<u>You may work in pairs or purely individually for this assignment.</u> Prepare your answers to the following questions in a plain ASCII text file or MS Word document. Submit your file to the Curator system by the posted deadline for this assignment. No late submissions will be accepted. If you work in pairs, list the names and email PIDs of both members at the beginning of the file, and submit your solution under only one PID. No other formats will be graded.

For this assignment, you may (and are encouraged to) work in pairs; if you do so, you must also write your solutions in such a way that it is clear how each member contributed to deriving the solution.

You will submit your answers to the Curator System (www.cs.vt.edu/curator) under the heading `OOC05`.

For each question below, the quality of your explanation of how you derived the answer will carry as least as much weight as whether you've stated a correct solution. For each problem, apply one of the heuristics discussed in the course notes, and explain how you applied it.

1. [20 points] Solve the following cryptarithm. Explain exactly how you deduced the solution.

```
TELL
TALE
TELL
TALE
----
LATE
```

Remember that each letter stands for a different digit (base-10), and that there are no leading zeros.

From the 1s column, we see that 2L + 2E = E + C0, or 2L + E = 10*C0, where C0 is or 1 or 2. Note that 0 is impossible since that would require that L and E both be 0, and 3 is impossible because 2L + E can be no larger than 26. This implies E must be even, but nothing else seems to follow immediately.

From the 10s column, we see that 4L + C0 = T + 10*C1, where C1 is 0 or 1 or 2 or 3 and C0 is either 1 or 2.

From the 1000s column, we see that 4T + C2 = L, where C2 is the carry from the 100s column. So, T can be no larger than 2 (or there would be a carry from the 1000s column), and T cannot be 0 since leading 0s are not allowed. So, T is either 1 or 2.

Returning to the observation from the 10s column, if T = 1 then 4L + C0 must be 1 or 11 or 21 or 31. If C0 = 1 then 4L + C0 must be 1 or 21, which implies L would be either 0 (impossible since L is a leading digit in the sum) or 5. If C0 = 2, then 4L + C0 cannot be any of 1, 11, 21 or 31. So, if T = 1 then L must be 5. On the other hand, if T = 2, then 4L + C0 must be 2 or 12 or 22 or 32, and a similar analysis shows that L would then have to be 5. So, L = 5 in any case.

But then, from the observations about the 1s column, since 2L + E = 10*C0, we must have that E = 10*(C0-1) and that's only possible if E = 0 and C0 = 1.

So, in the 10s column we must have 4*5 + 1 = T + 20, and so % = 1.

So now the original problem reduces to:

```
1055
1A50
1055
1A50
-----
5A10
```

Now, we see that the carry from the 100s column must be 1, and so in the 100s column we must have that 2 + 2A = A + 10, and so A = 8.

So, the solution is:

```
1055
1850
1055
1850
----
5810
```

In this analysis, I used the notion of a special feature in my analysis of the sum in each column.

2. [20 points] Solve the following classic puzzle from Sam Loyd (1841-1911). Explain your logic clearly.

> Here is a puzzle known as the Covent Garden Problem, which appeared in London half a century ago, accompanied by the somewhat surprising assertion that it had mystified the best mathematicians of England:
>
> Mrs. Smith and Mrs. Jones had equal number of apples but Mrs. Jones had larger fruits and was selling hers at the rate of two for a penny, while Mrs. Smith sold three of hers for a penny.
>
> Mrs. Smith was for some reason called away and asked Mrs. Jones to dispose of her stock. Upon accepting the responsibility of disposing her friend's stock, Mrs. Jones mixed them together and sold them of at the rate of five apples for two pence.
>
> When Mrs. Smith returned the next day the apples had all been disposed of, but when they came to divide the proceeds they found that they were just seven pence short, and it is this shortage in the apple or financial market which has disturbed the mathematical equilibrium for such a long period.
>
> Supposing that they divided the money equally, each taking one-half, the problem is to tell just how much money Mrs. Jones lost by the unfortunate partnership?

Note: one thing that may not be clear from the problem statement is that Mrs Smith was called away before either of them had sold any apples.

