Discrete Finite Probability

In these notes, I will consider only the finite discrete case.

That is, in every situation the possible outcomes are all distinct cases, which can be modeled by integers, and there are a finite number of such outcomes.

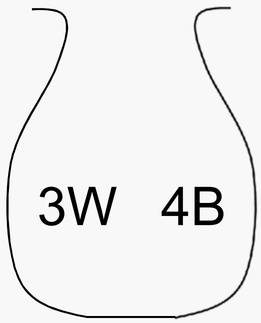
The discussion that follows was drawn from a number of sources, including:

Probability by G E Bates,
Addison-Wesley, 1965 0-201-00405-4

Puzzle-based Learning by Michalewicz & Michalewicz,
Hybrid, 2008 978-1-876462-5

Urns and Balls

Suppose we have an urn that contains 3 white balls and 4 black balls:



Now suppose we shake the urn to randomly (i.e., unpredictably) mix the balls together and then we draw out one ball.

How likely is it that we will draw out a white ball?

Suppose, instead, we draw out two balls.

How likely is it that we will draw out one white ball and one black ball?

Basic Concepts

A *probabilistic experiment* is any action (e.g., tossing a coin, rolling a die, drawing a ball from an urn) where the outcome of the action is not known in advance.

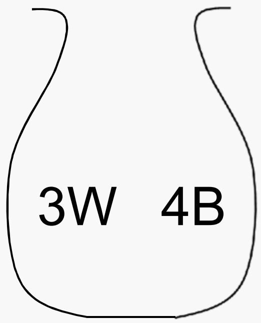
Such an experiment can lead to any of a number of results, called *outcomes*.

The set of all possible outcomes is called the *sample space*.

A subset of the sample space is called an *event*.

A subset of the sample space that contains a single outcome is called an *elementary event* or simply an *outcome*.

Suppose we have an urn that contains 3 white balls and 4 black balls:



Our first experiment is to draw a single ball from the urn.

There are 7 possible outcomes, since we could draw any of 3 white balls or any of 4 black balls, so the sample space is:

$$\{w_1, w_2, w_3, b_1, b_2, b_3, b_4\}$$

The event of interest is that we draw a white ball:

$$\{w_1, w_2, w_3\}$$

Naïve Probability

The likelihood that an outcome actually occurs is its *probability*.

An outcome that is impossible (e.g., drawing a green ball in our experiment) is assigned the probability 0 (or 0%).

An outcome that is inevitable (e.g., getting "heads" when flipping a two-headed coin) is assigned the probability 1 (or 100%).

Of course, most interesting outcomes are neither impossible nor inevitable; those outcomes have probabilities that are larger than 0 and less than 1.

But, how do we determine the probability of an outcome?

First, consider the simplest case: every possible outcome is just as likely to occur as any other possible outcome.

In this case, we say the outcomes are *equally likely* (or *equiprobable*).

Equally-likely Outcomes

If the balls in our urn are truly mixed in an unpredictable (or random) manner, it seems reasonable to declare that each ball is just as likely to be drawn as any other ball.

Since some outcome must occur, the sum of the probabilities of all the outcomes must equal 1.

So, the probability that any particular ball will be drawn from the urn is 1/7.

Probability of an Event

The probability of an event is the sum of the probabilities of the outcomes that belong to that event.

So, if the event of drawing a white ball is:

$$DrawWhite = \{w_1, w_2, w_3\}$$

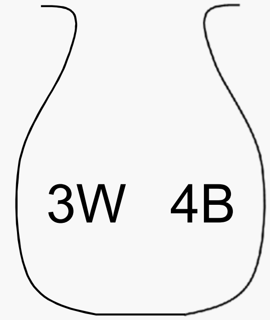
then the probability of drawing a white ball is:

$$P(DrawWhite) = P(w_1)+P(w_2) +P(w_3) = 1/7 + 1/7 + 1/7 = 3/7$$

Notice that if the outcomes are equally-likely then the probability of the event is just the number of outcomes in the event divided by the number of outcomes in the sample space.

