You may work in pairs or purely individually for this assignment. Prepare your answers to the following questions in a plain ASCII text file or MS Word document. Submit your file to the Curator system by the posted deadline for this assignment. No late submissions will be accepted. If you work in pairs, list the names and email PIDs of both members at the beginning of the file, and submit your solution under only one PID. No other formats will be graded.

For this assignment, you may (and are encouraged to) work in pairs; if you do so, you must also write your solutions in such a way that it is clear how each member contributed to deriving the solution.

You will submit your answers to the Curator System (www.cs.vt.edu/curator) under the heading OOC06.

1. [60 points] Solve each of the following recurrence relations:

a) $a_0 = 1,\ a_n = 3a_{n-1} + 4 \text{ for } n > 0$

This is a nonhomogeneous linear recurrence; apply the result from the first section of the mathematical notes to see that the solution will be given by:

$$a_n = 3^n\left(1 + \frac{4}{3} + \frac{4}{3^2} + \frac{4}{3^3} + \cdots + \frac{4}{3^n}\right)$$

Note: with reference to the notes, the coefficients of the linear relationship are constants, 3 and 4, respectively. Now we need to simplify this result:

$$\begin{aligned} a_n &= 3^n\left(1 + \frac{4}{3} + \frac{4}{3^2} + \frac{4}{3^3} + \cdots + \frac{4}{3^n}\right) \\ &= 3^n + 4\cdot 3^{n-1} + 4\cdot 3^{n-2} + 4\cdot 3^{n-3} + \cdots + 4 \\ &= 3^n + 4\left(3^{n-1} + 3^{n-2} + 3^{n-3} + \cdots + 1\right) \end{aligned}$$

Now we need a basic algebraic identity:

And so:

$$\begin{aligned} a_n &= 3^n + 4\frac{3^n - 1}{3 - 1} \\ &= 3^n + 2\cdot 3^n - 2 \\ &= 3^{n+1} - 2 \end{aligned}$$

for all n.

b) $b_0 = 1,\ b_1 = 1,\ b_n = 3b_{n-1} + 4b_{n-2}$ for $n > 1$

This is a homogeneous recurrence with constant coefficients; apply the result from the second section of the mathematical notes. The characteristic equation is:

$$\tau^2 = 3\tau + 4$$

Solving this, we get the distinct roots -1 and 4, and therefore the general solution of the recurrence will be of the form:

$$b_n = \alpha(-1)^n + \beta \cdot 4^n$$

Now we must find values for the coefficients, α and β, so that the initial conditions are also satisfied:

$$b_0 = \alpha(-1)^0 + \beta \cdot 4^0 = \alpha + \beta = 1 \text{ so } \beta = 1 - \alpha$$

And:

$$b_1 = \alpha(-1)^1 + \beta \cdot 4^1 = -\alpha + 4\beta = 1$$

Substituting yields:

$$-\alpha + 4(1-\alpha) = 1$$
$$-5\alpha = -3$$
$$\alpha = 3/5,\ \beta = 2/5$$

So, the specific solution is:

$$b_n = \frac{3}{5}(-1)^n + \frac{2}{5} \cdot 4^n$$

for all n.

c) $c_0 = 2,\ c_1 = 2,\ c_n = 2c_{n-1} - c_{n-2}$ for $n > 1$

This is also a homogeneous recurrence with constant coefficients; apply the result from the second section of the mathematical notes. The characteristic equation is:

$$\tau^2 = 2\tau - 1$$

Solving this, we get the repeated roots 1 and 1, and therefore the general solution of the recurrence will be of the form:

$$c_n = \alpha \cdot 1^n + \beta n \cdot 1^{n-1} = \alpha + \beta n$$

Now we must find values for the coefficients, α and β, so that the initial conditions are also satisfied:

$$c_0 = \alpha = 2 \text{ and } c_1 = \alpha + \beta = 2 + \beta = 2, \text{ so } \beta = 0$$

So, the specific solution is:

$$c_n = 2$$

for all n.

