Chapter 2
Scaling Laws

Amitabha Ghosh

Abstract It is a very well-known fact that when size diminishes to extremely small levels, Newtonian mechanics fails. In such situations one has to use quantum mechanics for studying physical systems of such extremely small dimensions. However, it is less readily recognized that even before reaching such extremely small dimensions leading to the breakdown of Newtonian mechanics there are many special aspects one needs to take care of for successful analysis and design of small systems. The counterintuitive features arise not due to breakdown of Newtonian mechanics but due to the changes in the order of predominance of physical phenomena caused by drastic reduction in size from the scales we are familiar with in our daily experience. In simple language this is called scaling effect and the laws which govern such effects are called scaling laws. While dealing with microsystems our normal engineering intuition fails and it needs to be replaced by special "microintuition" for developing microsystems which is becoming a very important engineering activity with the turn of the century. This chapter presents the rudiments of the common scaling laws and their importance.

Keywords Scaling · Microsystems · Micromechanics

1 Introduction

Once I was visiting the residence of one of my closest friends and colleague who had just returned from a summer trip abroad. He brought a toy train for his elder son who was about 11 years old. He had a younger son also, who received a battery-driven nice toy racing car as a gift. When I reached their home in the evening of a weekend, I found that his elder son was pestering his father for a clarification. The boy was observant and was asking his father why the toy train and the toy car were not destroyed when they derailed or collided whereas a real train or car gets destroyed completely on such occasions. I noticed with some amusement the

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problem my friend was facing to satisfy his son's query. He was a professor of economics. I could understand that he was intuitively getting the feel of why this happens, but could not satisfy his son's scientific mind. When I took a very small and a long piece of the same mild steel wire lying in the courtyard and demonstrated the much stiffer/stronger behavior of the smaller piece both the father and the son got some idea about what we call the scaling effect. From our daily experience we all have some intuition about the change of properties and characteristics of objects with drastically different sizes but of similar shapes. However, a formal study of the change of characteristics of objects and systems with change in scales is essential in modern engineering, particularly because of the emergence of microsystems technology.

Though the phenomenon of the effect of scaling on the properties and characteristics of objects and systems is known since a long time the emergence of "scaling laws" as a serious branch of systematic study is relatively recent. This is primarily due to the increased dependence of modern technology on miniaturization. At the same time using the scaling effects to predict the behavior and properties of a large system by experimenting on a small-sized scale model is a very useful tool. Physicists have also used the basic ideas behind "scaling laws" under a different name "dimensional analysis." Many times the characteristics or properties of a system can be expressed through combinations of various parameters so that each group is dimensionless. Thus, any change in size, i.e., scale, does not affect the magnitudes of these quantities and the performance of a system can be predicted from the results obtained on the performance of a similar system, but of different size. In more recent times the trend of miniaturization in technology has brought back "scaling laws" to the center stage of related design and fabrication activities.

1.1 Trend of Miniaturization

It may not be out of place to look into the reasons behind the ongoing technological revolution based on miniaturization. Figure 2.1 shows the current tendency to make systems and devices more intelligent and autonomous. To achieve higher degree of intelligence it is essential to drastically increase sensory data (by many orders of magnitude). This demands that the sensors be miniaturized so that a large number of these can be accommodated in small areas and at the same time neither the cost nor the energy consumption exceeds acceptable limits. Similarly for activating such machines/devices it is essential to employ a large number of actuators of miniaturized size (to effectively use the scaling effect of some physical laws as will be seen later) working in parallel.

Actuators in futuristic machines and devices will not only be numerous in number but also be distributed over the whole system instead of remaining confined to a few number of locations of the systems as is currently the practice. This will provide the required dexterity to the moving elements of the machine/system. In fact, the engineers and technologists are often taking lessons from the living world

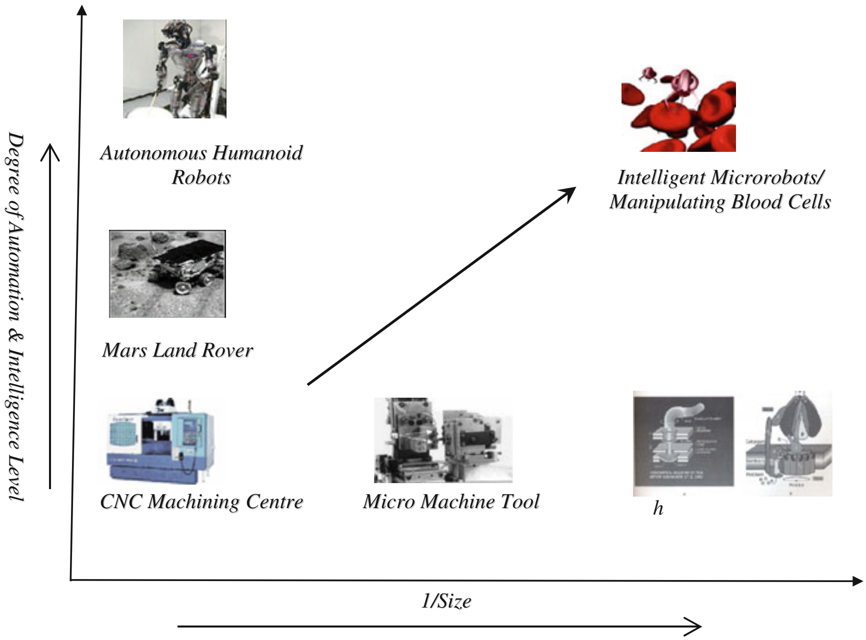


Fig. 2.1 Trend of miniaturization

for conceiving ultramodern machines. All living objects are dependent on miniaturization and massive parallelism for sensing and actuation. This also influences the technique of fabrication, and already there is a tendency to depend on "bottom-up" approaches instead of the traditional "top-down" approach. In the "bottom-up" approach the required shape, size, and characteristic features are achieved through material manipulation at the micro, nano, and even molecular levels. Such processes have been given a new name: "fabrionics." Quite often, therefore, it becomes necessary to utilize the phenomenon of "self-assembly" to make such fabrications technologically feasible and economically viable.

All these new and emerging revolutionary concepts are going to pave the way for the next turning point in human history – the Third Industrial Revolution. In all aspects of futuristic engineering, the principle of life science will play an extremely important role. In fact a completely new branch of engineering – "synthetic biology" – has slowly started to emerge where machines and devices will be artificially created following new emerging techniques of "fabrionics" but which will often function using the phenomena and principles of life science.

It is clear that understanding and developing such new era machines and devices can be possible only when one applies the knowledge and intuition acquired through experience in the macroscopic world keeping in mind the scaling laws.

1.2 Historical Background

As it happens in many areas of mechanics, the scientific study of scaling laws and scaling effects was first started by Galileo Galilei. However, it all started by a major mistake committed by him in his early life. Galileo was a student of medicine at the University of Pisa; but as he found that his major interest was in mathematics and natural philosophy (the old term used for what we call today "science") he left the university after a few years without completing his studies. He became a self-taught teacher of mathematics at Florence and was desperately looking forward to the position of professor of mathematics at Pisa. He was invited to deliver two public lectures when the chair of professor of mathematics at Pisa fell vacant. To impress upon the local ruler the Grand Duke of Tuscany and the audience in general, Galileo chose the topic of Dante's model of hell. The people of Florence loved to hear about the topic. One of the models was by Antonio Manetti, a former member of the Florentine Academy. A competing theory was by Alessandro Vellutello who was not a Florentine. Manetti's model of Dante's hell is shown in Fig. 2.2a where the hell is shown as a conical cavity with its apex at the center of the earth; the base was a circle with its center at Jerusalem. Vellutello's inferno is comparatively much smaller as indicated in Fig. 2.2b. Vellutello's one main reason

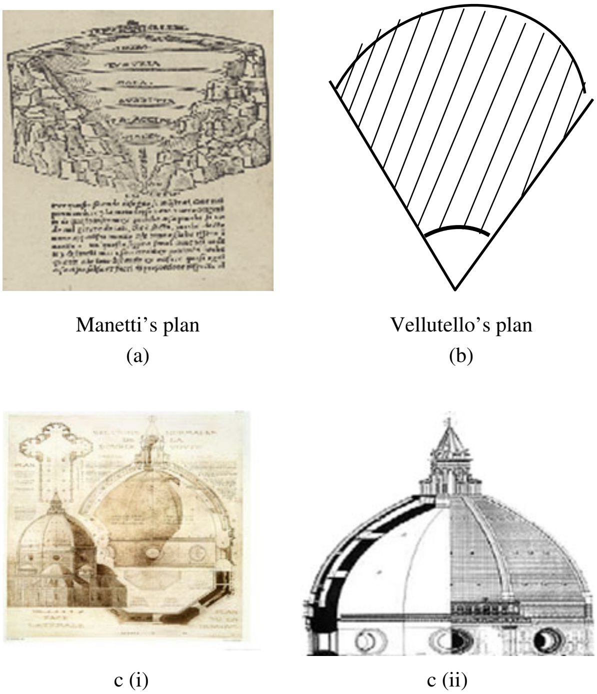


Fig. 2.2 Plan of hell in Galileo's lecture and Brunelleschi dome

for a smaller cavity was because he felt the cover of Manetti's hell was too thin. Galileo ridiculed Vellutello's model as too small to accommodate the sinners. He demolished Vellutello's objection to Manetti's model having a cover too thin to support itself by demonstrating the example of the Brunelleschi dome of Florentine Cathedral. He argued that relatively speaking the dome thickness (Fig. 2.2c) was equivalent to that of the cover of the inferno but it can also hold itself against falling under its own weight. This argument was based upon scale invariance of strength under self-weight. The audience went into rapture on seeing the rival model being ruthlessly demolished by Galileo. And, obviously, Galileo got the job of professor of mathematics at Pisa.

