Discrete Finite Probability

In these notes, I will consider only the finite discrete case.

That is, in every situation the possible outcomes are all distinct cases, which can be modeled by integers, and there are a finite number of such outcomes.

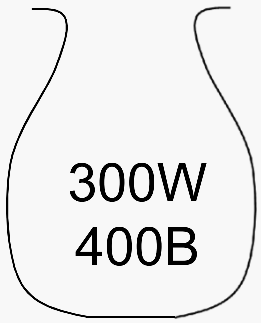
The discussion that follows was drawn from a number of sources, including:

Probability by G E Bates,
Addison-Wesley, 1965 0-201-00405-4

Puzzle-based Learning by Michalewicz & Michalewicz,
Hybrid, 2008 978-1-876462-5

A Challenge

Suppose we have an urn that contains 300 white balls and 400 black balls:



Suppose we draw 2 balls. The probability that we get 1 black ball and 1 white ball is:

$$\frac{\binom{400}{1}\binom{300}{1}}{\binom{700}{2}} = \frac{400 \cdot 300}{(700 \cdot 699)/2} \approx 0.49$$

Analysis

Here are the details of that last example:

Note the use of the Multiplication Rule here

$$\frac{\binom{\text{\# ways to choose}}{\text{1 black ball}}\binom{\text{\# ways to choose}}{\text{1 white ball}}}{\binom{\text{\# ways to choose 2 balls}}{\text{w/o restrictions}}} = \frac{\binom{400}{1}\binom{300}{1}}{\binom{700}{2}}$$

$$\binom{400}{1} = \frac{400!}{1!(400-1)!} = \frac{400!}{399!} = 400$$

$$\binom{700}{2} = \frac{700!}{2!(700-2)!} = \frac{700!}{2 \cdot 698!} = \frac{700 \cdot 699}{2} = 244650$$

Examples

With an urn that contains 300 white balls and 400 black balls:

Suppose we draw 5 balls. The probability that we get 2 black balls and 3 white balls is:

$$\frac{\binom{400}{2}\binom{300}{3}}{\binom{700}{5}} = \frac{\frac{400!}{2!398!}\frac{300!}{3!297!}}{\frac{700!}{5!695!}} = \frac{\frac{400\cdot 399}{2}\frac{300\cdot 299\cdot 298}{6}}{\frac{700\cdot 699\cdot 698\cdot 697\cdot 696}{120}}$$

$$= \frac{200\cdot 399\cdot 50\cdot 299\cdot 298}{35\cdot 699\cdot 698\cdot 697\cdot 116} \approx 0.257$$

Examples

With an urn that contains 300 white balls and 400 black balls:

Suppose we draw 5 balls. What's the probability that we get at least 3 black balls?

The key question is: in how many ways can we choose 5 balls and include at least 3 black balls?

So, we have three distinct (nonoverlapping) cases:

- 3B and 2W
- 4B and 1W
- 5B and 0W

Note the use of the Addition Rule here

$$\binom{400}{3}\binom{300}{2} + \binom{400}{4}\binom{300}{1} + \binom{400}{5}\binom{300}{0}$$

Examples

So, the probability that we get at least 3 black balls is:

$$\frac{\binom{400}{3}\binom{300}{2}+\binom{400}{4}\binom{300}{1}+\binom{400}{5}\binom{300}{0}}{\binom{700}{5}}$$

And, the probability that we get fewer than 3 black balls is:

$$1-\frac{\binom{400}{3}\binom{300}{2}+\binom{400}{4}\binom{300}{1}+\binom{400}{5}\binom{300}{0}}{\binom{700}{5}}$$