Here's Loyd's solution:

> **The mixed apples were sold of at the rate of five apples for two pence. So they must have had a multiple of five i.e. 5, 10, 15, 20, 25, 30,…, 60, 65,… etc apples.**
>
> **(WDM: however, if Mrs Smith was going to sell hers at 3 for a penny, the number of apples she had must have also been a multiple of 3; and similarly the number of apples Mrs Jones had must have also been a multiple of 2; and since they had the same number of apples, that must have been a multiple of both 2 and 3, and also of 5. The smallest number of apples each could have had is therefore 30.)**
>
> **But the minimum number of apples they could have together is 60; so that 30 would have been of Mrs. Smith's that would fetch her 10 (an integer) pence and the other 30 of Mrs. Jones's that would fetch her 15 (also an integer) pence.**
>
> **When sold separately it would fetch them 10+15=25 pence altogether. But when sold together it would fetch them 60X2/5=24 pence i.e. a loss of one (25-24=1) pence.**
>
> **Since they lost 7 pence altogether; they had altogether 60X7=420 apples that fetched them only 420X2/5=168 pence and they shared 84 pence each of them. But Mrs. Jones could sell her 420/2=210 apples for 210/2=105 pence so she lost "21 pence".**
>
> **I applied the notion of a special feature in taking advantage of the fact that the number of apples had to be a multiple of certain divisors, and the fact that each woman had the same number of apples.**

3. [20 points] A 32-page newspaper (think of a larger Collegiate Times) consists of 8 full sheets of newsprint. One of these sheets is selected at random, and the four page numbers that appear on it are added together. Can you determine the resulting sum? If so, what is it, and why?

The key is to recognize the relationship among the four page numbers that occur on each sheet. Consider the sheets beginning with the one containing page 1 of the paper:

1^{st} sheet: 1, 2, 31, 32
2^{nd} sheet: 3, 4, 29, 30
3^{rd} sheet: 5, 6, 27, 28

We see that the i-th sheet will have pages numbered $2i-1$, $2i$, $32-2i+1$ and $32-2i+2$.

So, if we add the four page numbers, the variable i cancels out and we obtain the sum 66.

I would say I took advantage of symmetry in this problem, since there is a natural symmetry between the smaller and larger page numbers that occur on each sheet.

4. [20 points] The CSI team has discovered a body in a storage unit whose interior dimensions are 10 feet by 10 feet by 10 feet. Due to the maturity of the corpse, there are 2001 flies inside the storage unit. Prove that, at any moment, there must be a group of three or more flies that would fit inside a sphere with radius 1 foot.

The storage unit can be viewed as a block of 10x10x10 1-foot cubes. So, there are 1000 cubical spaces within the storage unit. If each of them contained more than 2 flies then there could be no more than 2000 flies within the storage unit. Therefore, there must be a cube that contains three or more flies.

(This is just an application of the general form of the Pigeonhole Principle.)

Now, the diagonal of a 1-foot cube is sqrt(3) feet long, which is smaller than 2 feet. Therefore, a 1-foot cube will fit within a sphere of radius 1 foot (or diameter 2 feet).

So, there is a sphere of radius 1 foot that contains three or more flies.

In this case, I obviously applied the Pigeonhole Principle.

5. [20 points] The two-player game of Pargoo begins with two piles of small stones, one with 15 stones, and one with 20 stones. On each turn, the current player must choose a pile of stones and divide it into two nonempty smaller piles. Aside from the rule that a pile may not be empty, there are no restrictions on how many stones may be in each of the piles a player creates. The loser is the player who cannot carry out a valid move.

What strategy, if any, can the player who goes first use to guarantee that he wins? What strategy, if any, can the player who goes second use to guarantee that she wins?

Initially, there are 2 piles of stones.

No matter how a player makes a move, that move will increase the number of piles by exactly 1.

The losing player must be confronted with the situation that every pile contains exactly 1 stone, and therefore there must be 35 piles at that point in the game.

Since the number of stones is always odd after the first player makes a move, and always even after the second player makes a move, the first player must be the one who made the move that resulted in 35 piles of stones.

So, the first player always wins, no matter how she/he plays.

There is an invariant in this problem, namely 2 plus the number of moves that have been made, which will always equal the current number of piles. That, in turn, leads to a natural parity argument.