Later Galileo shifted to the university at Padua, very close to Venice, and his most active and productive years were spent there. The Republic of Venice had to depend upon a strong navy for obvious reasons and designing and building ships was an activity of prime importance. Being a very practical scientist it is conceivable that Galileo was in close collaboration with the Venetian arsenal and studied many aspects of ship design. This led him to study the basic aspects of strength of beams and structures. It was at that time he realized the major mistake he had made while demolishing the model of hell on the comparative argument based on the dome of the cathedral and the terrestrial cover of the hell. He kept his finding to his heart but started a thorough scientific study of the scaling laws. (This was perhaps to show his preparedness with the correct science in case someone else discovered the mistake in his lecture which was acclaimed enthusiastically and he got the job at Pisa.)

Galileo realized that large ships break under their own weight when out of water but a scale model of the same ship made of the same wood behaved much stronger. He also noticed that a thin square board can float in water (even though the board material is heavier than water) when the size of the piece is small enough. He rightly argued that when the piece is downsized its area (proportional to its weight) decreases faster than the rate at which the perimeter (which receives the support from waters' surface tension) decreases. Galileo also observed that animals cannot be simply scaled up. As the weight increases at third power of the size scale the bone's supporting cross-sectional area has to be disproportionately larger. Figure 2.3a explains the matter.

If system 1 is three times bigger than system 2 the weight ratio is $W_1/W_2 = 27$.

Hence the cross-sectional area of the supporting column is $A_1/A_2 = 27$

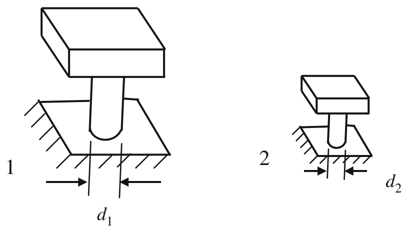


Fig. 2.3a Scaling of supporting columns

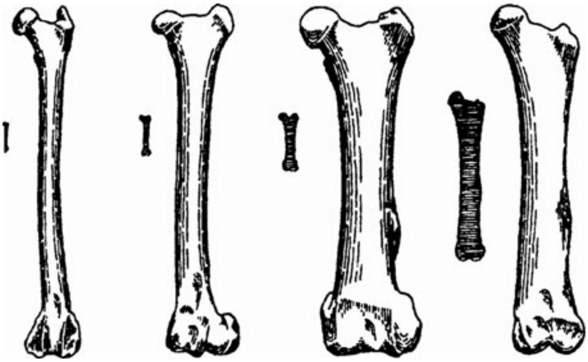


Fig. 2.3b Bone shapes with varying size

So, $A_1 = 27A_2$, since $A_1 \propto {d_1}^2$ and $A_2 \propto {d_2}^2$.

$$d_1 = 5.2d_2 > 3d_2$$

This way Galileo explained why the bones of larger animals are much thicker than those of small animals as shown in an illustration from his famous book *Dialogues on The New Sciences* (Fig. 2.3b).

2 Scaling Laws and Their Importance

The scaling laws are proportionality relations of any parameter associated with an object (or system) with its length scale. For example, the volume of an object varies as cubic length (i.e., as l^3); on the other hand, its surface area scales as l^2. Therefore, a smaller object possesses larger surface area to its volume when compared with a bigger object with similar geometrical shape. There are primarily two types of scaling laws. One is related to the scaling of physical size of objects. The other type is related to the scaling of a phenomenological behavior of an object/machine.

The first thing that a modern engineer or scientist requires for designing a miniaturized device is an understanding of the scaling laws. When all aspects of the device scale in a similar way the geometric integrity is maintained with size. Such type of scaling is called "isomorphic" (or "isometric") scaling. On the other hand, if different elements of a system with different functionalities do not scale in a similar way, the scaling is called "allometric" scaling. Scaling laws deal with the structural and functional consequences of changes in size or scale among otherwise similar structures/organisms; thus, only through the scaling laws a designer becomes aware of physical consequences of downscaling devices and systems. Human intuition is conditioned by the everyday observed phenomena around us within the common range of perception. This "macrointuition" may lead to erroneous designs of microsystems

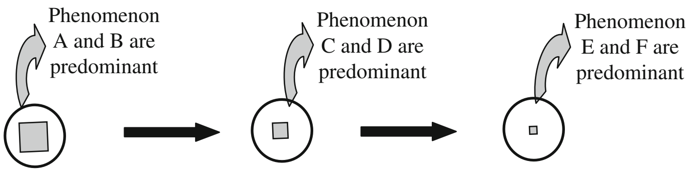


Fig. 2.4 Change in predominance of phenomena with varying scale

if the appropriate "scaling laws" are not taken into consideration, as these laws provide an idea about the system's performance at a totally different scale from that of an existing system at normal scales. Thus, the scaling laws help designers to develop a kind of "microintuition" that is essential for successful design of microsystems. In general, the performances of different subsystems of a system scale differently. This can lead to a different appearance of a system with much smaller size.

Another important aspect of scaling laws is very significant while conceiving, planning, and designing miniaturized devices and systems. When the size of an object is scaled down, different physical phenomena become predominant at different scales. Figure 2.4 shows this aspect of scaling laws.

This change of relative importance with size of a system is very important for successful design of a miniaturized system. For example, at macroscopic scale the weight of an object is predominant and it falls down under the influence of gravity. But when the same object becomes very small the weight (which scales as l^3) becomes relatively insignificant compared to air friction which depends on the surface area scaling as l^2. Therefore, even small air currents can keep it floating. When a glass full of water is overturned the water gets spilled; on the other hand, water confined to a capillary tube does not come out even when the tube is upturned. This is because the force due to surface tension becomes predominant at very small scales.

In complex systems scaling laws become relevant for understanding the interplay among various physical phenomena and geometric characteristics. Sometimes, relatively simple scaling laws, applicable to very complex systems, can provide clues to some fundamental aspects of the system. Thus, scaling laws are not only important for designing microsized systems but also very useful in understanding the basic physical principles involved in many complex phenomena.

3 Scaling Laws and Their Application

In this section important scaling laws and their use (and influence) will be taken up for discussion.

3.1 Geometric Scaling

The laws for geometric scaling are simple and well known. The scaling of various geometric parameters follows the laws given below:

Perimeter (P) $\propto l$
Area (A) $\propto l^2$
Volume (V) $\propto l^3$

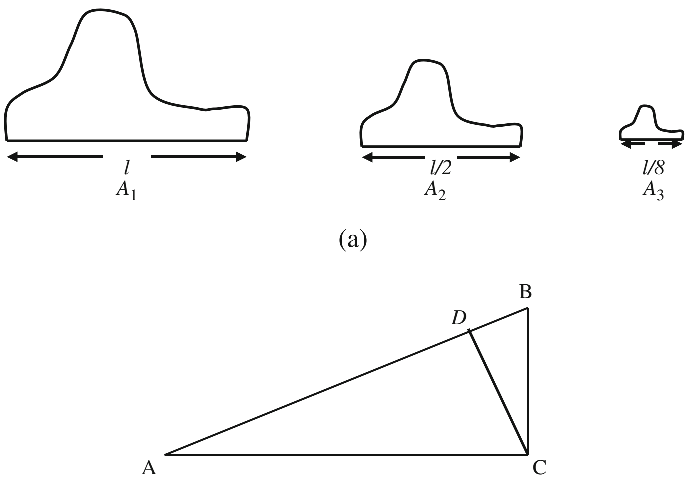
where l is the length scale.