The Second Experiment

Suppose we draw out two balls.



How likely is it that we will draw out one white ball and one black ball?

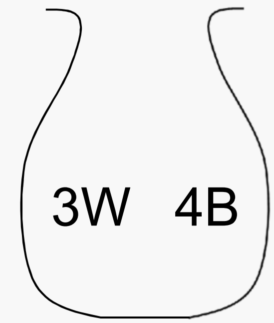
We need a different model:

- The possible outcomes are pairs of balls.
- There are a lot of such pairs:

  $\{w_1, w_2\}$ $\{w_1, w_3\}$ $\{w_2, w_3\}$
  $\{w_1, b_1\}$ $\{w_1, b_2\}$ $\{w_1, b_3\}$ $\{w_1, b_4\}$ $\{w_2, b_1\}$
  $\{w_2, b_2\}$ $\{w_2, b_3\}$ $\{w_2, b_4\}$
  $\{w_3, b_1\}$ $\{w_3, b_2\}$ $\{w_3, b_3\}$ $\{w_3, b_4\}$
  $\{b_1, b_2\}$ $\{b_1, b_3\}$ $\{b_1, b_4\}$
  $\{b_2, b_3\}$ $\{b_2, b_4\}$
  $\{b_3, b_4\}$
- The outcomes are equally-likely.

The Second Experiment

Suppose we draw out two balls.



How likely is it that we will draw out one white ball and one black ball?

The event is:

$\{ \{w_1, b_1\}, \{w_1, b_2\}, \{w_1, b_3\}, \{w_1, b_4\}, \{w_2, b_1\}, \{w_2, b_2\}, \{w_2, b_3\}, \{w_2, b_4\}, \{w_3, b_1\}, \{w_3, b_2\}, \{w_3, b_3\}, \{w_3, b_4\} \}$

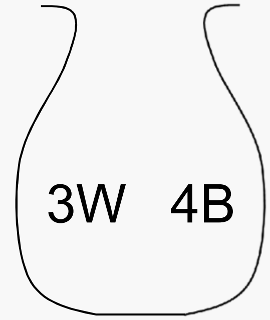
So, the probability of drawing one white ball and one black ball is:

$$P(1W1B) = 12/21 = 4/7$$

QTP: what is the probability of drawing two black balls?

The Third Experiment

Suppose we draw out two balls, <u>one at a time</u>.



Does that make any difference?

Sure. Now the sample space is the set of all ordered pairs of balls:

(w_1, w_2) (w_2, w_1) (w_2, w_3) (w_3, w_2)

. . .

So, we now have two different possible outcomes for each one we had when the balls were drawn together.

Interestingly, this doesn't change the probabilities of events that make sense in both experiments:

$$P(1W1B) = 24/42 = 4/7$$

QTP: what new events would this experiment entail?

Recap So Far

Given an experiment with possible outcomes $e_1, e_2, \ldots, e_n$:

- The set of all possible outcomes is the sample space.
- There is a probability function p() that assigns a real number to each of the possible outcomes in the sample space.
- For each outcome, e_k, $0 <= p(e_k) <= 1$.
- $p(e_1) + p(e_2) + p(e_3) + \ldots + p(e_n) = 1$
- An event is a subset of the sample space.
- If E is an event, then the probability of E is the sum of the probabilities of the individual outcomes in E.

A Pair of Bears

You see two bears, one black and one white, and wonder:

- What is the probability that both bears are males?
- What is the probability that both bears are males, IF you are correctly told that one of them is male?
- What is the probability that both bears are males, IF you are correctly told that the white bear is male?

Now, we cannot proceed unless we know something about bears: if we randomly select a bear from among all bears, what is the probability that bear will be male?

In the absence of any experts on the family Ursidae, we must make an assumption… we'll assume the probability our random bear is male is1/2.

A Pair of Bears – Q1

You see two bears, one black and one white, and wonder:

- What is the probability that both bears are males?