d) $d_0 = 1,\ d_1 = 2,\ d_n = 2d_{n-1} + 2d_{n-2}$ for $n > 1$

This is also a homogeneous recurrence with constant coefficients; apply the result from the second section of the mathematical notes. The characteristic equation is:

$$\tau^2 = 2\tau + 2$$

Solving this, we get the distinct roots:

$$\tau = \frac{2 \pm \sqrt{4+8}}{2} = 1 \pm \sqrt{3}$$

Therefore the general solution of the recurrence will be of the form:

$$d_n = \alpha\left(1-\sqrt{3}\right)^n + \beta\left(1+\sqrt{3}\right)^n$$

Now we must find values for the coefficients, α and β, so that the initial conditions are also satisfied:

$$\begin{aligned} d_0 &= \alpha + \beta = 1 \\ d_1 &= \alpha\left(1-\sqrt{3}\right)^1 + \beta\left(1+\sqrt{3}\right)^1 \\ &= \alpha\left(1-\sqrt{3}\right)^1 + (1-\alpha)\left(1+\sqrt{3}\right)^1 \\ &= -2\sqrt{3}\alpha + 1 + 2\sqrt{3} = 2 \end{aligned}$$

From this, we see that:

$$\alpha = \frac{\sqrt{3}-1}{2\sqrt{3}} \text{ and } \beta = \frac{\sqrt{3}+1}{2\sqrt{3}}$$

So, the specific solution is:

$$\begin{aligned} d_n &= \frac{\sqrt{3}-1}{2\sqrt{3}}\left(1-\sqrt{3}\right)^n + \frac{\sqrt{3}+1}{2\sqrt{3}}\left(1+\sqrt{3}\right)^n \\ &= -\frac{1}{2\sqrt{3}}\left(1-\sqrt{3}\right)^{n+1} + \frac{1}{2\sqrt{3}}\left(1+\sqrt{3}\right)^{n+1} \end{aligned}$$

for all n.

2. [20 points] Suppose that you deposit \$1000 in a savings account on January 1, 2013, and that you deposit an additional \$100 in to the account on each subsequent January 1. The bank pays a fixed annual rate of 5%, deposited at the end of each year. In other words, on December 31 of each year, the bank deposits 5% of the value of the account on the preceding January 1 (including your new deposit of \$100).

Find a recurrence relation for the value of the account, P_n, after n years. Then solve that recurrence relation to obtain a non-recursive formula for P_n. Then calculate P_{20}.

We are given directly that $P_0 = 1000$. And, from the description of how the account is updated we see that:

$$P_n = 1.05P_{n-1} + 100, \text{ for } n \geq 1$$

This is a linear recurrence with constant coefficients, similar to question 1a. Applying the same technique, we obtain:

$$\begin{aligned} P_n &= (1.05)^n \left(1000 + \frac{100}{1.05} + \frac{100}{1.05^2} + \frac{100}{1.05^3} + \cdots + \frac{100}{1.05^n}\right) \\ &= 1000(1.05)^n + 100\left(1.05^{n-1} + 1.05^{n-2} + 1.05^{n-3} + \cdots + 1\right) \\ &= 1000(1.05)^n + 100\frac{1.05^n - 1}{1.05 - 1} \\ &= 1000(1.05)^n + 2000(1.05)^n - 2000 \\ &= 3000(1.05)^n - 2000 \end{aligned}$$

From this, we can calculate that $P_{20} = 3000(1.05)^{20} - 2000 = 5959.89$.

3. [20 points] Find, but do not solve, a recurrence relation for the number of different ways to make a stack of *n* chips, using red, white, and blue chips, such that no two red chips are adjacent in the stack.

Let S_n be the number of ways form a stack of n chips that does not contain two adjacent red chips. Trivially, $S_0 = 0$, and $S_1 = 3$ since there are three ways to form a 1-chip stack, and none of those can possibly contain two adjacent red chips:

R B W

Now, what's S_2? We can easily enumerate the 8 possibilities:

W	B	R	B	W	W	B	R
R	R	W	W	W	B	B	B

Now, is there a pattern? Yes. We can take any stack of 1 chip and extend it to a stack of 2 chips by adding either a B or a W chip to the top, without violating the rules. And, we can take any stack of 0 chips and extend it to a stack of 2 chips by adding either of the following combinations to the top, without violating the rules and without duplicating any of the 2-chip stacks we created in the first manner:

R	R
W	B

Note that when we extend a stack of n-2 chips, the result always has a red chip on top, and when we extend a stack of n-1 chips, the result never has a red chip on top.

Hence, we can create two valid stacks of n chips from every valid stack of n-1 chips, and create two valid stacks of n chips from every valid stack of n-2 chips, without creating any duplicates.

So, we get the following recurrence:

$$S_n = \begin{cases} 1 & n = 0 \\ 3 & n = 1 \\ 2a_{n-1} + 2a_{n-2} & n > 1 \end{cases}$$