The scaling law $A \propto l^2$ can be used in geometry. This law states that the area of a geometric figure shown in Fig. 2.5a scales as l^2. If keeping the geometric shape unchanged the size is changed the area of the figure will change in proportion to the square of the length scale. So, $A_1{:}A_2{:}A_3 = 1{:}1/4{:}1/64$.

Figure 2.5b shows how the scaling law can be used to prove the Pythagoras theorem. ABC is a right-angled triangle and CD is the normal to the hypotenuse AB. It is very easy to show that the three triangles ΔABC, ΔACD, and ΔBCD are similar. So, their areas will be proportional to the square of any characteristic length (let it be the hypotenuse of each triangle).

Thus, the area of $\Delta ABC = \lambda AB^2$, the area of $\Delta ACD = \lambda AC^2$ and the area of $\Delta BCD = \lambda CB^2$, where λ is a constant of proportionality. It is obvious that

Area of ΔACD + Area of ΔBCD = Area of ΔABC
or, $\lambda AC^2 + \lambda BC^2 = \lambda AB^2$
or, $AC^2 + BC^2 = AB^2$



(b)

Fig. 2.5 Scaling of area and proof of Pythagoras theorem

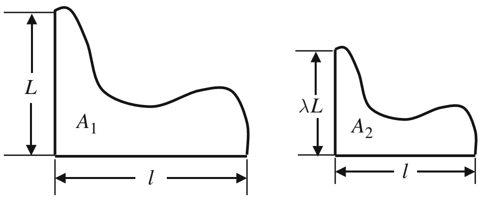


Fig. 2.6a Unsymmetrical scaling of area

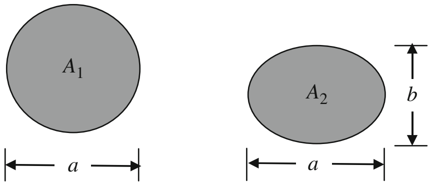


Fig. 2.6b Area of an ellipse by scaling law

If only one dimension is scaled the area will also scale as linear power of that length scale. Figure 2.6a shows two figures where only the vertical dimension is changed. So, if A_1 and A_2 be the areas of the two figures, $A_1{:}A_2 = 1{:}\lambda$.

Figure 2.6b shows a circle with diameter a and area $A_1(= \pi a^2/4)$. So, when the vertical scale is reduced and a is reduced to b the area of this is flattened circle (an ellipse with major axis a and minor axis b) A_2 will be in proportion to the vertical scale. So, $A_1{:}A_2 = 1{:}\lambda$.

where $b = \lambda a$ or $\lambda = b/a$. Thus

$$(\pi/4)a^2 : A_2 = 1{:}b/a$$

or,

$$A_2/[(\pi/4)a^2] = (b/a)/1$$

or,

$$A_2 = (\pi/4)ab$$

It is also quite obvious that Pythagoras theorem is valid for not only squares drawn on the three sides of a right-angled triangle but figures with any shape as indicated in Fig. 2.6c, where areas A, B, and C satisfy the condition $A = B + C$.

The quantities P/V and A/V scale as l^{-2} and l^{-1}, respectively. These two ratios control many important aspects. An object (like a piece of board) floating on a liquid surface experiences upward force due to the surface tension proportional to the length of the perimeter (Fig. 2.7a). So, the upward force F can be written as βP. On the other hand, the downward force due to gravity w will be ςA where ς is the weight of the sheet per unit area.

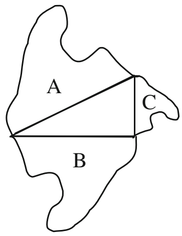


Fig. 2.6c Pythagoras theorem for arbitrary geometrical shape

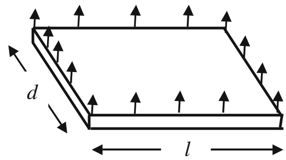


Fig. 2.7a Surface tension on the perimeter of a floating square plate

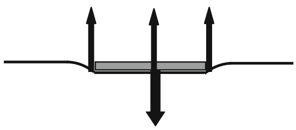


Fig. 2.7b Forces on a floating plate

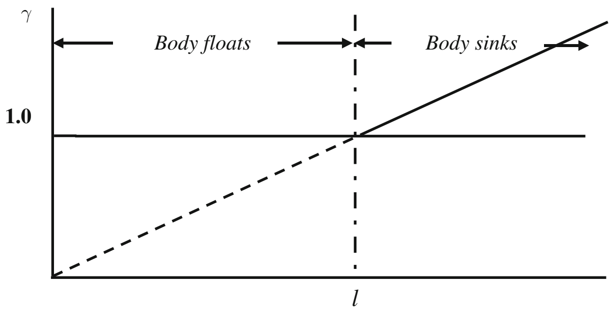


Fig. 2.7c Condition for a board to float

The ratio "γ" of the downward weight *W* and upward force *F*, *W/F*, scales as l^2/l, i.e., *l*.

Figure 2.7c shows that when the size reduces and γ becomes less than 1 the body floats. This was also discovered by Galileo.

3.2 Scaling in Mechanics

Scaling effects on problems of mechanics are very important and needs to be considered while designing systems and devices at microscopic scales. This will be elaborated with the help of a few standard problems in mechanics. To begin with, problems in solid mechanics will be taken up.

3.2.1 Cantilever Beam

Figure 2.8 shows a typical cantilever beam of length L, width b, and thickness h. If a force F acts at the tip of the beam the resulting deflection of the tip is δ. Thus the stiffness of the beam can be represented by the quantity $k = F/\delta$. It is known that

$$\delta = (FL^3)/(3EI) \tag{2.1}$$

where $I = (1/12)bh^3$. Therefore,

$$k = F/\delta = 3EI/L^3 = Ebh^3/4L^3$$

If the material remains the same, the stiffness of a cantilever beam scales as the length scale (l) of the beam. So,

$$k \propto l$$

If one has to find out the stiffness property under the beam's own weight, the deflecting force will be proportional to its weight that scales as l^3. Since I scales as l^4,

$$\delta \propto l^3 \times l^3 \times l^{-4} \propto l^2 \tag{2.2}$$

Thus, smaller beams behave stiffer than the larger ones.

Next, let the problem of the beam's ability to prevent breakage under self-weight be considered. The bending moment is maximum at the fixed end and its magnitude is $(1/2)\rho bhL^2$, where ρ is the density of the beam material. The maximum stress σ_{max} at this end is given by

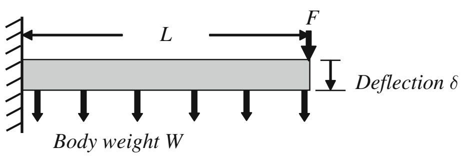


Fig. 2.8 Forces on a cantilever beam

$$\sigma_{\max} = (3\rho L^2)/h \tag{2.3}$$

Hence,

$$\sigma_{\max} \propto l$$

Figure 2.9a shows the nature of dependence of the maximum stress developed on the length scale (i.e., the size of the beam keeping the proportions among various dimensions unchanged). It is seen that beyond a particular size the beam breaks under self-weight as $\sigma_m \geq \sigma_u$, the limiting stress the beam material can withstand.

3.2.2 Simply Supported Beam

A similar analysis for a simply supported beam (shown in Fig. 2.9b) results in $M_{\max} = (1/8)\chi bhL^2$ at the middle of the beam. And the corresponding maximum stress is given by

$$\sigma_{\max} = (3/4)\rho L^2/h \propto l \tag{2.4}$$

In this case also when $\sigma_{\max}$ reaches the breaking stress of the material with the increase in the size of the beam, it breaks under self-weight. This is the reason why a real ship breaks when it is brought out of water though a scale model behaves strongly.

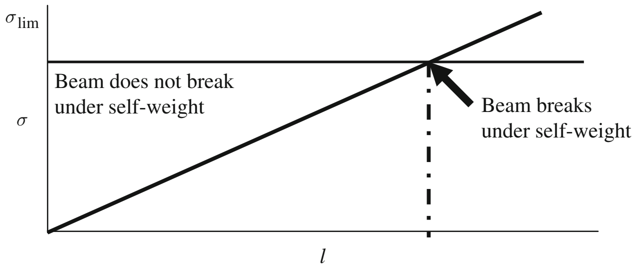


Fig. 2.9a Dependence of maximum stress due to self-weight of a cantilever beam on size

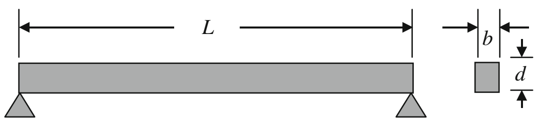


Fig. 2.9b A simply supported beam

3.2.3 Columns

In both the above cases a single phenomenon, bending of beams, played the predominant role. Sometimes, with scale, the deciding factor for failure may shift from one phenomenon to another.