Our experiment is to determine the genders of the two bears.

The possible outcomes (listing the black bear first) are:

FF FM MF MM

Now, under our assumption, each of these outcomes is equally-likely, and therefore the answer to our first question is 1/4.

QTP: what is the probability that the bears are not both males?

A Pair of Bears

You see two bears, one black and one white, and wonder:

- What is the probability that both bears are males, IF you are correctly told that one of them is male?

- What is the probability that both bears are males, IF you are correctly told that the white bear is male?

Do you see, intuitively, why these two questions lead to different answers?

A Pair of Bears – Q2

You see two bears, one black and one white, and wonder:

- What is the probability that both bears are males, IF you are correctly told that (at least) one of them is male?

Given the fact that (at least) one of the bears is male, we find that the sample space has changed:

FM MF MM

Again, under our assumption these are equally-likely, and so the probability that both bears are males is 1/3.

QTP: what if we were told that at most one of the bears was male?

A Pair of Bears – Q3

You see two bears, one black and one white, and wonder:

- What is the probability that both bears are males, IF you are correctly told that the white bear is male?

Now, we see that the sample space is:

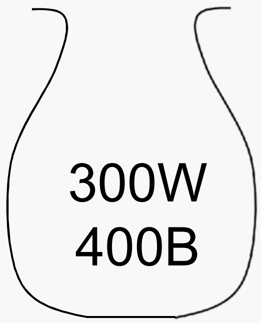
FM MM

(Remember we are listing the black bear first.)

So, now the probability both bears are males is 1/2.

A Challenge

Suppose we have an urn that contains 300 white balls and 400 black balls:



We certainly would not want to have to list all the outcomes that corresponded to the event of drawing 1 white ball and 1 black ball now.

We need a more effective way to count…

Addition Rule

Suppose you are supposed to select and carry out one of a collection of N tasks, and there are T_K different ways to carry out task K.

Then the number of different ways to select and carry out one of the tasks is just the sum of the numbers of ways to carry out the individual tasks.

That is, the number of ways to complete the sequence of tasks is:

$$\sum_{K=1}^{N} T_K = T_1 + T_2 + \cdots + T_N$$

Multiplication Rule

Suppose you are supposed to carry out a sequence of *N* tasks, and there are T_K different ways to carry out task *K* no matter how the other tasks are carried out.

Then the number of different ways to complete the sequence of tasks is just the product of the numbers of ways to carry out the individual tasks.

That is, the number of ways to complete the sequence of tasks is:

$$\prod_{K=1}^{N} T_K = T_1 \cdot T_2 \cdot \cdots \cdot T_N$$

Counting Permutations

Suppose you are given a collection of *N* different objects (you can tell them apart).

Then an arrangement of all *N* of the objects in a row is called a *permutation* of the set of objects.

The number of different permutations of *N* things equals:

$$\prod_{K=1}^{N} K = 1 \cdot 2 \cdots N = N!$$

The last expression is read "*N* factorial", and is very convenient shorthand for the expressions that precede it.

Counting Permutations

Suppose you are given a collection of *N* different objects (you can tell them apart).

Then an arrangement of a subset of *R* of the objects in a row is called a *permutation of the N objects, taken R at a time*.

The number of different permutations of *N* things taken *R* at a time equals:

$$P(N,R) = N \cdot (N-1) \cdot \cdots \cdot (N-R+1) = \frac{N!}{(N-R)!}$$

Counting Combinations

Suppose you are given a collection of *N* different objects (you can tell them apart).

Then a selection of a subset of *R* of the *N* objects is called a *combination of the N objects taken R at a time*.

The number of different combinations of *N* things taken *R* at a time equals:

$$C(N,R) = \frac{P(N,R)}{R!} = \frac{N!}{R!(N-R)!}$$

This makes intuitive sense because there are *R!* permutations of *R* things, and each of those corresponds to the same combination.

The more common notation is:

$$C(N,R) = \binom{N}{R}$$