Problems for CS 2104 Homework Assignment 8
October 28, 2008
50 Points

Problem 1. A bar of candy often comes organized into an m by n grid of rectangles so that it can be broken into smaller pieces. See the top of Figure 1, where the candy bar is a 3 by 4 grid of rectangles. In one move, you can pick up any piece of candy that has at least two rectangles and break it into two pieces along a horizontal or vertical line. See the two sample moves in Figure 1.

Prove that, no matter how you move, it will take the same number of moves to break the candy bar down to its individual rectangles. If the bar consists of $m \times n$ rectangles, what is that number of moves?

Problem 2. Let O be the origin of the plane. Imagine that five points, not equal to O, are placed on the plane. Prove that there are two of those points, P and Q, such that the angle $\angle POQ$ is acute.

View this problem as an application of the pigeonhole principle. What are the pigeonholes? What are the pigeons? Complete the argument.

Problem 3. Six people enter a room. Either there are three people who know each other or there are three people who are strangers to each other.

Prove the above statement. Where does a variant of the pigeonhole principle come in?

Problem 4. You are given a matrix of real numbers, such as

$$M = \begin{pmatrix} -4 & 0 & -9 & 17 & -10 \\ 2 & -2 & -5 & 6 & -7 \end{pmatrix}.$$

In one move, you can select one row or one column of M and invert all the signs in that row or column. For example, choosing the first row results in

$$M' = \begin{pmatrix} 4 & 0 & 9 & -17 & 10 \\ 2 & -2 & -5 & 6 & -7 \end{pmatrix}.$$

Prove that, for any $m \times n$ matrix M, there is a sequence of moves that will result in a matrix with every row sum and every column sum non-negative. What heuristic(s) did you use?

Problem 5. A man walks one mile due north, then one mile due east, and finally one mile due south. In class, using the going-to-extremes heuristic, we decided that the man might have started at the south pole.

Again use the going-to-extremes heuristic. Show that there is an infinite set of points where the man might have started. Once you have that set of points, show that there is a second infinite set of points where the man might have started!

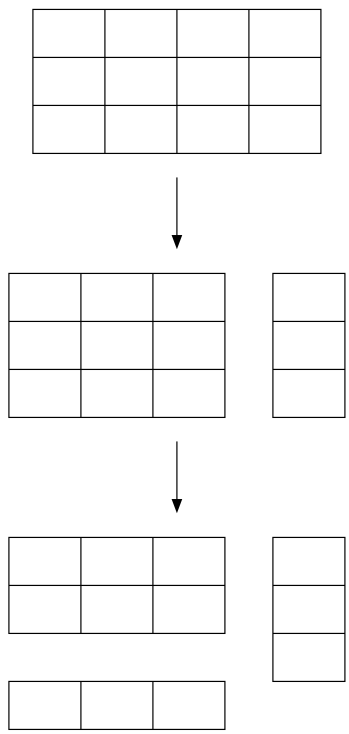

Figure 1: Sample moves for breaking a bar of candy.