Figure 2.10 shows a solid spherical object of diameter D mounted on a circular column of height h and diameter d. If the diameter of the column be just right to withstand the compressive load then

$$\sigma_u = \{\rho(1/6)\pi D^3\}/\{(\pi/4)d^2\} = (2/3)\rho(D^3/d^2)$$

Hence,

$$d^2 = (2/3)(\rho/\sigma_u)D^3$$

or,

$$d = \sqrt{(2/3)(\rho/\sigma_u)D^3} \propto l^{3/2} \tag{2.5}$$

Thus d has to scale as $l^{3/2}$ to prevent compression failure. When designing such a system it is noticed that with scale the relative dimensions change. For example, in this case D and h may scale as l but d has to scale as $l^{3/2}$, i.e., for larger systems the column is disproportionately thicker as shown in Fig. 2.11.

This is noticed in the case of many trees. The stems of saplings are relatively much slender compared to the trunk of the tree when it grows and becomes big. But there is another important point that needs to be noted. With the change in size even the mode of failure may change from compressive failure of the column to the buckling or instability of the column.

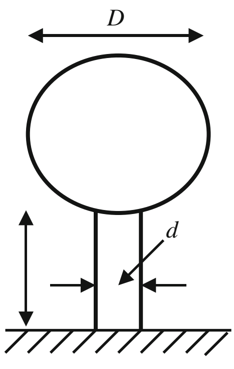


Fig. 2.10 A spherical object supported by a column of circular cross section

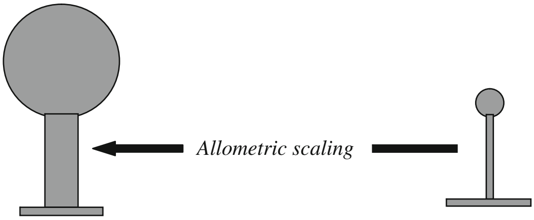


Fig. 2.11 Allometric scaling of a sphere on a column

The condition for buckling of a column, whose upper end is free and the lower end is fixed, is given by $F_b = (\pi^2 EI)/4h^2$, where F_b is the vertical force on the tip of the column to cause instability and buckling failure. Thus $F_b \propto l^2$.

But actual load acting F_a at the tip is the weight of the sphere that is proportional to its volume. So, $F_a \propto l^3$. Hence, Fig. 2.12a indicates how the failure mode may change from compression failure to buckling as the length scale reaches a critical value l^* (h may be also taken as a measure of the length scale).

If the height of the column h be χD then column diameter d_b to prevent buckling is obtained from the fallen equation:

$$\{\pi^2 E(1/64)\pi d_b^4\}/(4\alpha\chi^2 D^2) = (1/6)\pi D^3 \rho$$

or,

$$d_b^4 = \{(128\pi^2/3)(\chi^2 \rho/E)\}D^5 \tag{2.6}$$

Again the diameter of the column d_c to prevent compressive failure is obtained from

$$[\{(1/6)\pi D^3 \rho\}/\{(\pi/4)d_c^2\}] = \sigma_u$$

or,

$$d_c^2 = \{(2/3)(\rho/\sigma_u)\}D^3 \tag{2.7}$$

so, $d_b \propto D^{5/4}$ and $d_c \propto D^{3/2}$ (Fig. 2.12b).

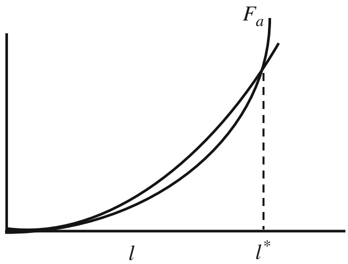


Fig. 2.12a Change of failure mode

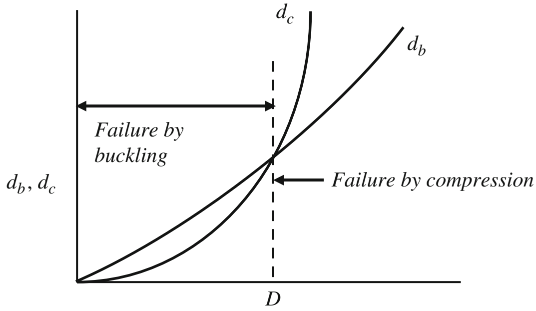


Fig. 2.12b Critical column diameter for change of failure mode

Therefore, at very small scales stability of the supporting columns becomes a problem instead of compressive failure.

3.3 Micromechanisms

In the microscopic scales, mechanisms and machines are generally of monolithic construction. This is primarily to facilitate fabrication. At that small scale it is next to impossible to assemble micron-sized parts and develop revolute hinges and prismatic joints without increasing the cost of the device to prohibitive levels. For providing the capacity of relative motion among various members localized compliances are used. Figure 2.13 shows how a hinge can be replaced by a localized compliance. Though in the second case the range of relative motion is somewhat limited in most situations it does not impose any serious problems or limitation.

Figure 2.14 shows the different types of arrangements normally used to simulate revolute, prismatic, and spherical joints. The maximum possible deflection δ_{max} achievable with a particular material as a function of the dimensions is also given in the figure. To proceed further, some relationships are assigned to the dimensions as follows:

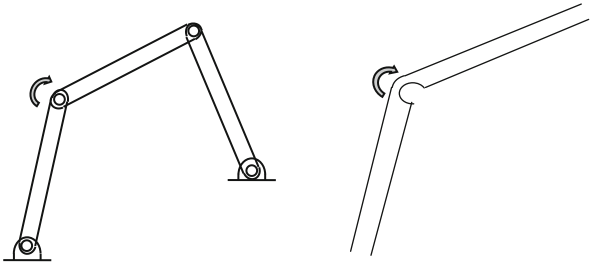


Fig. 2.13 Replacement of a hinge by localized compliance

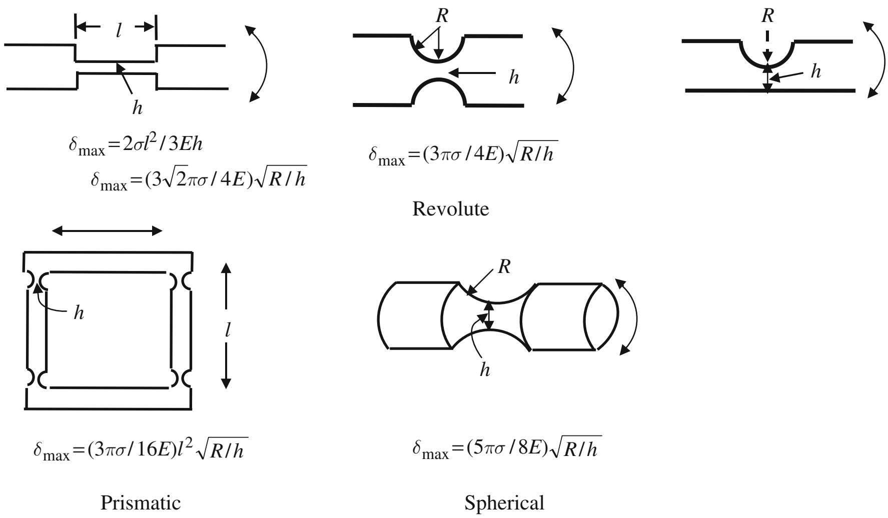


Fig. 2.14 Different types of localized compliances

$$h = \lambda_1 l \text{ in the first case}$$
$$h = \lambda_2 R \text{ in all the other cases}$$
$$h = \lambda_3 l \text{ in the case of the prismatic joint}$$

The parameters λ_1, λ_2, and λ_3 specify the geometric shape independent of the size. Using this dependence on the size scale δ_{max} is found as follows:

$$\delta_{max} = (2\sigma/3\lambda_1 E)l \propto l^1 \text{ first case}$$
$$\delta_{max} = (3\pi\sigma/4E\sqrt{\lambda_2}) \propto l^0 \text{ second case}$$
$$\delta_{max} = (3\sqrt{2}\pi\sigma/4E\sqrt{\lambda_2}) \propto l^0 \text{ third case}$$
$$\delta_{max} = (3\pi\sigma/16E)(\sqrt{3/\lambda_2\lambda_3})l^2 \propto l^2 \text{ fourth case}$$
$$\delta_{max} = (5\pi\sigma/8E)\sqrt{\lambda_2} \propto l^0 \text{ fifth case}$$

Thus, it is clear that except for the first and fourth cases the deflection is independent of the size. Both in the first and fourth cases the deflection increases with size and the systems behave much stiffer in the microscopic scales.

