Review of Mathematical Prerequisites

- Sets, relations, and functions
- Cardinality — countable and uncountable sets
- Recursive or inductive definitions
- Proof by induction
- Directed graphs
Reading the Text

• Notational conventions in preface

• Defined terms are **boldface**

• Understanding definitions
  – Cyclic and acyclic graphs

• Understanding examples

• Theorems and proofs

• Asking questions
Sets

A **set** is a collection of objects. A set may be described by

- List
- Property
- Operations on other sets
- Construction

The **power set** of $X$ is the set of all subsets of $X$:

\[ \mathcal{P}(X) = \{Y \mid Y \subseteq X\}. \]

This is also written

\[ \mathcal{P}(X) = 2^X. \]
Relations

The cartesian product of sets $X_1, X_2, \ldots X_n$ is

$$X_1 \times X_2 \times \cdots \times X_n$$

$$= \{ (x_1, x_2, \ldots x_n) \mid x_1 \in X_1, \quad x_2 \in X_2, \ldots, x_n \in X_n \}.$$ 

The ordered list $(x_1, x_2, \ldots x_n)$ or $[x_1, x_2, \ldots x_n]$ is an $n$-tuple.

A relation on $X_1, X_2, \ldots X_n$ is a subset of $X_1 \times X_2 \times \cdots \times X_n$.

- Binary relations

- Ordering relations

- Equivalence relations
Functions

A function $f : X \rightarrow Y$ is a relation on $X$ and $Y$ such that if $x \in X$, $y_1, y_2 \in Y$, and $(x, y_1), (x, y_2) \in f$, then $y_1 = y_2$.

We write $f(x) = y_1$.

If $f(x)$ is defined for all $x \in X$, then $f$ is a total function. (This is what we typically think of as a function.) The default is a total function.

If $f$ is not total, then it is a partial function.

EXAMPLE. What is an example of a partial function?
Cardinality

Number of elements in a set

If there is a one-to-one, onto function $f: \{1, 2, \ldots, n\} \rightarrow X$, then $X$ has cardinality $n$, written $\text{card}(X) = n$. $X$ is finite.

If $Y$ is not finite, then there is a one-to-one function $g: Y \rightarrow Y$ that is not onto. $Y$ is infinite.

If there is a one-to-one, onto function $f: X \rightarrow Y$, then $X$ and $Y$ have the same cardinality, written $\text{card}(X) = \text{card}(Y)$.

If there is a one-to-one function $f: X \rightarrow Y$, then write $\text{card}(X) \leq \text{card}(Y)$.

**Schröder-Bernstein Theorem.** If $\text{card}(X) \leq \text{card}(Y)$ and $\text{card}(Y) \leq \text{card}(X)$, then $\text{card}(X) = \text{card}(Y)$. 
Countable and Uncountable

A set that has the same cardinality as the natural numbers

\[ N = \{0, 1, 2, \ldots, \} \]

is denumerable.

Every infinite subset of \( N \) is denumerable.

\( X \) is countable if it is finite or denumerable.

\( Y \) is uncountable if it not countable.

EXAMPLE. The set of real numbers is uncountable.
Proof of Countability

The cartesian product $\mathbb{N} \times \mathbb{N}$ is countable.

\begin{center}
\begin{tabular}{cccccccc}
 & & & & & & & \\
 & & & & & & & \\
5 & & & & & & & \\
&  & & & & & & \\
4 & & & & & & & \\
&  & & & & & & \\
3 & & & & & & & \\
&  & & & & & & \\
2 & & & & & & & \\
&  & & & & & & \\
1 & & & & & & & \\
&  & & & & & & \\
0 & & & & & & & \\
&  & & & & & & \\
0 & 1 & 2 & 3 & 4 & 5 & \\
& & & & & & \\
\end{tabular}
\end{center}

**Technique:** Construct a one-to-one, onto function $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$.

More generally, the cartesian product of a finite number of countable sets is countable.
Other Countable Sets

Theorem 1.4.4 (iii). The set of finite subsets of a countable set is countable.

Similar result. Suppose $\Sigma$ is a finite alphabet (e.g., ASCII). Let $\Sigma^*$ be the set of all finite strings of characters from $\Sigma$. Then $\Sigma^*$ is countable.
Proof of Uncountability

The set of functions

\[ N^N = \{ f : N \to N \} \]

is uncountable.

Technique. Proof by Diagonalization:

- Assume \( \text{card}(N) = \text{card}(N^N) \).

