | 16 |
| 6 | 2 | 1 | 2 | 5 | 10 |
| 7 | 1 | 1 | 1 | 6 | 6 |
| 8 | 1 | 1 | 1 | 7 | 7 |

(I am treating the last couple as a special case and assuming they will have a boy as their $8^{th}$ child; if not, then we would see an increase in the number of girls and no change in the number of boys.)

So the total number of boys equals 128 (one per couple, after all) and the total number of girls equals 127, so they are roughly equal.

It should be clear, intuitively, that the pattern will continue if we increase the number of couples. We will see a relatively small proportion of families having lots of girls and one boy, balanced by half the couples having a single boy and no girls.

A more careful analysis (requiring the use of some interesting summation formulas) shows that if the number of couples is $2^N$, then the number of boys will be $2^N$ and the number of girls will be $2^N - 1$.

**Details:**

**We need a couple of summation formulas:**

$$\sum_{k=0}^{N} 2^k = 2^{N+1} - 1 \text{ and } \sum_{k=1}^{N} k2^{k-1} = (N-1)2^N + 1$$

**We will have $2^{N-1}$ couples with 1 boy, $2^{N-2}$ with 1 boy and 1 girl, $2^{N-3}$ with 1 boy and 2 girls, and so forth. In general, for each k from 0 to N-1, we will have $2^k$ couples, each with 1 boy, plus one more boy for the final couple. So, if we let $B_N$ be the number of boys, then:**

$$B_N = 1 + \sum_{k=0}^{N-1} 2^k = 1 + \left(2^N - 1\right) = 2^N$$

**And, in general, for each k from 0 to N-1, we will have $2^k$ couples, each with N-k-1 girls, and the N more girls for the last couple. So, if we let $G_N$ be the number of girls, then:**

$$\begin{aligned} G_N &= N + \sum_{k=0}^{N-1} (N-k-1)2^k \\ &= N + N\sum_{k=0}^{N-1} 2^k - 2\sum_{k=0}^{N-1} k2^{k-1} - \sum_{k=0}^{N-1} 2^k \\ &= N + (N-1)(2^N - 1) - 2\left((N-2)2^{N-1} + 1\right) \\ &= 2^N - 1 \end{aligned}$$

**Now, we see that the ratio of boys to girls is almost 1 and closer to 1 as N increases.**