3.4 Scaling in Dynamics

In dynamics the scaling effects can be determined using the laws of motion. The important quantities which are involved in dynamical problems are discussed below.

The displacement scales as l, velocity as lt^{-1} and acceleration as lt^{-2}. Since mass scales as l^3 and force is mass times the acceleration, dynamic force scales as l^4t^{-2}.

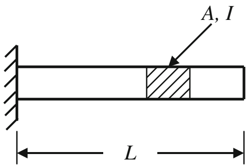


Fig. 2.15 A vibrating cantilever beam

Energy and work done by a force have the same dimension which is force times displacement. Hence, energy scales as l^5t^{-2}. Mass moment of inertia has dimension of mass times length square; thus it scales as l^5. Angular displacement is dimensionless; angular velocity and acceleration have dimensions of t^{-1} and t^{-2}, respectively. Moment, being mass moment of inertia times angular acceleration, scales as l^5t^{-2}. As mass moment of inertia scales as l^5 it is extremely small in microsystems and that is why micromotors can be both started and stopped almost instantaneously. Both mass and mass moment of inertia depend very heavily on the scale and, therefore, their importance in systems of very small size can be ignored. As a result a kinetostatic analysis is good enough in case of micromachines and micromechanisms. Such systems are under static equilibrium conditions at all instants.

Elastic bodies' dynamic characteristics are also influenced by scaling. The example of free vibration of a uniform, prismatic cantilever beam is presented here.

Figure 2.15 shows such a beam of length L, cross-sectional area A, and second moment of area I. If the modulus of elasticity and the density of the material be E and ρ, respectively, the natural frequency of transverse vibration is

$$\omega_{\mathrm{n}} \propto (1/L^2)\sqrt{EI/A\rho} \tag{2.8}$$

So, if the material is kept unchanged and the size of the beam is changed (keeping dimensional proportions intact) it can be seen that $\omega_{\mathrm{n}} \propto l^{-1}$.

As a result the system's natural frequency increases with miniaturization. Typical frequency of microsystems and devices can be in MHz (or even GHz) range. The response time of mechanical systems in the microrange can be as low as electronic devices. This is an important point to be remembered while designing microsystems.

3.5 Scaling in Fluid Mechanics

Matters related to fluid flow are severely affected by scaling effects. Very common daily occurrences like a speck of dust floating around without falling to the ground due to gravity and the reluctance of a fluid in a capillary tube to come out due to gravity, when the tube is kept vertical, are all examples of these scaling effects. The force due to surface tension becomes predominant at microscales as it scales as l in comparison with weight that scales as l^3.

In micron-sized channels the Reynolds number for a fluid flow is very low and may be around 1; it is known that

$$Re \propto vd\rho/\eta$$

where ρ and η are the density and viscosity of the fluid, d is the diameter of the channel, and v is the velocity of the fluid. Since v scales as l it is clear that

$$Re \propto l^2 \tag{2.9}$$

At such low Reynolds number the flow is extremely laminar and it becomes very difficult to mix fluids in microchannels. The viscous drag on a body offered by a fluid limits the terminal velocity v_{lim}

$$v_{\mathrm{lim}} \propto 4gd^2\rho/18\eta \propto l^2 \tag{2.10}$$

As a result v_{lim} is very small for microparticles and this causes the particles to float with the moving air instead of falling.

From Hagen–Poiseuille's equation it is known that the volumetric fluid flow rate satisfies the following equation:

$$Q = (\pi r^4 \Delta p)/(8\eta L) \tag{2.11}$$

where the quantities are explained in Fig. 2.16. So, for a given pressure drop rate (i.e., $\Delta p/L$) $Q \propto l^4$. Hence for a given volume flow rate, the pressure drop rate $(\Delta p/L) \propto l^{-4}$.

So, it is extremely difficult to push fluid through microchannels using pressure drop. This makes conventional pressure-driven pumping difficult for microchannels. Surface driving forces are more suitable as such forces scale favorably.

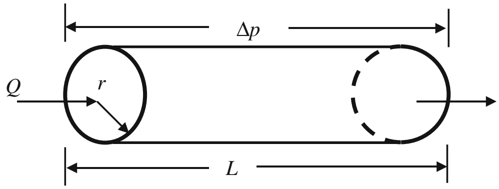


Fig. 2.16 Fluid flow in a circular pipe

3.6 Scaling of Common Forces

Solving dynamical problems involves various types of forces. The most common types of forces and their scaling are tabulated below and Fig. 2.17 shows the nature of some common types of forces.

Force	Scaling
Surface tension	l^1
Fluid force/electrostatic force	l^2
Weight/inertia force/electromagnetic	l^3
Electromagnetic force (constant current density)	l^4

For handling different types of forces in a compact manner, William Trimmer proposed a matrix representation for force scaling. This column matrix, called the "force scaling vector F" is defined as follows:

$$F = [l^F] = \begin{bmatrix} l^1 \\ l^2 \\ l^3 \\ l^4 \end{bmatrix} \tag{2.12}$$

The above matrix represents four different cases of force laws (given in the table above) which scale differently. Using this nomenclature, scaling of different parameters can be determined as explained below.

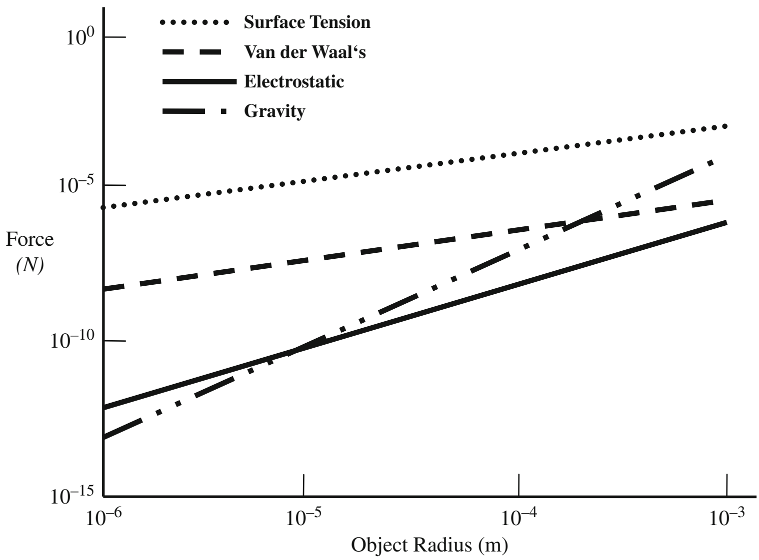


Fig. 2.17 Scaling of different types of forces

3.7 Scaling of Acceleration

From the second law of motion, $F = ma$, with m as mass and a as acceleration. So, $a = F/m$ and its scaling for different types of force can be expressed in a compact form as follows:

$$[a] = \left[l^F\right] \times \left[l^3\right]^{-1} = \begin{bmatrix} l^1 \\ l^2 \\ l^3 \\ l^4 \end{bmatrix} \times \left[l^{-3}\right] = \begin{bmatrix} l^{-2} \\ l^{-1} \\ l^0 \\ l^1 \end{bmatrix} \qquad (2.13)$$

because m scales as volume, i.e., as l^{-3}.

3.8 Scaling of Time

If s be the displacement in time t (starting from rest)

$$s = (1/2)at^2$$

So,

$$t = (2s/a)^{1/2} = (2sm/F)^{1/2}$$

Thus,

$$[t] = [s]^{1/2}[m]^{1/2}[F]^{-1/2} = \left[l^1\right]^{1/2}\left[l^3\right]^{1/2}\begin{bmatrix} l^1 \\ l^2 \\ l^3 \\ l^4 \end{bmatrix}^{-1/2} = \begin{bmatrix} l^{1.5} \\ l^1 \\ l^{0.5} \\ l^0 \end{bmatrix} \qquad (2.14)$$

3.9 Power Density

Power density is defined as the power supply per unit volume of the device. This is a very important parameter for designing a device. Too little of it can result in inactivity and too much of it can damage a device. The scaling of power density for different types of forces involved can be found as given below.

Let $\gamma = P/V$ where $\gamma =$ power density, $P =$ power, and $V =$ volume of the device under consideration. Now $P =$ (work/time) $= Fs/t$ and $\gamma = Fs/Vt$.

