Quantification

Quantifier --- a type of determiner, such as *all* or *some* or *many*, that indicates quantity.

All men are mortal.

Every prime integer larger than 2 is odd.

Some headlines exhibit ambiguity.

Most mammals are terrestrial rather than aquatic.

A few birds are naturally flightless.

Quantification: Domains

When a quantifier is used, there should usually be an associated *domain*; that is, we should know the collection of "things" that are the object of the quantifier.

> Some student correctly answered every question ~~on the assessment test~~.

In the absence of a specific domain, the sentence may be ambiguous or meaningless.

> Every x is prime.

Quantification: Order Matters

Every question on the assessment test was answered correctly by some student in the class.

Some student in the class correctly answered every question on the assessment test.

For every question on the assessment test, there exists a student in the class who answered that question correctly.

There exists a student in the class who answered every question on the assessment test correctly.

Quantification: Order Matters

For every integer x, there is some integer y such that x > y.

There is some integer y such that, for every integer x, x > y.

Verification: existential quantifiers

In order to show that an existentially-quantified statement is true, we merely must find some value in the domain for which the statement is true:

> There is an integer that equals twice the sum of its digits.

Consider 18.

Of course, finding a confirming example is not always trivial.

Verification: universal quantifiers

To show a universally-quantified statement is true, we must verify that the statement is true for every value in the domain:

> Every sentence on this slide contains an odd number of words.

Consider each sentence and count the words.

Of course, checking every value is not always possible.

Verification: universal quantifiers

In order to show that a universally-quantified statement is true, we must verify that the statement is true for every value in the domain:

For every even integer m, m*m + 1 is odd.

There are infinitely many even integers, so we cannot check them all.

Instead, we must make a formal argument (proof)… but that will be later.

Falsification: universal quantifiers

In order to show that a universally-quantified statement is false, we only need to find one instance of values from the domain for which the statement is false:

> Every prime integer x is odd.

This is false because 2 is a prime integer and 2 is not odd.

Falsification: existential quantifiers

In order to show that an existentially-quantified statement is false, we must show that for every value from the domain, the statement is false:

> There is a volume in the 1980 edition of the Encyclopedia Britannica that has less than 100 pages.

I can show this is false by examining my copy of the 1980 edition of the Encyclopedia Britannica and verifying that every volume has (far) more than 100 pages… assuming that all copies of this edition are the same, which seems a fair assumption.

Falsification: existential quantifiers

But sometimes the domain is not finite:

> There is an integer larger than 10 that is equal to the product of its digits.

Now, this is harder.

We obviously cannot check every value in the domain.

This would call for a formal proof that the assumption such an integer exists implies some contradiction…

… or perhaps the statement is, in fact, true.

(A quick check with a short C program verifies no integer less than one billion has the property… but that proves nothing.)

Quantification: Negation

What is the negation of:

Every prime integer is odd.

Clearly, the negation must deny the assertion that every prime integer has the stated property.

That means the negation must merely claim that there is some prime integer that does not have the stated property:

There is a prime integer that is not odd.

Quantification: Negation

What is the negation of:

Some three-digit integer equals the product of its digits.

Clearly, the negation must deny the assertion that there is an integer with three digits has the stated property.

That means the negation must merely claim that there is no integer with three digits that does have the stated property:

No three-digit integer equals the product of its digits.

Alternatively:

Every three-digit integer is unequal to the product of its digits.

Quantification: Negation and the Domain

Note that negating does not alter the domain:

All dogs are endotherms.

Some dogs are not endotherms.

You cannot deny the truth of the first statement by talking about the wrong set of things:

Some iguanas are not endotherms.

Quantification: Formal Negation

So:

"not (every x in D has property P)" is
"some x in D does not have property P"

and

"not (some x in D has property P)" is
"no x in D has property P"

Examples

Of course, it gets more interesting; negate these:

For every natural number N, there is a natural number M such that M < N.

There is a natural number M such that, for every natural number N, M ≤ N.

For every integer N, if N is a multiple of 2 then $\log_2(N)$ is an integer.

For every real number ε > 0, there is a real number N > 0 such that whenever x is a real number such that x > N, 1/x < ε.

Answers

The negations:

There is a natural number N such that for every natural number M, M ≥ N.

For every natural number M, there is a natural number N such that M > N.

There is an integer N such that N is a multiple of 2 and $\log_2(N)$ is not an integer.

There is a real number ε > 0 such that, for every real number N > 0, there is a real number x such that x > N, but 1/x ≥ ε.