You may work in pairs or purely individually for this assignment. Prepare your answers to the following questions in a plain ASCII text file or MS Word document. Submit your file to the Curator system by the posted deadline for this assignment. No late submissions will be accepted. If you work in pairs, list the names and email PIDs of both members at the beginning of the file, and submit your solution under only one PID. No other formats will be graded.

For this assignment, you may (and are encouraged to) work in pairs; if you do so, you must also write your solutions in such a way that it is clear how each member contributed to deriving the solution.

You will submit your answers to the Curator System (www.cs.vt.edu/curator) under the heading `OOC05`.

1. [25 points each] Solve each of the following recurrence relations:

a) $a_0 = 0, a_1 = 10 \; a_n = a_{n-1} + 6a_{n-2}$ for $n > 1$

The characteristic equation is: $\tau^2 - \tau - 6 = 0$

So, we can factor and obtain the roots: $\tau = -2$ and $\tau = 3$

This implies the general solution would be: $a_n = c_1(-2)^n + c_2 \cdot 3^n$ for $n \geq 0$

Now, using the initial conditions we get:

$a_0 = c_1 + c_2 = 0 \quad \Rightarrow \quad c_2 = -c_1$

$a_1 = -2c_1 + 3c_2 = 10 \quad \Rightarrow \quad c_1 = -2, c_2 = 2$

So, the specific solution is: $a_n = (-2)^{n+1} + 2 \cdot 3^n$ for $n \geq 0$

b) $b_0 = 1, \; b_n = 5b_{n-1} + 4$ for $n \geq 1$ (Hint: look for example in the notes.)

This is a nonhomogeneous equation; we reduce it to a homogeneous equation by performing an algebraic "trick" shown in the notes:

$$b_{n+1} = 5b_n + 4$$
$$b_n = 5b_{n-1} + 4$$

Subtracting yields: $b_{n+1} - b_n = 5b_n - 5b_{n-1}$

Or: $b_{n+1} = 6b_n - 5b_{n-1}$

So the characteristic equation is $\tau^2 - 6\tau + 5 = 0$ and the roots are $\tau = 1$ and $\tau = 5$.

The general solution is $b_n = c_1 \cdot 1^n + c_2 \cdot 5^n$ and we can find the coefficients by using the initial condition and value for b_1 from the original equations:

$$b_0 = c_1 + c_2 = 1 \quad \Rightarrow \quad c_2 = 1 - c_1$$
$$b_1 = c_1 + 5c_2 = 9 \quad \Rightarrow \quad c_1 = -1 \text{ and } c_2 = 2$$

Finally, the specific solution is: $b_n = -1 + 2 \cdot 5^n$ for $n \geq 0$

2. [25 points] The characteristic polynomial for a linear homogeneous recurrence relation with constant coefficients has the roots 1, 1 and 3. What is the general solution of the recurrence relation.

A repeated root (1 in this case) yields multiple terms in the general solution; see the notes.

The general solution in this case would look like:

$$c_n = c_1 \cdot 1^n + c_2 \cdot n1^n + c_3 \cdot 3^n = c_n = c_1 + c_2 \cdot n + c_3 \cdot 3^n$$

3. [25 points] You have an unlimited number of red and blue cubes. Find a recurrence relation for the number of different ways to build a vertical stack of n blocks, for n > 0, such that there are never two adjacent red blocks. Explain your logic, but do not solve the recurrence relation.

Let S(n) be the number of ways to create a stack of n blocks so that no two adjacent blocks are red. Obviously S(1) = 1, since we can create a stack of 1 red block or of 1 blue block. And, S(2) = 3 since we have the following possible stacks that meet the restriction:

RB (stacks are drawn horizontally, red block on bottom, blue block on top)
BR
BB

For n = 3, we could create the following 5 stacks:

RBB
BRB
BBB
RBR
BBR

And there's a pattern here. The stacks that have a blue block on top are obtained by placing a single blue block on top of a stack of height 2; the stacks that have a red block on top are obtained by placing a pair of blocks, BR, on top of a stack of height 1. And, not only does that process always yield valid stacks, it also accounts for every possible valid stack of height 3, since:

Height-3 stacks with a red on top must have a blue block beneath it, and hence trace back to a valid stack that's two block shorter.
Height-3 stacks with a blue on top must trace back to a valid stack that's one block shorter.

So, this suggests that S(n) = S(n-1) + S(n-2) if n ≥ 3. We can check our logic by considering valid blocks of height 4:

RBBB // adding a blue block to a height-3 stack
BRBB
BBBB
RBRB
BBRB

RBBR // adding a BR pair to a height-2 stack
BRBR
BBBR

That gives us 8 valid stacks, which is consistent with the recurrence relation, and it's clear that there aren't any other valid stacks of height 4.

So, we have the recurrence:

$$S_1 = 1, S_2 = 3,$$
$$S_n = S_{n-1} + S_{n-2}, \text{ for } n \geq 3$$