Therefore, scaling of γ can be determined as outlined below:

$$\left[\gamma\right] = [F] \times [s] \times [V]^{-1} \times [t]^{-1}$$

Using (2.12) and (2.14) in the R.H.S. of the above equation and noting that $[s] = [l^1]$ and $[V] = [l^3]$ the following equation is obtained:

$$[\gamma] = \begin{bmatrix} l^1 \\ l^2 \\ l^3 \\ l^4 \end{bmatrix} [l^1] \times [l^3]^{-1} \begin{bmatrix} l^{1.5} \\ l^1 \\ l^{0.5} \\ l^0 \end{bmatrix} = \begin{bmatrix} l^{-2.5} \\ l^{-1} \\ l^{0.5} \\ l^0 \end{bmatrix} \tag{2.15}$$

The table below presents the important parameters for four different types of force law:

Force scaling	Acceleration scaling	Time scaling	Power density scaling
l^1	l^{-2}	$l^{1.5}$	$l^{-2.5}$
l^2	l^{-1}	l^1	l^{-1}
l^3	l^0	$l^{0.5}$	$l^{0.5}$
l^4	l^1	l^0	l^2

So, it is clear that devices operated by electrostatic actuators tend to have higher power density at small scales whereas for electromagnetic motors power density is low for small sizes. Thus miniaturized machines should be operated by electrostatic forces instead of electromagnetic drives.

3.10 Scaling of Electrical Parameters

Besides mechanics scaling laws have important consequences for electrical systems also. The three passive electrical elements, resistors, capacitors, and inductors, are taken up for determining the scaling effects.

3.10.1 Resistance

The resistance of a given conductor (Fig. 2.18a) of length L, cross-sectional area A, and specific resistivity ρ is given by

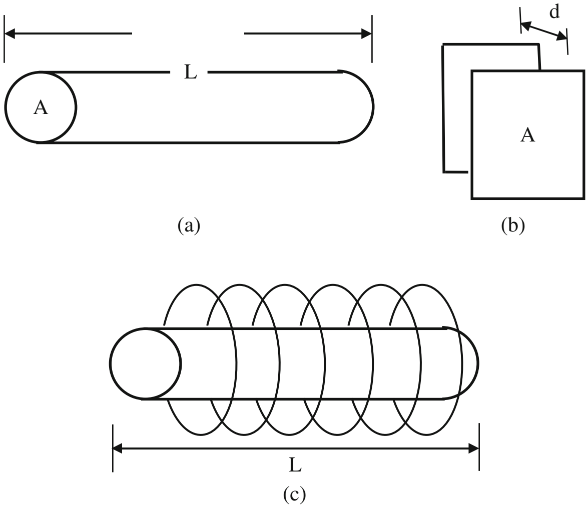
$$R = L\rho/A = [l^1] \times [l^2]^{-1} = l^{-1}$$

So, for a given material electrical resistance scales as l^{-1}.

3.10.2 Capacitance

For a parallel plate capacitor (Fig. 2.18b) of plate area A, plate gap d, and ε as the permittivity of gap insulation material the capacitance

Fig. 2.18 Basic electrical elements



$$C = \varepsilon A/d = \left[l^2\right] \times \left[l^1\right]^{-1} = l^1$$

So, capacitance scales as l^1.

3.10.3 Inductance

With N as the number of coils per unit length with a coil area A and L as the length of the inductor (Fig. 2.18c), the inductance $\mathcal{L}$ is given by

$$\mathcal{L} = \mu N^2 A/L = \left[l^2\right] \times \left[l^1\right]^{-1} = l^1$$

where μ is the permeability of the material between the coils. Hence, the inductance scales as l^1.

In an electrical circuit combinations of these basic units govern the characteristics of a system. For example, the time constant of a circuit is dependent on the product RC and governs the behavior when the voltage varies. To understand how it will scale the following expression helps:

$$RC = [R] \times [C] = \left[l^{-1}\right] \times \left[l^1\right] = \left[l^0\right]$$

So time constant is independent of scaling.

3.11 Some Important Scaling Laws

A table containing the scaling exponent of different physical quantities is given for ready reference and use. Scaling laws for more complex parameters can be found out from these. If the parameter/quantity be P then scaling will be $[P] = [l^n]$

Physical quantity, P	Scaling exponent "n"
Bending stiffness	1
Mass	3
Mass moment of inertia	5
Second moment of area	4
Strength	2
Shear stiffness	1
Natural frequency	−1
Reynolds number	2
Electrical resistance	−1
Electrical capacitance	1
Inductance	1
Surface tension, van der Walls force	1
Fluid force	2
Inertia force	3, 4
Kinetic energy	3 (Const. speed), 5
Potential energy (gravitational)	4
Elastic potential energy	2
Strength to weight ratio	−1
Resistance power loss	1
Thermal time constant	2
Heat capacity	3
Electric field energy	−2
Available power	3
Power loss/power available	−2
Electromagnetic force	3
Electrostatic force	2

3.12 Scaling in Electromagnetic and Electrostatic Phenomena

When two parallel electrically conducting plates of area "A" at a distance "d" and a voltage "V" are applied as shown in Fig. 2.19, the corresponding potential energy stored in the capacitor that is formed is given by $U = -1/2CV^2$, where C is the capacitance. The capacitance C for the system is proportional to the area "A" and inversely proportional to the gap between the plates "d." Thus

$$U \propto AV^2/d \tag{2.16}$$

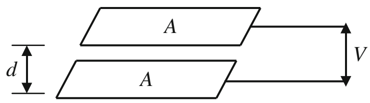


Fig. 2.19 A charged capacitor

The area A scales as l^2 and the gap d scales as l^1. To find out the scaling effect of the voltage V it may be done by considering the breakdown voltage given by Paschen's effect as shown in Fig. 2.20.

Approximately, the breakdown voltage can be taken as proportional to the gap d when $d > 10$ μm. Hence, the voltage V can be considered to scale as l^1. Hence, the electrostatic potential energy

$$U \propto l^2/l^1(l^1)^2 \propto l^3$$

Scaling of electrostatic force can be found out as shown in Fig. 2.21.

At the symmetric position F_L and F_B will be zero. But when the plates are misaligned

$$F_L = -\partial U/\partial L \propto l^3/l^1 \propto l^2$$
$$F_B = -\partial U/\partial B \propto l^3/l^1 \propto l^2$$
$$F_d = -\partial U/\partial d \propto l^3/l^1 \propto l^2$$

Thus, electrostatic force scales as l^2.

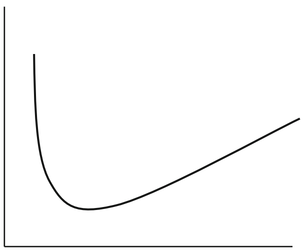


Fig. 2.20 Paschen's effect

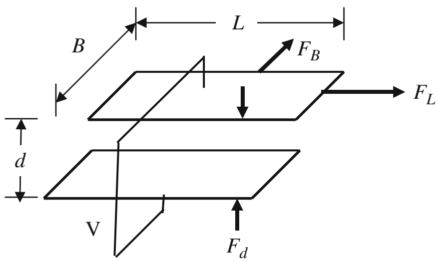


Fig. 2.21 Scaling of electrostatic force from a charged capacitor

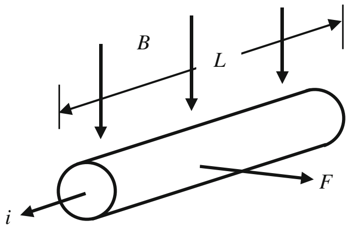


Fig. 2.22 Current-carrying wire in a magnetic field

Figure 2.22 shows a wire of length L carrying a current i in a magnetic field B. The force F on the wire is given by

$$F = iLB \tag{2.17}$$

For a given current density the current i is proportional to the cross-sectional area of the wire. So,

$$i \propto l^2$$

Again $B = \mu_0(n/L_C)i_C$, where μ_0 is the permeability, n is the number of turns in the wire, L_C is the total length of the coil, and i_C is the current through the coil producing the magnetic field. As i_C scales as the cross-sectional area of the coil area it scales as l^2 and L_C scales as l^1. Thus B scales as l.

Therefore, as $L \propto l$, $F \propto l^4$. Thus, electromagnetic force scales as l^4.

3.13 Scaling Laws Related to Surface/Volume Ratio

As the surface area (S) of an object scales as l^2 and the volume (V) as l^3 the ratio S/V scales as l^{-1}. This simple scaling law has many profound implications both in the living and nonliving worlds. It decides many important matters in living organisms, thermal phenomena, dynamics of particles in fluids, etc. Figure 2.23 shows the way S/V ratio varies with size.

