Algebraic Graph Theory on Hypergraphs

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Introduction

• What is Algebraic Graph Theory? Why do we care?
• Graph structure vs. Graph algorithms
Spectral Graph Theory- Simple Graphs

• Adjacency Matrix
  – Characteristic Polynomial
  – Trace-Eigenvalue Proof
  – Use of eigenvalues to quickly study graph structure
Linear Algebra and Hypergraphs

• No clear definition for adjacency matrix
• Module over Ring $R$ (called $R$-Module)
  – Two operations: $+$ and $\ast$
  – Abelian Group on $+$
  – $\ast$: $M \times R \to M$
  – Multiplication distributes and is associative
• Tensor Algebra
  – Given $M$ and $N$ as modules over commutative rings $R$ and $S$ containing $1_R$ and $1_S$, one can form a third module $P$ such that, given $m$ in $M$ and $n$ in $N$, $mn$ is in $P$. 
Hypermatrices

- Hypermatrix - Tensor over specific basis
- Matrix $M_{nxm}$ in $F^{nxm}$, while vector $m_{nm}$ in $F^{nm}$
- Hyper matrix of form $F^{a \times b \times c \times \ldots \times n}$
- $[A_1 \mid A_2 \mid A_3 \mid \ldots \mid A_n]$
Adjacency Hypermatrix

- Requires k-uniform hypergraph
- Dimension: $|V|^k$
- Analogous to a multi-dimensional array
  - MatrixA[1][1][3] = 1: Edge containing exactly \{v_1, v_3\}
  - MatrixB[2][5][7][6] = 0: No edge containing exactly \{v_2, v_5, v_6, v_7\}
- Symmetry- Elements given by permutations of index set have same value.
- Analogue to Square Matrices- Cubical (think $Q_n$)
Hyperdeterminant and Eigenvalues

• Over field $\mathbb{C}$, $\lambda$ is an eigenvalue of hypermatrix $A$ if the following is satisfied: $\det(A - \lambda I) = 0$.
• The $\det$ function is defined as a hyperdeterminant.
• Analogue of Eigenvectors ($Ax = \lambda x$): $Ax^{k-1} = \lambda x^{j,k-1}$.
• Think of $x^{k-1}$ as a basis of cardinality $|k-1|$ (i.e., $x^i$ is an index for a vector coordinate).
• Rather than single equation as in linear algebra, multilinear analogue is system of $(k-1)$ equations.