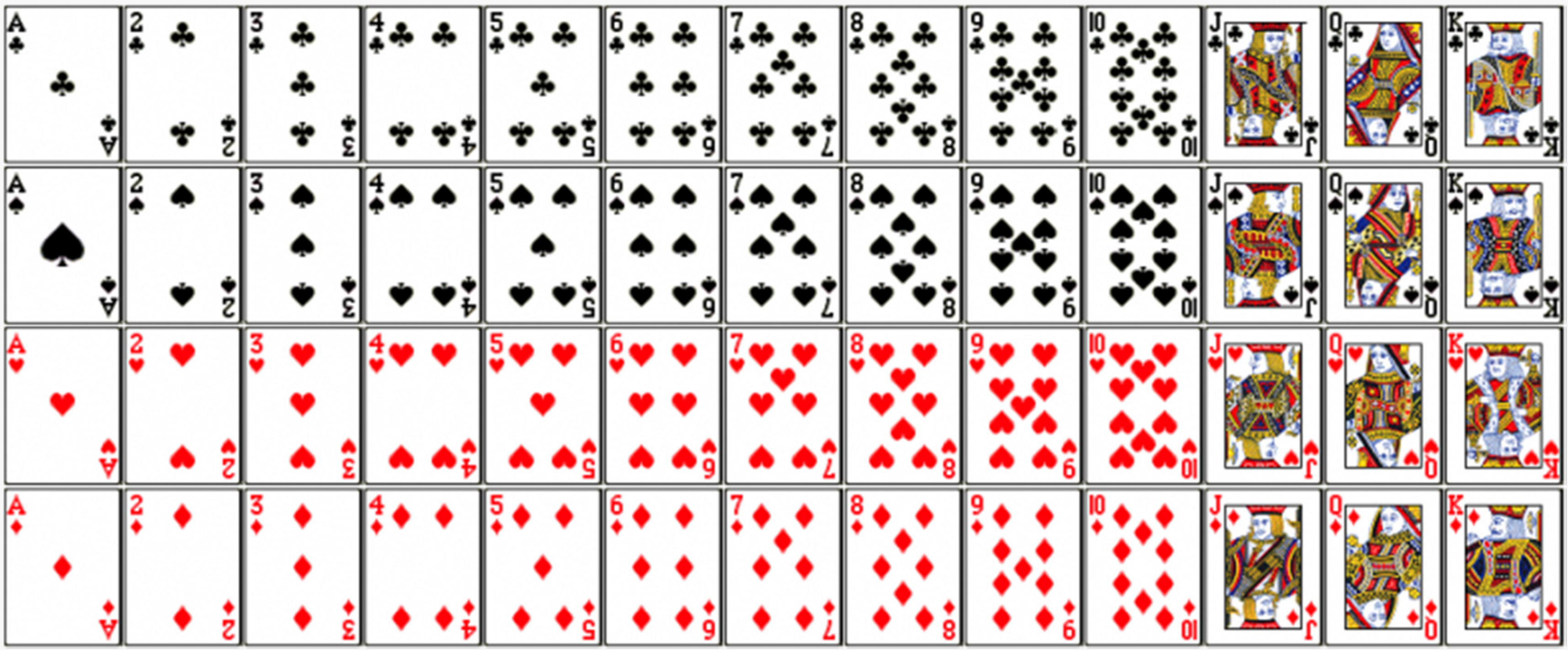
Poker Deck

A standard poker deck:

contains 52 cards,

divided into 4 suites (spades, clubs, hearts, diamonds),

with 13 values in each suite (Ace, 2, 3, …, 9, 10, Jack, Queen, King)



Counting Poker Hands

A *full house* is a poker hand that consists of 3 cards of one value and 2 cards of another value.

For example:

3ofHearts, 3ofClubs,3ofDiamonds,JackofSpades,JackofClubs

So, how many different full houses are there? (Not all at once.)

Number of Full Houses

We can construct a full house by carrying out the following sequence of tasks:

- choose the value for the three-of-a-kind
- choose 3 cards of that value
- choose a different value for the pair
- choose 2 cards of that value

Hence, using the Multiplication Rule:

$$\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}$$

Number of Two Pairs

This one's tricky. If you pick the values for the two pairs separately, you will double-count:

- choose a value for one pair
- choose 2 cards of that value
- choose a different value for the other pair
- choose 2 cards of that value
- choose a different value for the fifth card
- choose 1 card of that value

$$\binom{13}{1}\binom{4}{2}\binom{12}{1}\binom{4}{2}\binom{11}{1}\binom{4}{1}$$

That's incorrect… but the error is subtle.

Number of Two Pairs

What if you try to pick the value for the higher pair first, then pick the value for the lower pair?

- choose a value for the higher pair
- choose 2 cards of that value
- choose a lower value for the other pair
- choose 2 cards of that value
- choose a different value for the fifth card
- choose 1 card of that value

$$\binom{12}{1}\binom{4}{2} ?$$

The problem is that we don't know how many possible values we have left to choose from unless we know what the higher value was…

Number of Two Pairs

Two pairs consists of two cards of one value, two cards of another value, and a fifth card of a third value.

We can construct such a hand by:

- choose the values for the two pairs
- choose 2 cards of the higher value
- choose 2 cards of the lower value
- choose a different value for the fifth card
- choose 1 card of that value

Hence, using the Multiplication Rule:

$$\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{11}{1}\binom{4}{1}$$

A Puzzle

There are three cards in a bag.

One card has the symbol **X** written on both sides; one card has the symbol **O** written on both sides; one card has the symbol **X** on one side and the symbol **O** on the other side.

You draw one card at random and examine one side of the card.

You see the symbol **X** on that side of the card.

What is the probability that there is also an **X** on the other side?

Solution

There are three cards in the bag:

XX **OO** **XO**

Now, we know we did not draw the card **OO**.

So, the probability is 1/2, right?

Nope. (Although we're right to eliminate **OO**.)

We could be seeing any of three different **X**s:

XX **XO**

If we're seeing either of the first two, then there's an **X** on the other side.

If we're seeing the third **X**, there's an **O** on the other side.

So, the probability is 2/3, not 1/2.

Solution

A clearer way of saying this is that we are thinking of the sample space (the set of all possible outcomes) incorrectly:

XX **XO**

We are looking at one side of a card, and we know that's an **X**.

A precise description of the sample space is that when we have the following possible cases:

X on front of another **X**

X on back of another **X**

X on front of an **O**

So, in 2 cases out of 3, we're looking at the card with 2 **X**'s.

Expectation

Suppose we have a probabilistic experiment that has a sample space

$$\{e_1, e_2, \ldots, e_N\}$$

and that for each outcome in the sample space we have a probability $p(e_K)$.

And, suppose that corresponding to each outcome in the sample space we have an associated value v_K.

Then the *expectation* or *expected value* of the experiment is the weighted average of the values, using the outcome probabilities as weights:

$$\sum_{K=1}^{N} p(e_K) v_K$$

Example

A game is played by drawing balls out of an urn; the players gain $2 when a green ball is drawn and lose $1 when a red ball is drawn.

Suppose the urn contains 30 green balls and 70 red balls.

What is the expected payoff for a round of this game?

Solution

The probability of drawing a green ball is 0.3 and the probability of drawing a red ball is 0.7.

The value of drawing a green ball is +2 and the value of drawing a red ball is -1.

So, the expectation for a single draw would be

$$0.3 * 2 + 0.7 * -1 = -0.1$$

Hence, you'd expect, on average, to lose 10 cents each time you play the game.