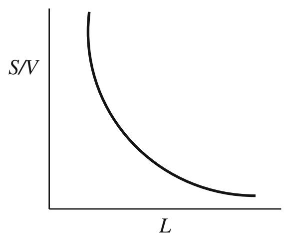


Fig. 2.23 Variation of *S/V* ratio with size

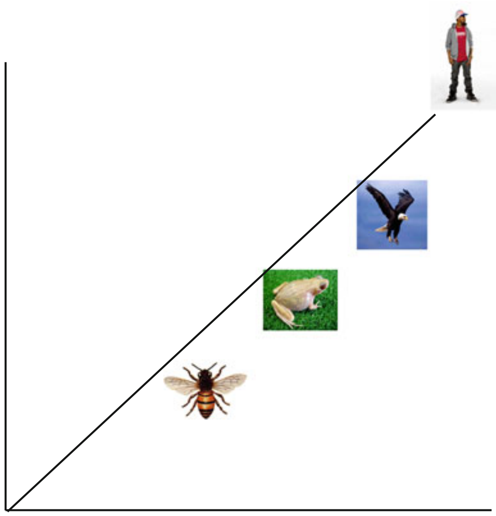


Fig. 2.24 Effect of *S/V* ratio on living objects

If one considers the phenomena of heating and cooling the influence of S/V ratio becomes very apparent. When a microsized chip comes out of girding operation it is white hot and brilliantly luminous. But by the time it flies away from the machining area taking only a fraction of a second it cools down to almost room temperature. Similarly during hailstorm the small-sized hails melt away very quickly but big-sized chunks of ice from ice factories take a long time to melt away. Both are effects of the scaling law applied to the S/V ratio. Heating or cooling rate of an object depends on the surface area but the amount of heat in the body at a given temperature is proportional to the volume. The S/V ratio scales as l^{-1} and, therefore, is much larger for smaller objects compared to a big-sized body. Thus, small objects (like the grinding chips) cool down (or lose the heat of the body) very fast.

In the living world also the S/V ratio plays a critical role. Eating food generates heat through metabolism. So, to maintain a specific temperature small-sized mammals have to eat more frequently as due to larger S/V ratio they tend to lose heat at a relatively faster rate. Figure 2.24 shows this phenomenon. For still smaller living organisms like insects it is not possible to maintain a fixed body temperature.

It was already mentioned how the small dust particles float around in air current (even if of small speeds). This is because of the fact that the viscous drag with air is proportional to surface area but the gravitational pull depends on the volume.

3.14 Allometric Scaling Laws in Biology

There is nothing that is more complex than life. However, despite the extreme complexity and diversity, living organisms obey certain very simple scaling laws. The general equation that represents the scaling behavior of living organisms spanning a mass range of over 21 orders of magnitude (smallest microbe of 10^{-13} g to the largest mammals and plants of mass 10^8 g) can be written as follows:

$$X = X_0 M^{\lambda} \tag{2.18}$$

where X is some observable (and quantifiable) biological parameter, X_0 is a normalizing constant, M is the mass of the organism, and λ is an exponent. An equation of this type is called allometric scaling law and λ is the allometric exponent. The table below shows various parameters along with the corresponding exponent.

Sl. No.	Parameter, X	Exponent, λ
1	Metabolic rate	3/4
2	Lifespan	1/4
3	Growth rate	−1/4
4	Heart beat rate	−1/4
5	Length of aorta, height of trees	1/4
6	Radii of aorta, radii of tree trunks	3/8

A very interesting point is that λ takes values which are simple multiplies of ¼. Figure 2.25 shows the plot of metabolic rate against body mass for the whole spectrum of living organisms. It is also interesting to note from the above table that the fundamental principles of life provide for certain invariant quantities like the total number of heart beats in a lifetime which is approximately equal to 1.5×10^9, irrespective of the size of the organism. Another interesting point regarding the metabolic rate scaling must be discussed here. It was mentioned earlier that the rate of heat loss through the surface scales as l^2. But the metabolic rate, which is proportional to the heat generation rate, scales as l^3 and is proportional to mass M. Thus, to maintain a constant temperature the rate of heat generation should be equal to the rate of heat loss. M scales as volume, i.e., l^3, and the surface area S scales as l^2, given as $(M^{1/3})^2$ or $M^{2/3}$. Thus, the metabolic rate should scale as $M^{2/3}$

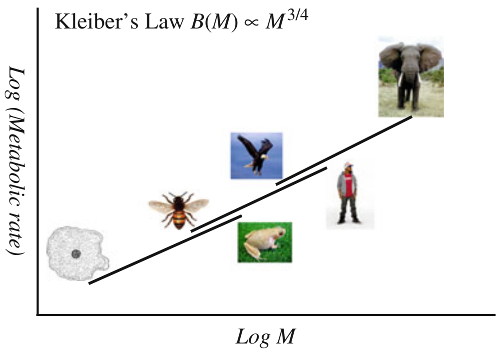


Fig. 2.25 Kleiber's law

for maintaining a fixed body temperature. But research over a long period by many researchers has established that

$$\text{Metabolic rate} = B \propto M^{3/4} \tag{2.19}$$

This is known as Kleiber's law. This controversial issue was finally explained by West more recently and the fractal nature of the fluid transportation system (blood circulatory system in animals) is the true reason behind the exponent being 3/4 instead of 2/3 as expected according to the scaling of surface to volume ratio.

As mentioned before this scaling law gave a deeper understanding of the life forms at all scales and the fractal nature of the fluid distribution system got unraveled as the most efficient form of transport.

Certain other features related to the living world also demonstrate the significance of scaling laws. The strength and ability to jump always appear to be disproportionately higher for smaller living objects. The strength always appears to be more when the size scales down as shown earlier. Certain other abilities can also be scaled.

3.14.1 Jumping

Figure 2.26 shows an animal of body mass M jumping vertically up through a height "h." So, the potential energy gained is equal to "Mgh." This energy is supplied by the muscles of the limbs and is proportional to the mass of the muscles "m." So, work done (= energy gained due to the height gained)

$$W \propto m$$

But the mass of the muscles "m" is proportional to the overall body mass "M." Hence

$$W \propto m \propto M \tag{2.20}$$

Again W = gain is potential energy = Mgh. Hence,

$$Mgh \propto M \tag{2.21}$$

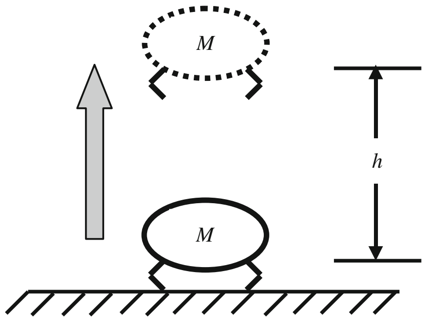


Fig. 2.26 Jumping of an animal

Equation (2.21) leads to the conclusion that "h" is independent of M. Or, in other words, the ability to gain height by leaping is independent of the size of the animal. So, smaller animals appear to have relativity higher capability to jump.

3.14.2 Running

The number of steps "n" an animal has to take for covering a given distance is inversely proportional to the size of limbs. Or,

$$n \propto L^{-1} \tag{2.22}$$

where L represents the limb size. Thus, "n" scales as l^{-1}.

Again the body mass M depends on the volume and so M scales as l^3. Or,

$$\begin{aligned} & M \propto L^3 \\ \therefore \quad & L \propto M^{1/3} \end{aligned} \tag{2.23}$$

And using (2.23) in (2.22)

$$n \propto M^{-1/3} \tag{2.24}$$

The work done per step is proportional to the body mass M. So, for covering a given distance (using "n" steps) the work done

$$W \propto nM \propto M^{1/3} M$$

or,

$$W \propto M^{2/3} \tag{2.25}$$

For comparing the situations using a common norm the work done to move a given distance per unit body mass (specific energy of running)

$$w = W/M \propto M^{2/3}/M \propto M^{-1/3} \tag{2.26}$$

The above equation shows that larger the animal, less is the specific energy of running. So, as the energy available for an animal is proportional to its body mass M a bigger animal can cover a larger distance.

3.14.3 Swimming

The speed of swimming or cruising for an animal or a vessel depends on the power of the entity and the fluid resistance it has to overcome. Now work done per unit time (i.e., power) is

$$W \propto \text{distance covered in unit time} \propto v$$

Again $W \propto$ fluid resistance and fluid resistance $\propto$ surface area (S). Furthermore, fluid resistance is also proportional to speed (V).