- Let \( g : N \to N^N \) be a one-to-one, onto function.

- Construct a function \( f \) that is not in the range of \( g \).

- This is a contradiction. Conclude that \( \text{card}(N) \neq \text{card}(N^N) \).
Using Diagonalization

Corollary. There are more functions than there are programs to compute them.

APPLICATION. Prove that, for any set $X$,

$$\text{card}(X) < \text{card}(\mathcal{P}(X)).$$
Recursive Definitions

• Method used to define a set $X$

• Explains how to *generate* an element of $X$

• Especially useful if $X$ is infinite

• Also called *inductive definitions*
Form of a Recursive Definition

Three components:

1. **Basis:** A finite set $X_0$ of “basic” elements of $X$.

2. **Recursive (or inductive) step:** Operations that can be used to construct new elements of $X$ from known elements of $X$.

3. **Closure (disclaimer):** The only elements of $X$ are those in $X_0$ and those that can be gotten from $X_0$ by a finite number of applications of the operations in component 2.
Example of a Recursive Definition

Define

\[ Y = \{3n \mid n \in \mathbb{N}\} \]

recursively.

1. **Basis:** The minimum basis set consists of 0:

\[ Y_0 = \{0\}. \]

2. **Recursive step:** If \( y \in Y \), then \( y + 3 \in Y \).

3. **Closure:** The only elements of \( Y \) are those in \( Y_0 \) and those that can be gotten from \( Y_0 \) by a finite number of applications of the recursive step.
Another Example

First Definition of a Tree. An undirected tree is an undirected graph that is connected and that contains no cycle.

Second Definition of a Tree. The set $Z$ of undirected trees is defined recursively by

1. **Basis:** The basis set $Z_0$ consists of every undirected graph having a single vertex and no edges.

2. **Recursive step:** If $T$ is a tree, then the addition of a new vertex $v$ and an edge from $v$ to any vertex of $T$ results in a tree.

3. **Closure:** The only elements of $Z$ are those in $Z_0$ and those that can be gotten from $Z_0$ by a finite number of applications of the recursive step.
Nested Sequence of Sets

For each $i \in \mathbb{N}$, define $X_i$ to be the set of elements that can be gotten from $X_0$ by $i$ or fewer applications of the recursive step.

Then, for $i > 0$,

$$X_i = X_{i-1} \cup \{x \mid x \text{ can be obtained from elements in } X_{i-1} \text{ by one application of the recursive step}\}.$$

Clearly,

$$X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_{i-1} \subset X_i \subset \cdots$$

and

$$X = \bigcup_{i=0}^{\infty} X_i.$$

**EXERCISE.** What are the nested sequences for each of the previous two examples?
Proof by Induction

Prove a property $P$ of the elements of a set $X$ that is defined recursively.

An **inductive argument** is structured as follows:

1. **Basis:** Prove $P$ for all $x \in X_0$.

2. **Inductive hypothesis:** Assume that $P$ is true for all $x$ in $X_0, X_1, \ldots, X_i$, where $i \geq 0$.

3. **Inductive step:** Prove that the inductive hypothesis implies that $P$ holds for all $x \in X_{i+1}$.

By the Principle of Mathematical Induction, $P$ holds for all $x \in X$. 

CS 4114 Lecture Notes
Example of Inductive Proof

Show that the two definitions of $Y$ given on slide 14 are equivalent.
Another Example

Show that the two definitions of $\mathbb{Z}$ given on slide 15 are equivalent.
Directed Graphs

A directed graph $G = (V, A)$ consists of a set of nodes $V$ and a set of arcs (or edges) $A \subset V \times V$.

So a directed graph is just a binary relation on $V$.

**EXAMPLE.** The graph

$$G = (\{a, b, c\},$$

$$\{(a, a), (b, c), (b, a), (a, c), (c, a)\})$$

might be drawn as follows:

![Directed Graph Diagram](image-url)
Common Graph Terminology

- Indegree and outdegree
- Path
- Cycle
- Acyclic
Ordered Trees

A directed tree \( T = (V, A) \) is a directed graph having a unique node \( r \) such that every node is reached from \( r \) by a unique path.

Every node either has one or more children or is a leaf.

\( T \) is an ordered tree if the children of each internal node (non-leaf) are given a fixed order.

**EXAMPLE.**

![Diagram of an ordered tree](image-url)
Concluding Exercise

**EXERCISE.** Give a recursive definition of an ordered tree.