Hence compiling the above relations

$$P \propto V^2 S \tag{2.27}$$

Now the power depends on the body mass, i.e., body volume ($\propto L^3$) and scales as l^3. But, the surface area S scales as l^2 ($\propto L^2$) Thus, from (2.27) one gets

$$V^2 \propto P/S \propto L \tag{2.28}$$

And $V \propto \sqrt{L}$. Hence, swimming/cruising speed scales as $l^{1/2}$. This is known as Froude's law.

3.14.4 Flying

The capability to fly (for birds and planes) also follows nice scaling laws. If Mg be the body weight, S the wing area, V the speed, and α the lift, then

$$Mg = \text{Lift} = \alpha \propto SV^2$$

or,

$$M \propto SV^2 \tag{2.29}$$

Now if L represents a characteristic length of the flying object, $M \propto L^3$ and $S \propto L^2$. Thus, $L \propto M^{1/3}$ and hence $S \propto M^{2/3}$. Using this in (2.29)

$$M \propto M^{2/3} V^2$$

or,

$$M^{1/3} \propto V^2$$

or,

$$V \propto M^{1/6} \tag{2.30}$$

Figure 2.27 shows the cruising speed for flying objects over a range of their sizes.

3.15 Some Other Interesting Scaling Laws

Simple power laws $y(x) = y_0 x^\lambda$ are abundant in nature and such laws are self-similar. Thus, if "x" is scaled by multiplying it by a factor "a," $y(x)$ will be $(y_0 a^\lambda) x^\lambda$ which is a similar power law.

Many phenomena of nature occur in different sizes or intensities and in most cases they follow simple scaling laws.

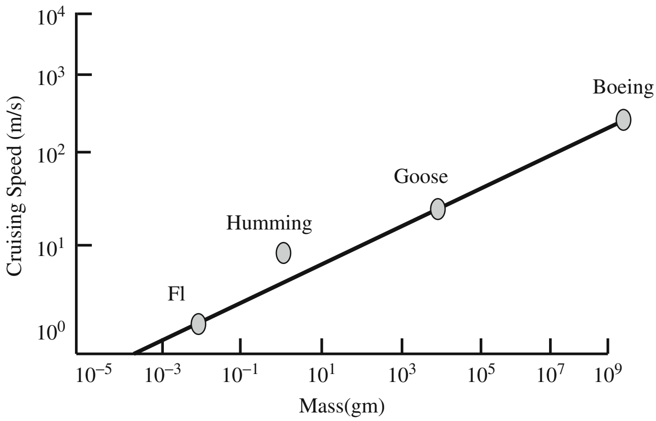
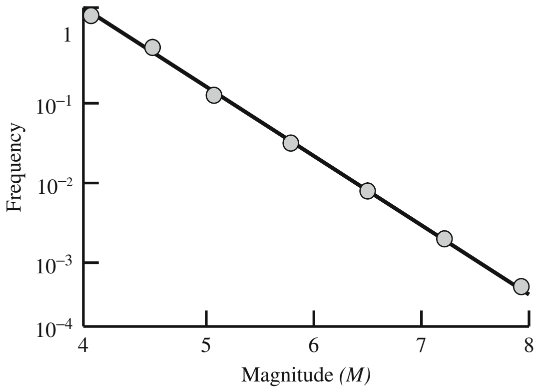


Fig. 2.27 Scaling of cruising speed

Fig. 2.28 Scaling of earthquake frequency



The frequency of occurrence of earthquakes and their respective intensities follow a very nice power law and it is known as Gutenberg–Richter scaling law as given by $\log N = a - bM$, where N is the frequency of occurrence and M is the magnitude in Richter scale. Figure 2.28 shows the situation.

The size of meteors falling on the earth and their frequency of appearance also follow the nice scaling law shown in Figure 2.29.

Another important phenomenon, number of species versus their size, also follows a scaling law to some extent but not exactly (Fig. 2.30).

3.16 Importance of Scaling Laws in Microactuation

It has already been shown in Fig. 2.17 how different types of forces scale. In the macroscopic scale mostly electromagnetic actuators are employed. Electromagnetic

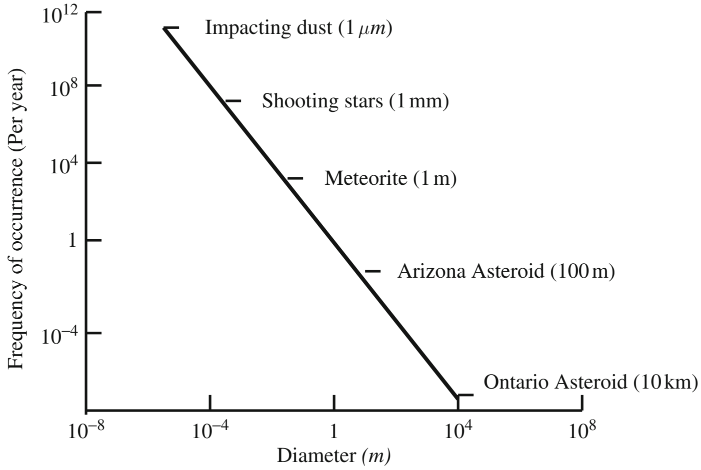


Fig. 2.29 Scaling of objects falling on the earth

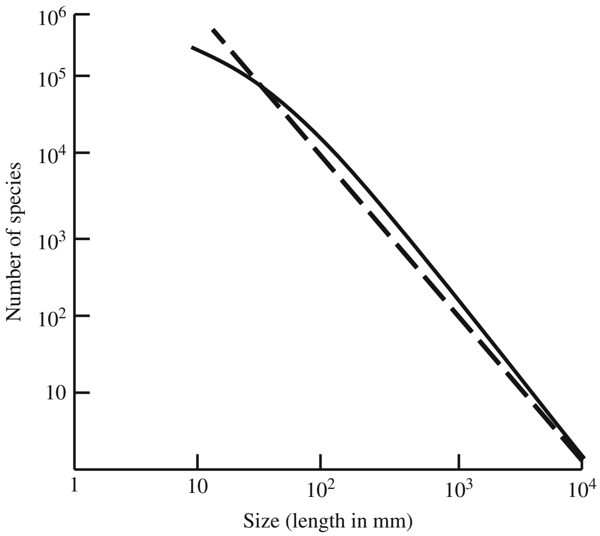


Fig. 2.30 Number of species with size

force scales as either l^3 or l^4. Hence, when the size of the device is very small actuation principle based upon electromagnetic force becomes ineffective. On the other hand, electrostatic force scales as l^2 and, therefore, actuators based upon electrostatic force are more suitable for microscopic devices.

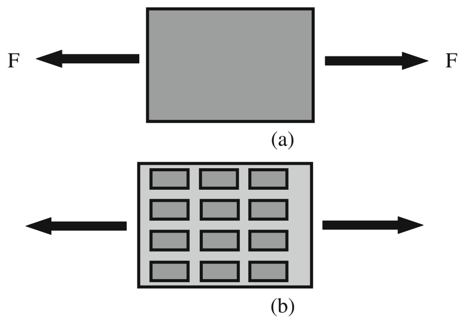


Fig. 2.31 Massively parallel miniaturized actuator

The scaling law can be very nicely used even for actuators of macroscopic size. One example can explain the principle. Figure 2.31a shows an actuator with a characteristic length L based upon electrostatic force and let the force it can develop be F. Now the actuator is miniaturized and scaled down by a factor n. Since length scale is reduced by a factor n the volume of the miniaturized device will be n^{-3} times the volume of the original actuator. Next n^3 numbers of these miniaturized actuators are placed inside a volume equal to that of the original device and arrangement is made so that all these n^3 numbers of actuators act in parallel.

Since the size of each device is reduced by a factor n each one will develop a force F/n^2 as electrostatic force scales as l^2. The n^3 numbers of these actuators, which act in parallel, will develop a total force

$$n^3 \times F/n^2 = nF$$

When $n \sim 1000$ almost 1000 times larger force can be generated by the composite actuator. This principle of miniaturized and massively parallel systems is employed by nature in many situations and the muscles are nothing but massively parallel miniaturized protein molecules acting on the basis of electrostatic force.

4 Concluding Remarks

Scaling laws play an extremely important role in designing microsized systems and devices. Macrointuition can often mislead a designer as intuition develops based on their experience in the macroworld. Scaling laws can help a designer to develop a kind of microintuition. It should also be remembered that completely different concepts may be essential for designing miniaturized systems and guidance to that can come from an understanding of the scaling laws. Since miniaturization is going be a very important phenomenon leading the world toward the eagerly awaited Third Industrial Revolution, study of scaling laws is going to acquire primary importance in the coming years.

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